Equidistribution and counting

in convex projective geometry

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joint work w/ Pierre-Louis BLAYAC

$$\Gamma \overset{\text{isometric}}{\curvearrowright} X$$

f.g. group   space

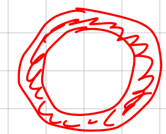
e.g. $\pi_1(M) \curvearrowright \tilde{M} \cong \mathbb{H}^2$

<u>Q</u> How fast do orbits grow?

$$\#\{\gamma \in \Gamma \mid d(o, \gamma.o) \le R\} \sim f(R) \ ?$$

$$\#\{\gamma \in \Gamma \mid d(o, \gamma.o) \le R\} \sim f(R)$$

Intuition: $f(R) = \dfrac{\text{area of } R\text{-ball}}{\text{area of fundamental domain}}$

Problem: Sometimes, edge effects dominate (e.g. in $\mathbb{H}^2$ )

Solution: On average, edge effects cancel out!

To make precise, use dynamics.

e.g.  $\underbrace{\pi_1(M)}_{\Gamma} \curvearrowright X = \widetilde{M} \cong \mathbb{H}^2$

Thm (Margulis, ... *) $\exists$ $(\mu_x)_{x \in \widetilde{M}}$ finite measures on $\partial \widetilde{M} = \partial X$ s.t.

$$\sum_{\substack{\gamma \in \Gamma \\ d(x, \gamma y) \leq R}} D_{\gamma y} \otimes D_{\gamma x} \xrightarrow[R \to \infty]{} \frac{e^R}{\mathrm{vol}(M)} (\mu_x \otimes \mu_y)$$

in $C^*(\overline{X} \times \overline{X})$

* ... Patterson, Sullivan, ..., Roblin

Cor. $\#\{\gamma \in \Gamma \mid d(o, \gamma \cdot o) \leq R\} \sim \frac{\|\mu_o\|^2}{\mathrm{vol}(M)} e^R$

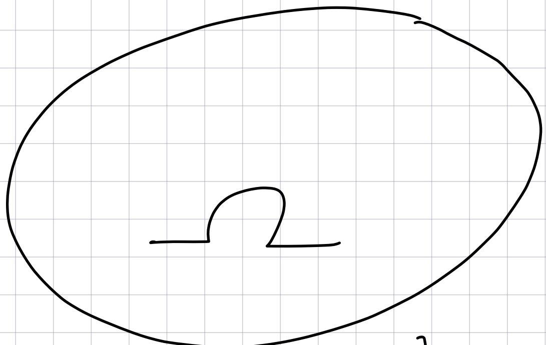
Bonus!

Q How many closed geodesics (of length up to L)?

$$\# \{ [\gamma] \mid \ell(\gamma) \leq L \} \sim g(L) \ ?$$

Thm (Margulis, ..., Roblin) $g(L) = \dfrac{e^L}{L}$

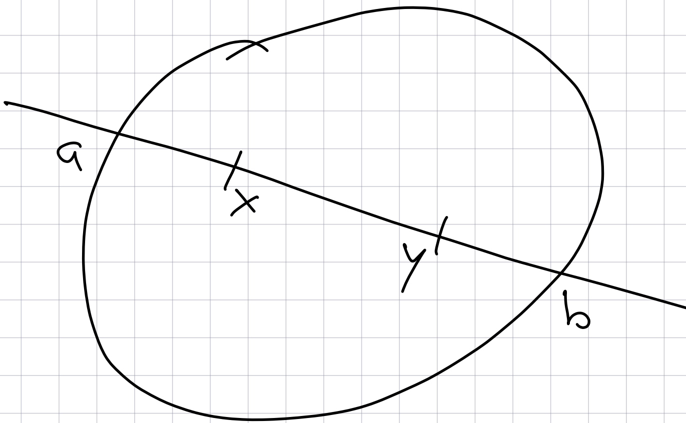
E ora, qualcosa di diverso



$\subset \mathbb{RP}^n = \mathbb{P}(\mathbb{R}^{n+1})$

properly convex domain

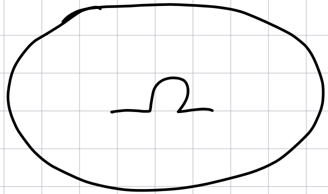
$$\mathrm{Aut}(\Omega) = \{\gamma \in PGL(\mathbb{R}^{n+1}) \mid \gamma(\Omega) = \Omega\}$$



$$d_\Omega(x, y) = \frac{1}{2} \log CR(a, x, y, b)$$

$\mathrm{Aut}(\Omega) \curvearrowright (\Omega, d_\Omega)$ isometrically

$\Gamma < \mathrm{Aut}(\Omega)$ discrete, torsion-free

$\Longrightarrow$ $\Gamma \backslash \Omega$ manifold w/ prop. convex projective structure e.g. 

= ellipsoid

$$\mathrm{Aut}(\Omega) \cong \mathrm{Isom}(\mathbb{H}^n)$$

d_Ω = hyperbolic metric

(Ω, d_Ω) = projective model of hyperbolic space

+ more!

- deformations of hyperbolic structures
- convex projective structures on nonhyperbolisable manifolds
- closely related to Anosov reps (Danciger-Guéritaud-Kassel, Zimmer)

NOT CAT(K) for any K in general,
but exhibit features of nonpositive curvature

more "negatively-curved" when

(*) $\begin{cases} \bullet\ \Omega \text{ is round} \\ \text{OR} \\ \bullet\ \Gamma \text{ is nonelementary rank-one} \end{cases}$ [M. ISLAM, A. ZIMMER]

cf. rank-one subgroup of Isom(CAT(0) space)

Given a domain Ω, $\Gamma < \mathrm{Aut}(\Omega)$ satisfying $(*)$, get

- $(\mu_x)_{x \in \Omega}$ finite measures on $\partial\Omega$
- m_Γ a measure on $\Gamma \backslash \Omega$

Thm (Blayac–Z.) If m_Γ is finite, $\exists\, \delta > 0$ s.t.

- $$\|m_\Gamma\| \sum_{\substack{\gamma \in \Gamma \\ d_\Omega(x, \gamma y) \leq R}} D_{\gamma y} \otimes D_{\gamma^{-1} x} \xrightarrow[R \to \infty]{} \frac{e^{\delta R}}{\delta} (\mu_x \otimes \mu_y)$$
- $$\#\{\gamma \in \Gamma \mid d_\Omega(o, \gamma \cdot o) \leq R\} \sim \frac{\|\mu_o\|^2}{\delta \|m_\Gamma\|} e^{\delta R}$$
- If $\Gamma \curvearrowright \Omega$ geometrically finite,
  $$\#\{\text{closed } d_\Omega\text{-geodesics of length} \leq L\} \sim \frac{e^{\delta L}}{\delta L}$$

Props all folks!

Happy to
talk more
later