Coarse Groups

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Slogan: A coarse group is "a group where the operation is defined coarsely"

I.e. $(G, *, d)$ where $*$ is only defined up to uniformly bounded error
metric

Example 0: If $(G, \cdot)$ is a group and d is a metric on G we wish to let $* := [\cdot]$

$\boxed{\textbf{Issue} \text{ This only makes sense if } d \text{ is bi-invariant}}$

Easy examples of coarse groups:

$(G,+)$ abelian group, d left-invariant

$\rightsquigarrow$ $(G, [+], d)$ is a coarse group

Non-abelian examples

$(G, \cdot)$ finitely generated group $G = \langle S \rangle$ finite

$\boxed{d_{bw} := (\text{word metric w.r.t } \overline{S})}$ normal closure

$\rightsquigarrow$ $(G, [\cdot], d_{bw})$ is a coarse group

Observation: up to coarse equivalence, d_{bw} does not depend on S

$\Rightarrow$ $\boxed{(G, [\cdot], d_{bw}) \text{ is a canonical coarsification of } G}$

Non-group examples

$(G, \cdot)$ finitely generated group

$X \subseteq G$ approximate subgroup

$(X, [\cdot], d_{bw})$ is a coarse group

the restriction to X
is coarsely well-defined

Coarse Homomorphisms

$(G, *, d)$ and $(G', *', d')$ coarse groups

Def $\varphi : G \longrightarrow G'$ is a coarse homomorphism if

$$\varphi(g_1 * g_2) \approx \varphi(g_1) *' \varphi(g_2)$$

Observation: $(G, \cdot)$ fin. gen group

$$\varphi : (G, [\cdot], d) \longrightarrow (\mathbb{R}, [+], \|\cdot\|)$$

is a coarse hom if.f. it is a quasimorphism

Coarse Automorphisms

$$\mathrm{Aut}_{crs}(G,*,d) := \left\{ \begin{array}{c} \text{coarsely invertible coarse hom} \\ \varphi : (G,*,d) \to (G,*,d) \end{array} \right\} \Big/ \approx$$

Observation $(G,\cdot)$ group, d bi-invariant

$$\begin{tikzcd} \mathrm{Aut}(G) \arrow[r] \arrow[d] & \mathrm{Aut}_{crs}(G,[\cdot],d) \\ \mathrm{Out}(G) \arrow[ur] & \end{tikzcd}$$

Examples

1)
$$\begin{tikzcd} \mathrm{Out}(\mathbb{Z}^n) \arrow[d] \arrow[r, phantom, "\cong"] & GL(n,\mathbb{Z}) \arrow[d, bend right] \\ \mathrm{Aut}_{crs}(\mathbb{Z}^n, [+], \|\cdot\|) \arrow[r, phantom, "\cong"] & GL(n,\mathbb{R}) \end{tikzcd}$$

2) F_n free group

$$\mathrm{Out}(F_n) \longrightarrow \mathrm{Aut}_{crs}(F_n, [\cdot], d_{bw})$$

injective [Hartnick-Schweitzer]

Soft Question: What is $\mathrm{Aut}_{crs}(F_n, [\cdot], d_{bw})$?