Lecture 3. Enumeration of meanders and volumes of moduli spaces of quadratic differentials

Anton Zorich

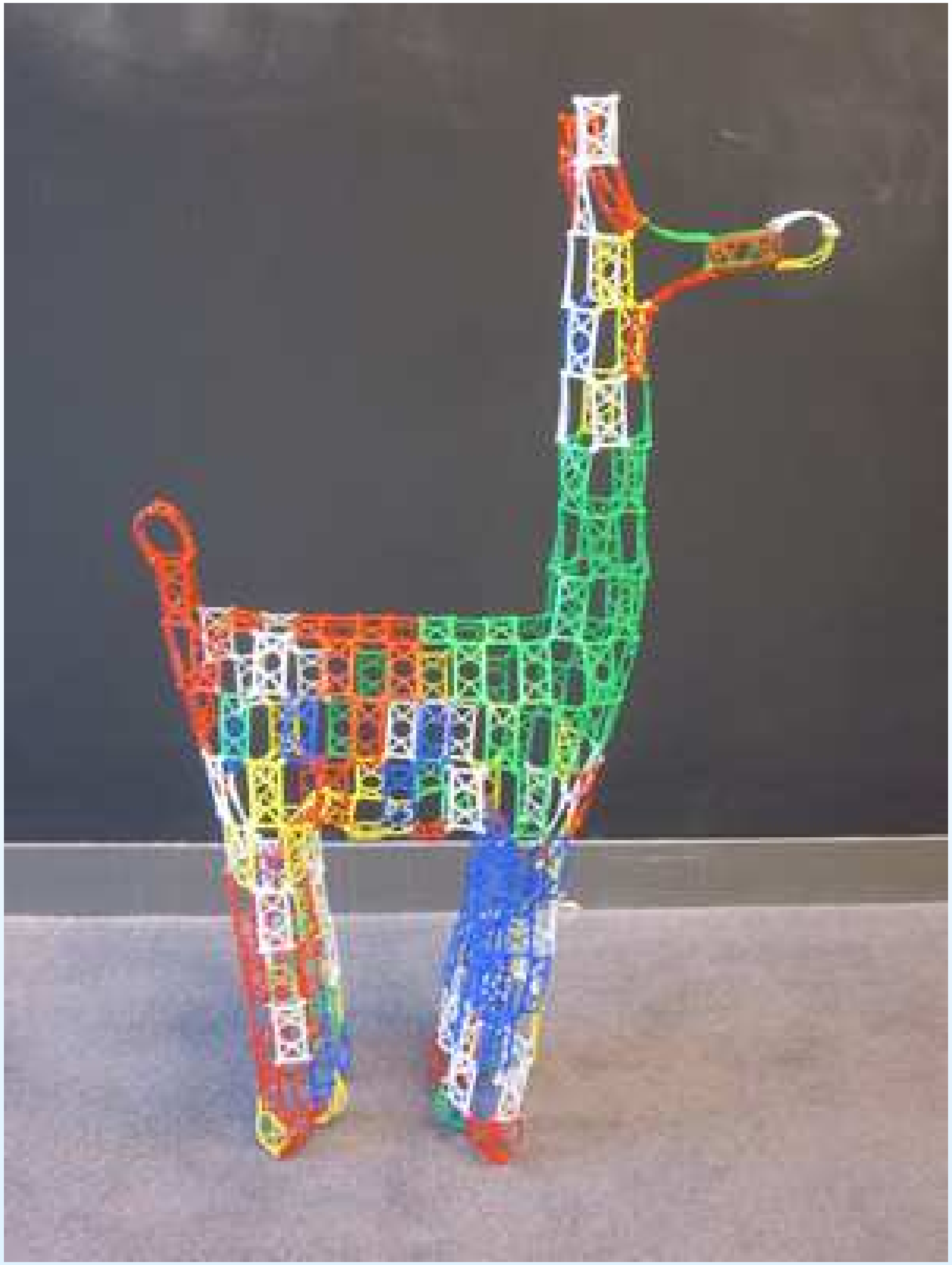
(after a joint work with V. Delecroix, E. Goujard and P. Zograf; partly published in *Forum of Mathematics Pi*, **8:4** (2020)
`arXiv:1705.05190`).

Counting Problems

Ventotene, September 10, 2021

Frequencies of k-cylinder square-tiled surfaces. Non-correlation of horizontal and vertical distributions

Reminder: a square-tiled surface

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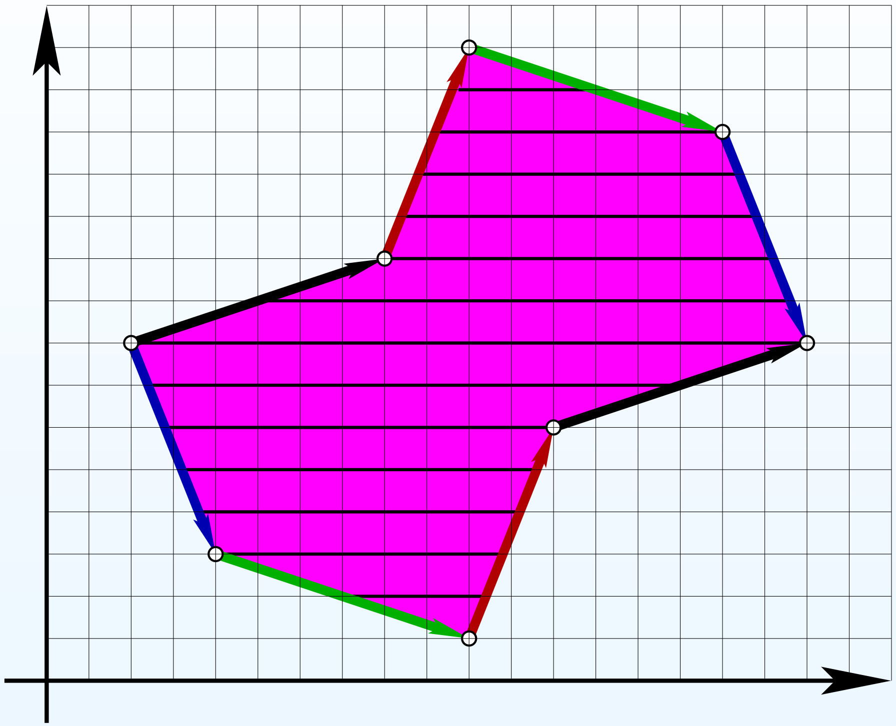
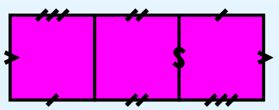
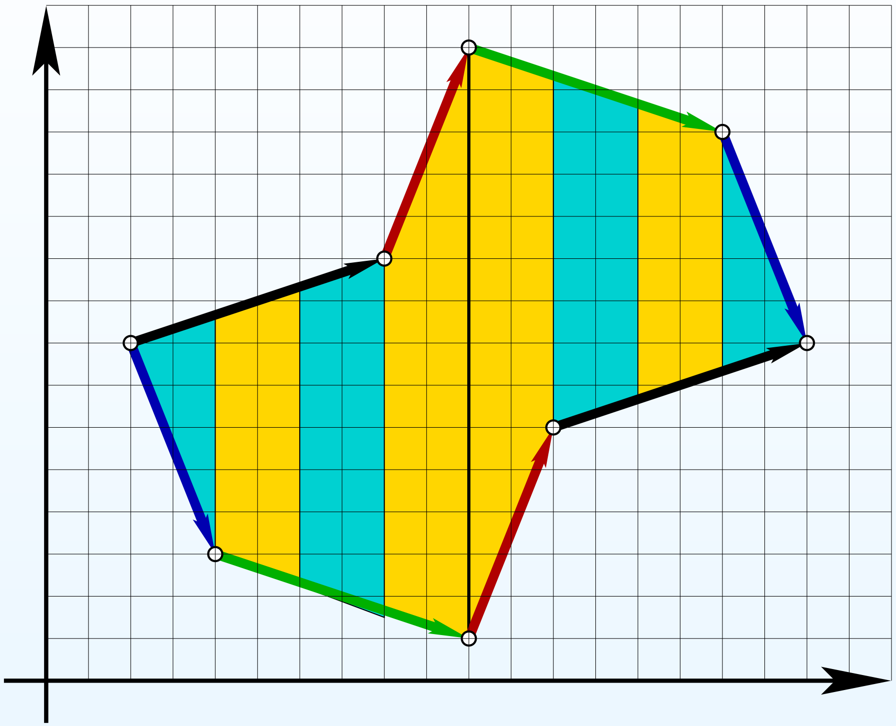
Horizontal and vertical decompositions of a square-tiled surface

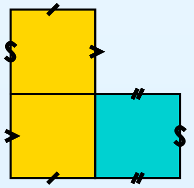


Single maximal horizontal cylinder
as in

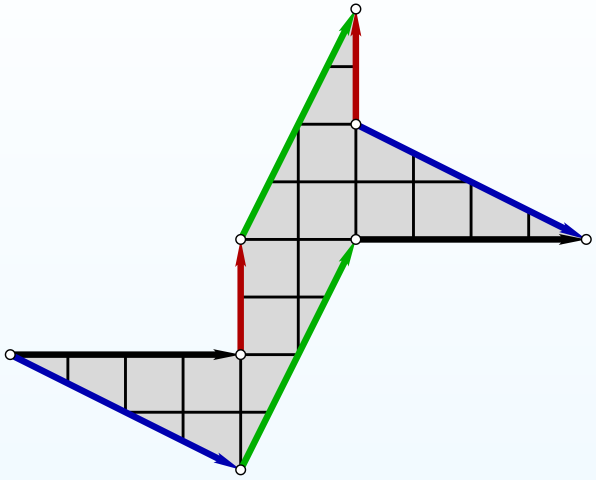




two maximal vertical cylinders.
as in



Another way to see horizontal (vertical) maximal cylinders



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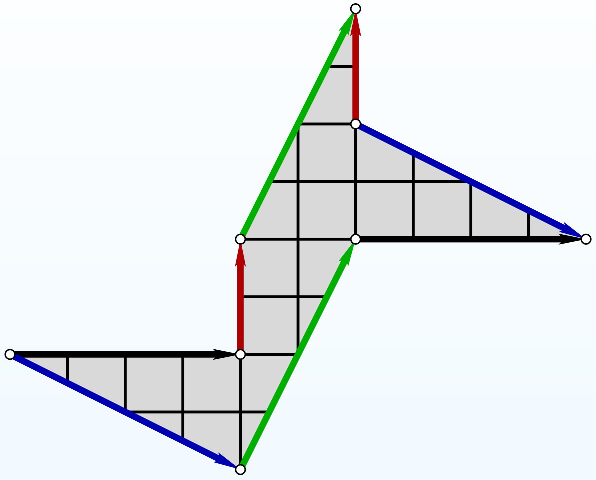

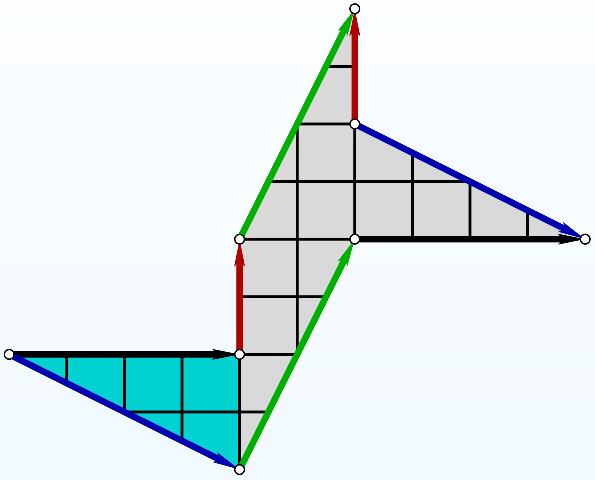

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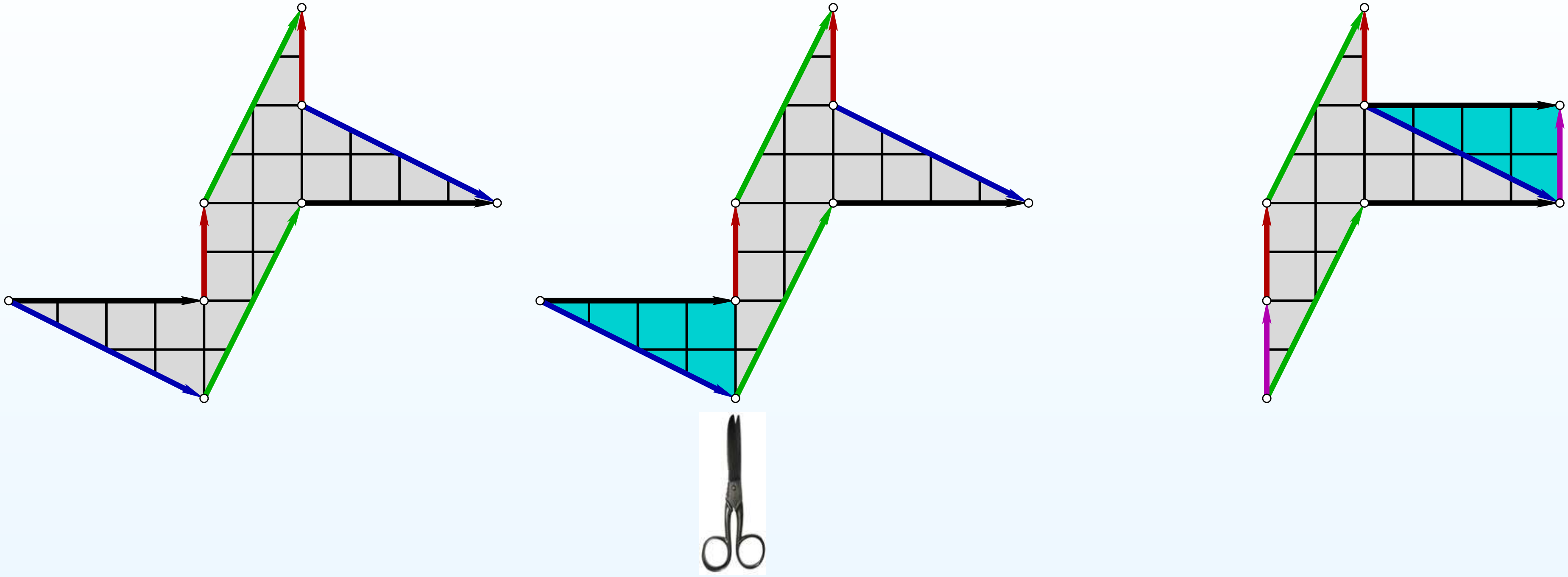

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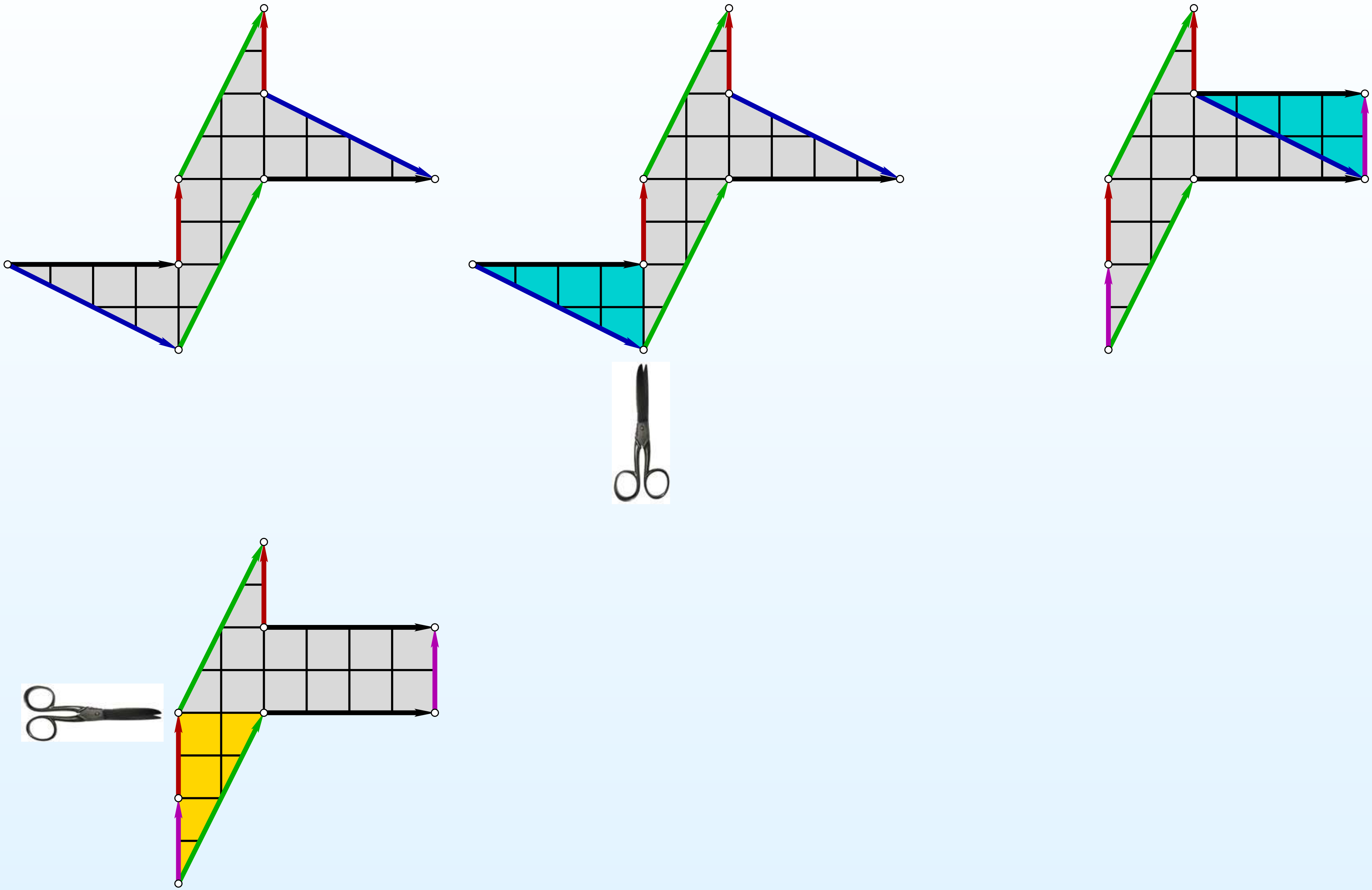

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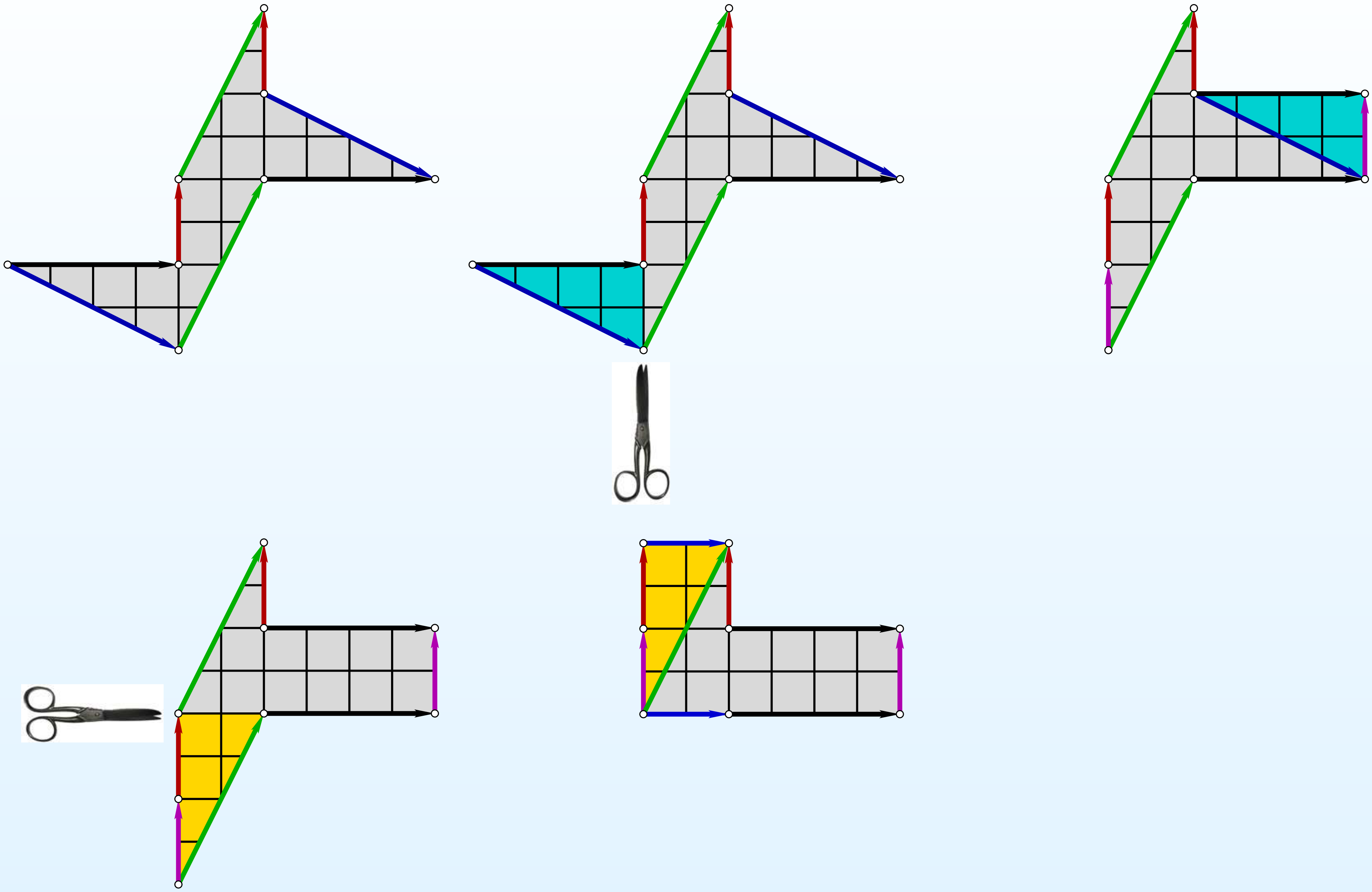

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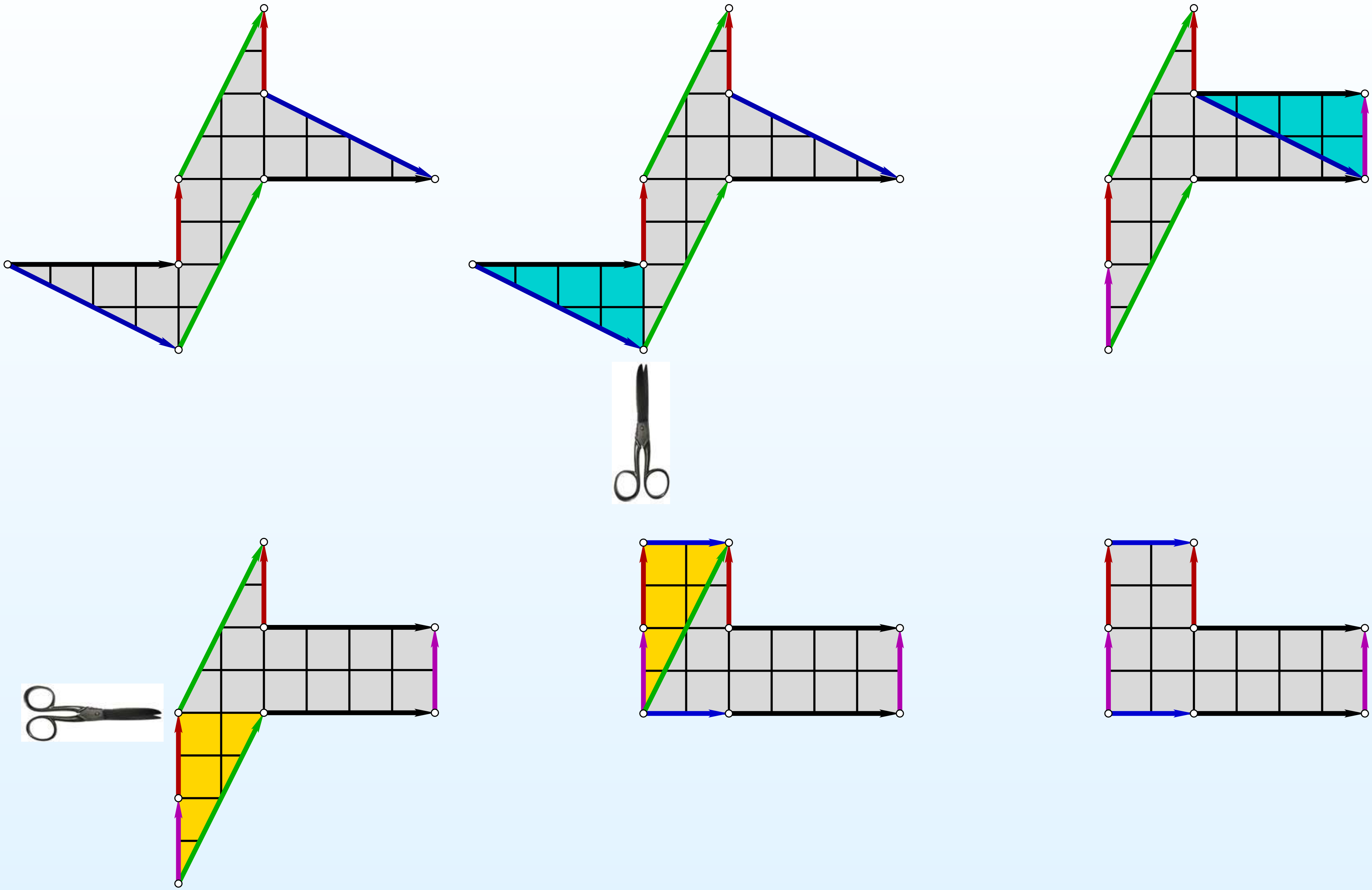


Contribution of k-cylinder square-tiled surfaces to $\operatorname{Vol}\mathcal{H}(3,1)$

$$0.19 \approx p_1(\mathcal{H}(3,1)) = \frac{3\,\zeta(7)}{16\,\zeta(6)}$$ ← the only quantity which is easy to compute

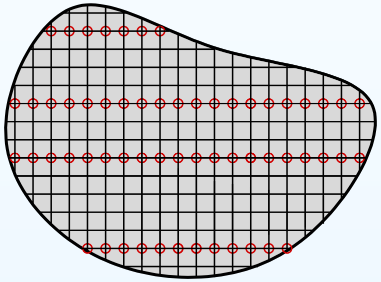
$$0.47 \approx p_2(\mathcal{H}(3,1)) = \frac{55\,\zeta(1,6) + 29\,\zeta(2,5) + 15\,\zeta(3,4) + 8\,\zeta(4,3) + 4\,\zeta(5,2)}{16\,\zeta(6)}$$

$$\begin{aligned}0.30 \approx p_3(\mathcal{H}(3,1)) = \frac{1}{32\,\zeta(6)}\Big(& 12\,\zeta(6) - 12\,\zeta(7) + 48\,\zeta(4)\,\zeta(1,2) + 48\,\zeta(3)\,\zeta(1,3)\\ &+ 24\,\zeta(2)\,\zeta(1,4) + 6\,\zeta(1,5) - 250\,\zeta(1,6) - 6\,\zeta(3)\,\zeta(2,2)\\ &- 5\,\zeta(2)\,\zeta(2,3) + 6\,\zeta(2,4) - 52\,\zeta(2,5) + 6\,\zeta(3,3) - 82\,\zeta(3,4)\\ &+ 6\,\zeta(4,2) - 54\,\zeta(4,3) + 6\,\zeta(5,2) + 120\,\zeta(1,1,5) - 30\,\zeta(1,2,4)\\ &- 120\,\zeta(1,3,3) - 120\,\zeta(1,4,2) - 54\,\zeta(2,1,4) - 34\,\zeta(2,2,3)\\ &- 29\,\zeta(2,3,2) - 88\,\zeta(3,1,3) - 34\,\zeta(3,2,2) - 48\,\zeta(4,1,2)\Big)\end{aligned}$$

$$0.04 \approx p_4(\mathcal{H}(3,1)) = \frac{\zeta(2)}{8\,\zeta(6)}\Big(\zeta(4) - \zeta(5) + \zeta(1,3) + \zeta(2,2) - \zeta(2,3) - \zeta(3,2)\Big)\,.$$

Equidistribution Theorems

Theorem. *The asymptotic proportion $p_k(\mathcal{L})$ of square-tiled surfaces tiled with tiny $\varepsilon \times \varepsilon$-squares and having exactly k maximal horizontal cylinders among all such square-tiled surfaces living inside an open set $B \subset \mathcal{L}$ in a stratum $\mathcal{L}$ of Abelian or quadratic differentials does not depend on B.*



Let $c_k(\mathcal{L})$ be the contribution of horizontally k-cylinder square-tiled surfaces (pillowcase covers) to the Masur–Veech volume of the stratum $\mathcal{L}$, so that $c_1(\mathcal{L}) + c_2(\mathcal{L}) + \cdots = \operatorname{Vol}\mathcal{L}$, and $p_k(\mathcal{L}) = c_k(\mathcal{L})/\operatorname{Vol}(\mathcal{L})$. Let $c_{k,j}(\mathcal{L})$ be the contribution of horizontally k-cylinder and vertically j-cylinder ones.

Theorem. *There is no correlation between statistics of the number of horizontal and vertical maximal cylinders:*

$$\frac{c_k(\mathcal{L})}{\operatorname{Vol}(\mathcal{L})} = \frac{c_{kj}(\mathcal{L})}{c_j(\mathcal{L})}\,.$$

This formula is an *asymptotic* formula! Proof: Moore ergodicity theorem.

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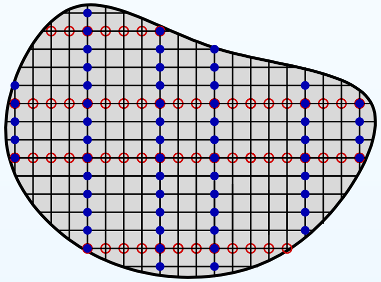

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Contribution of 1-cylinder diagrams

Theorem. *The contribution c_1 of 1-cylinder square-tiled surfaces to the volume $\operatorname{Vol}\mathcal{H}_1(m_1,\dots,m_n)$ of any nohyperelliptic stratum of Abelian differentials satisfies*

$$\frac{\zeta(d)}{d+1}\cdot\frac{4}{(m_1+1)\dots(m_n+1)}\le c_1\le\frac{\zeta(d)}{d-\frac{10}{29}}\cdot\frac{4}{(m_1+1)\dots(m_n+1)}\,,$$

where $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.

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where $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.

Corollary. *The contribution $c_1(\mathcal{H}(m_1,\dots,m_n)$ to the Masur–Veech volume $\operatorname{Vol}(\mathcal{H}_1(m_1,\dots,m_n))$ coming from 1-cylinder square-tiled surfaces satisfies*

$$d\cdot\frac{c_1(\mathcal{H}(m_1,\dots,m_n))}{\operatorname{Vol}(\mathcal{H}_1(m_1,\dots,m_n))}\to 1\ \text{ as }\ g\to+\infty\,,$$

where convergence is uniform for all strata in genus g.

Here we use the results on volume asymptotics conjectured by A. Eskin and A. Zorich twenty years ago and recently proved by D. Chen, M. Möller, A. Sauvaget, D. Zagier and independently by A. Aggarwal.

Pairs of transverse multicurves as square-tiled surfaces. Meanders count

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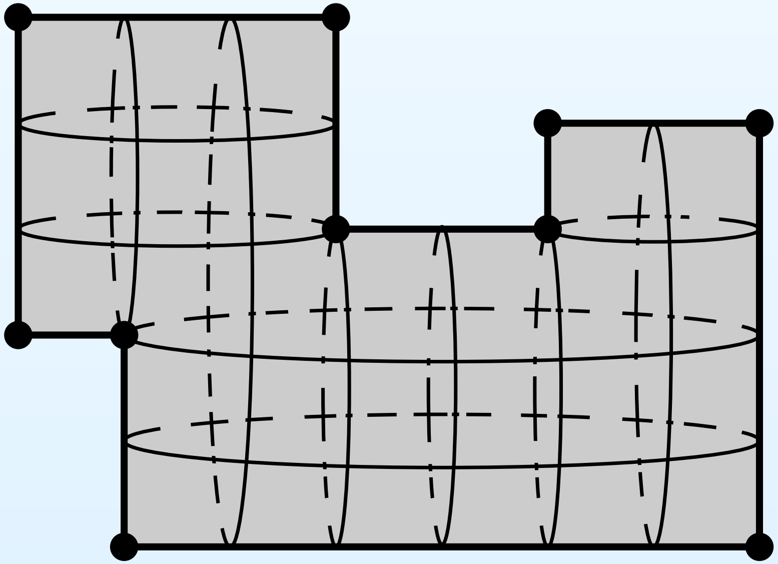
Consider a square-tiled sphere.



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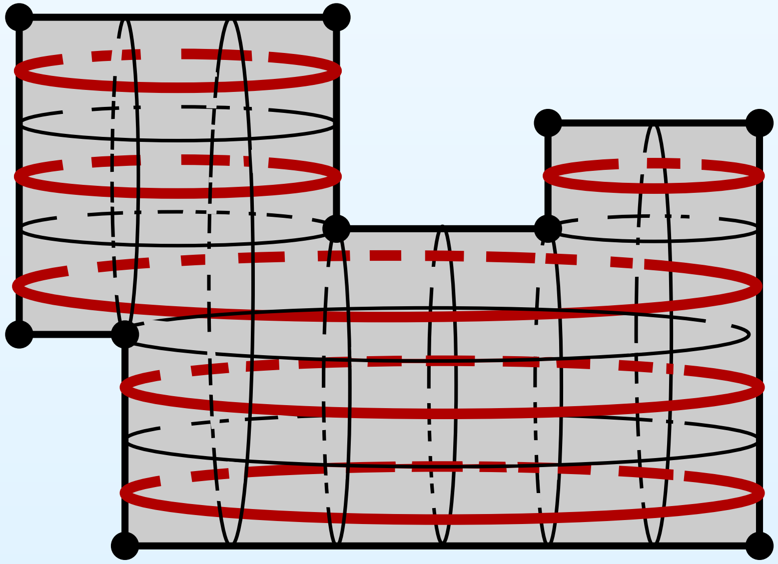

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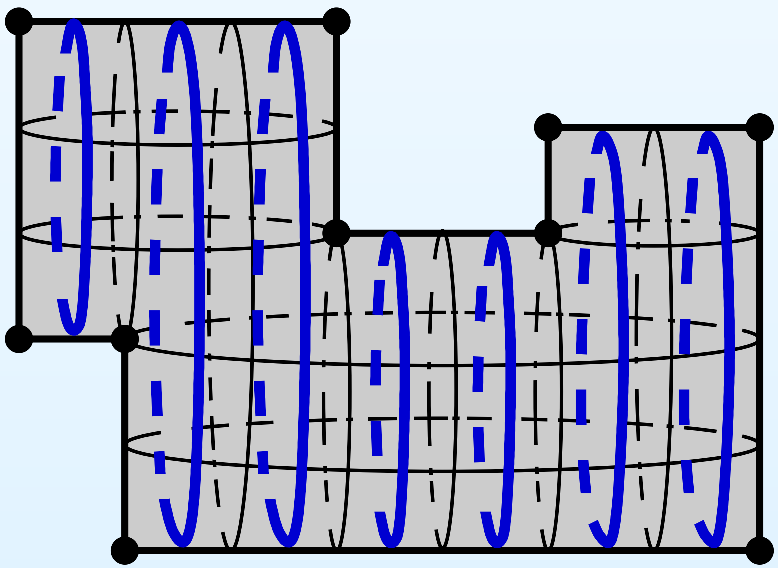

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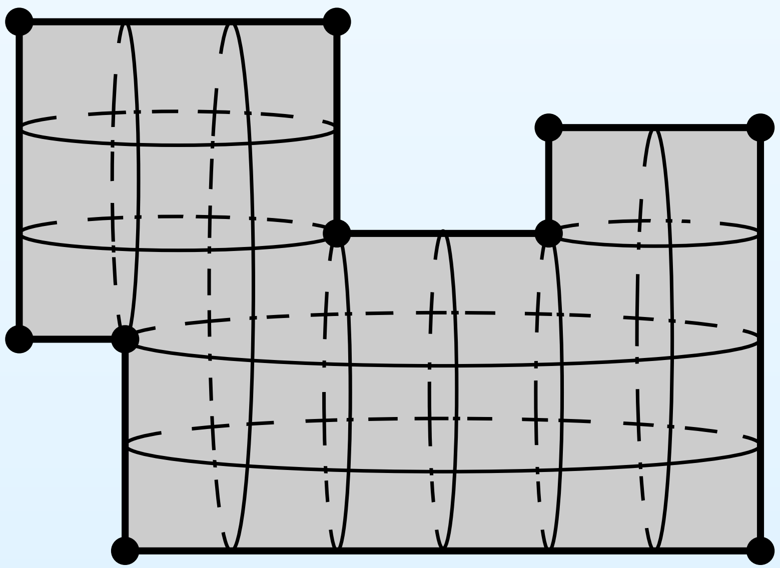
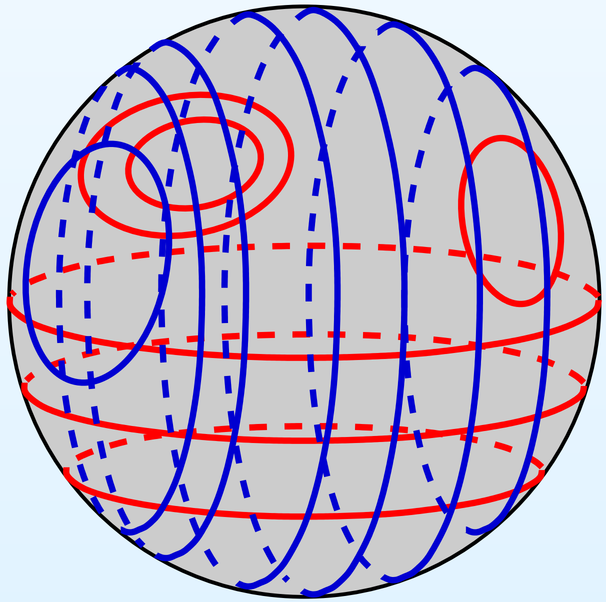
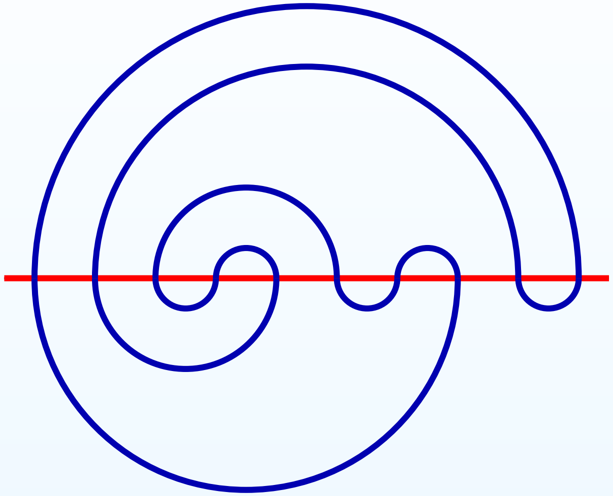




Meanders and arc systems

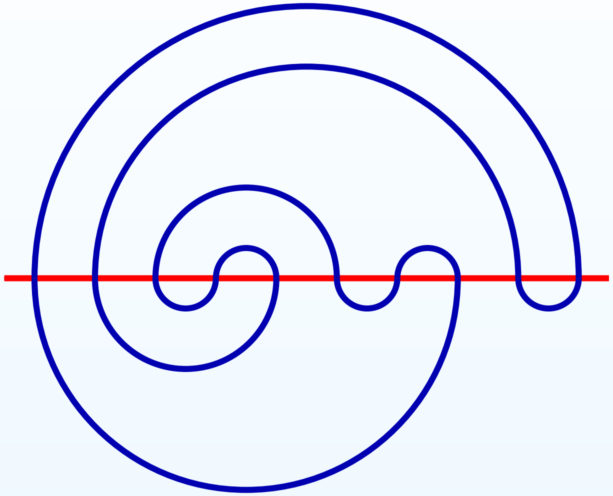


A closed *meander* is a smooth simple closed curve in the plane transversally intersecting the horizontal line.

According to S. Lando and A. Zvonkin the notion “meander” was suggested by V. Arnold though meanders were studied already by H. Poincaré.

Meanders appear in various contexts, in particular in mathematics, physics and biology.

Meanders and arc systems

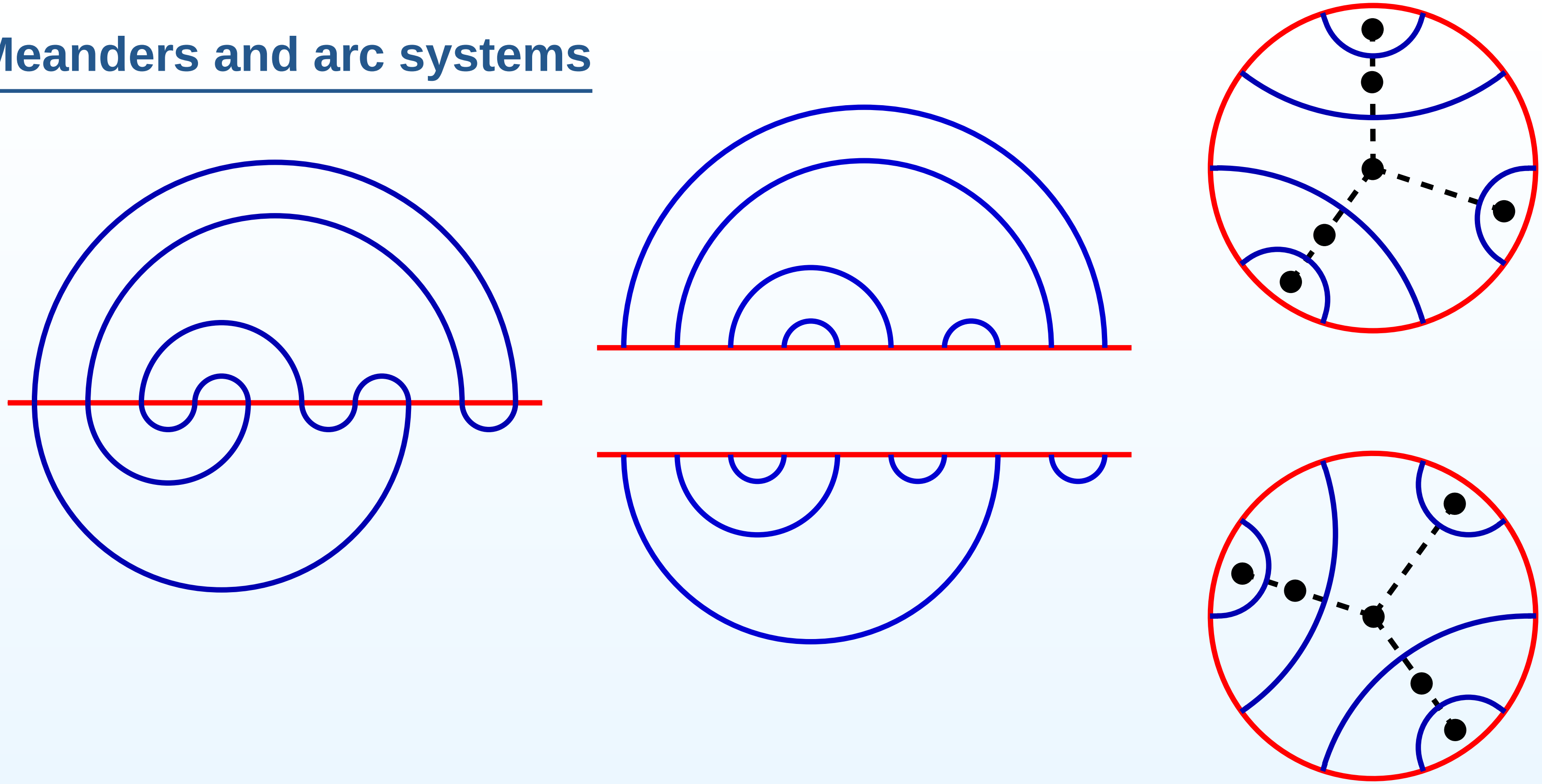


Conjecture (P. Di Francesco, O. Golinelli, E. Guitter, 1997). *The number of meanders with $2N$ crossings is asymptotic to*

$$const \cdot R^{2N} \cdot N^{\alpha} \,,$$

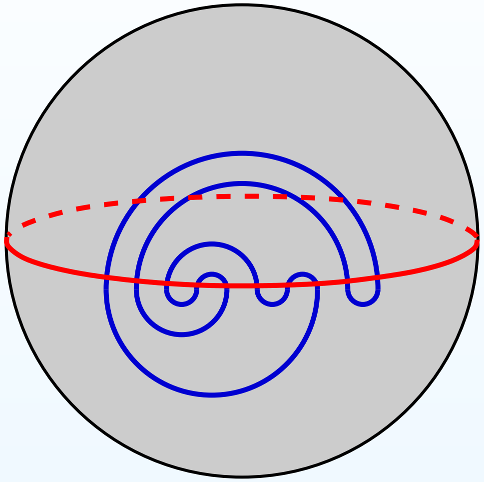
where $R^2 \approx 12.26$ (value is due to I. Jensen) and $\alpha = -\frac{29+\sqrt{145}}{12}$.

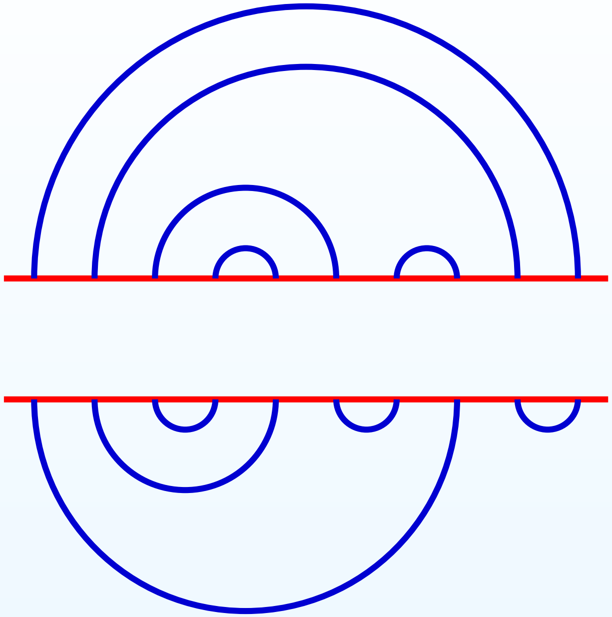
Meanders and arc systems

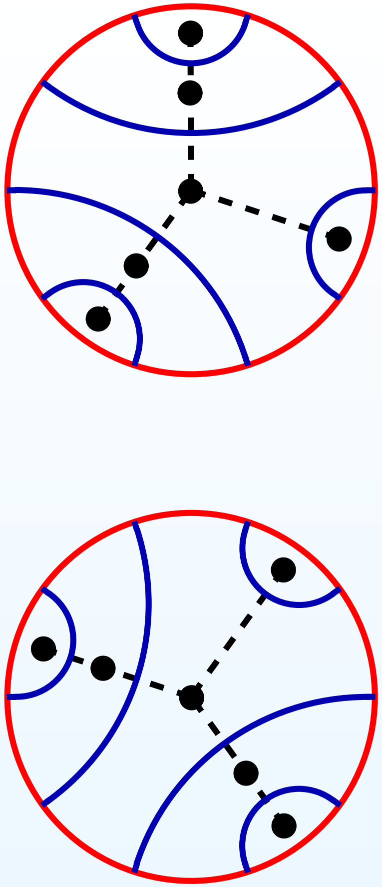


A closed meander on the left. The associated pair of arc systems in the middle. The same arc systems on the discs and the associated dual graphs on the right. We usually erase vertices of valence 2 from dual trees.

Meanders and arc systems





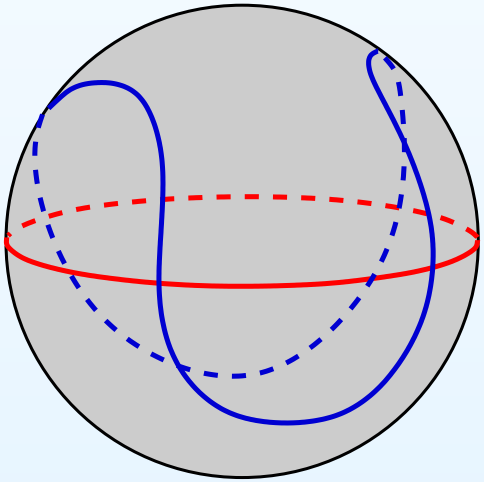


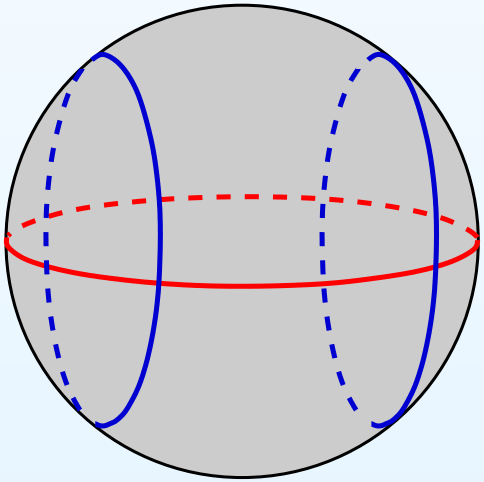
A closed meander on the left. The associated pair of arc systems in the middle. The same arc systems on the discs and the associated dual graphs on the right. We usually erase vertices of valence 2 from dual trees.

Compactifying the plane (left picture) with one point at infinity, or gluing together arc systems on the two discs (right picture) we get an ordered pair of smooth simple transversally intersecting closed curves on the sphere.

Meanders versus multicurves

It is much easier to count arc systems (for example, arc systems sharing the same reduced dual tree). However, this does not simplify counting meanders since identifying a pair of arc systems with the same number of arcs by the common equator, we sometimes get a meander and sometimes — a *multicurve*, i.e. a curve with several connected components.





Attaching arc systems on a pair of hemispheres along the common equator we might get a single simple closed curve (as on the left picture) or a multicurve with several connected components (as on the right picture).

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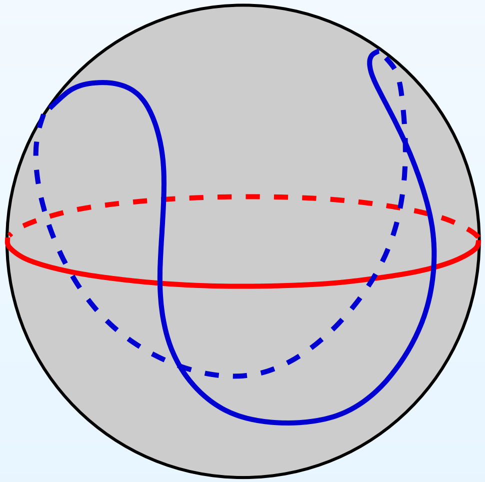

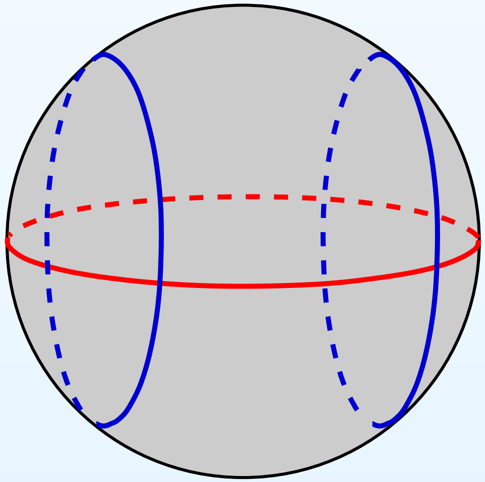

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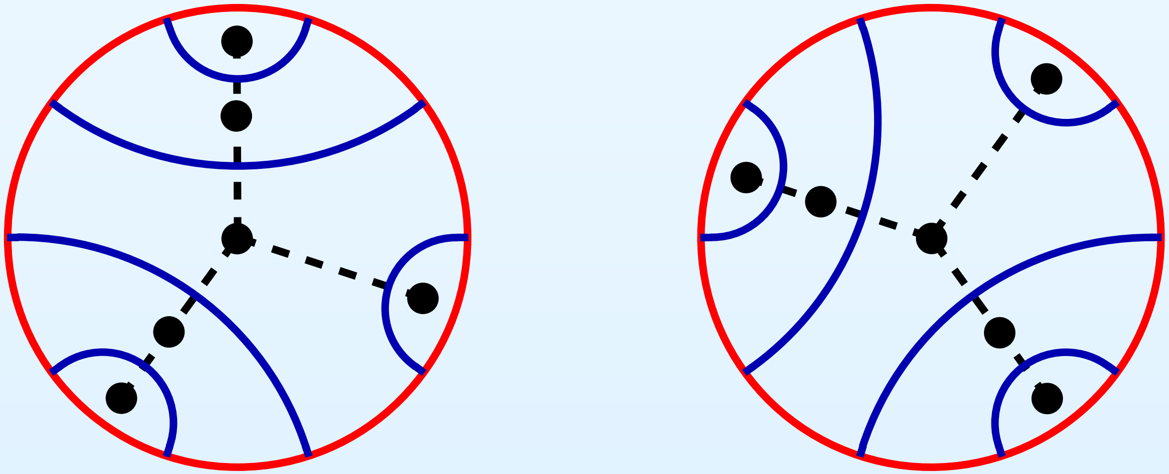
Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



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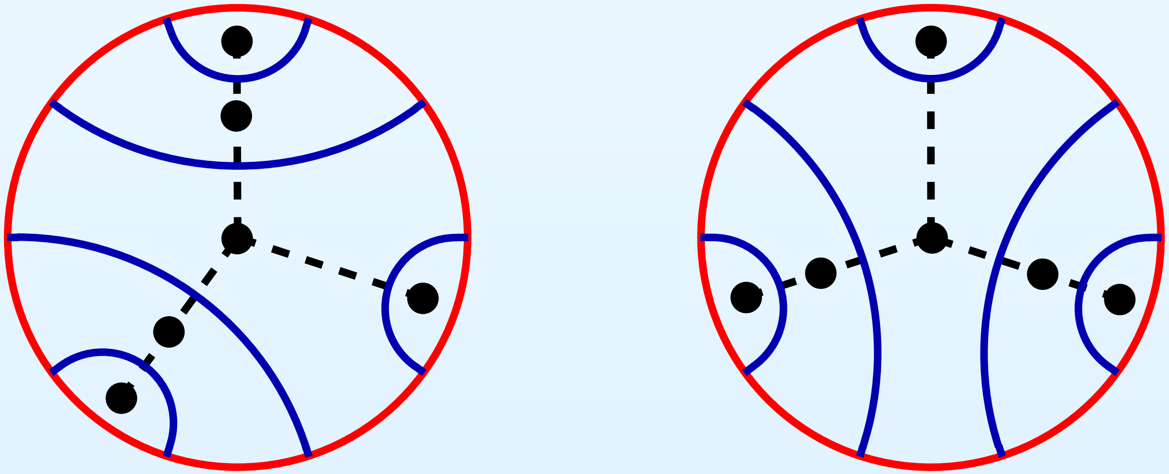

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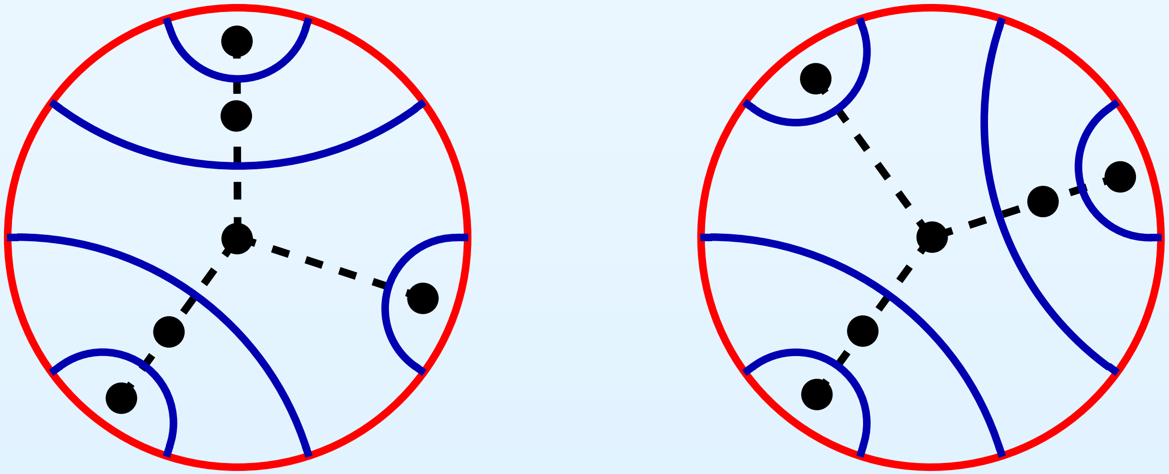

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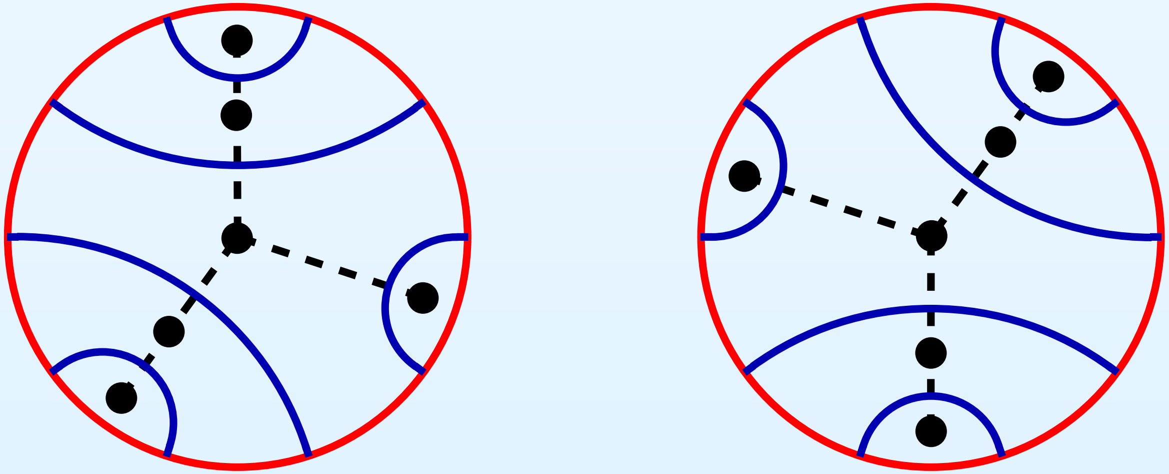

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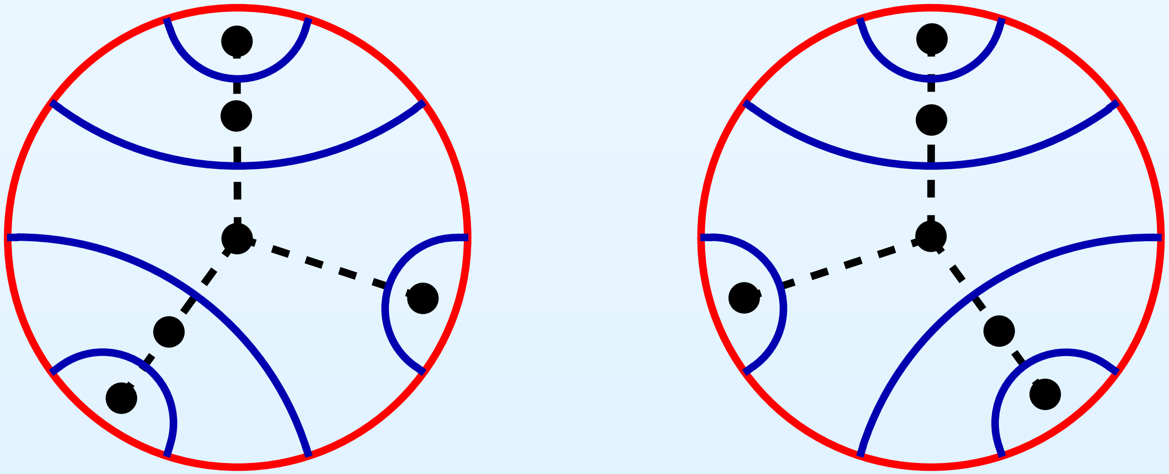

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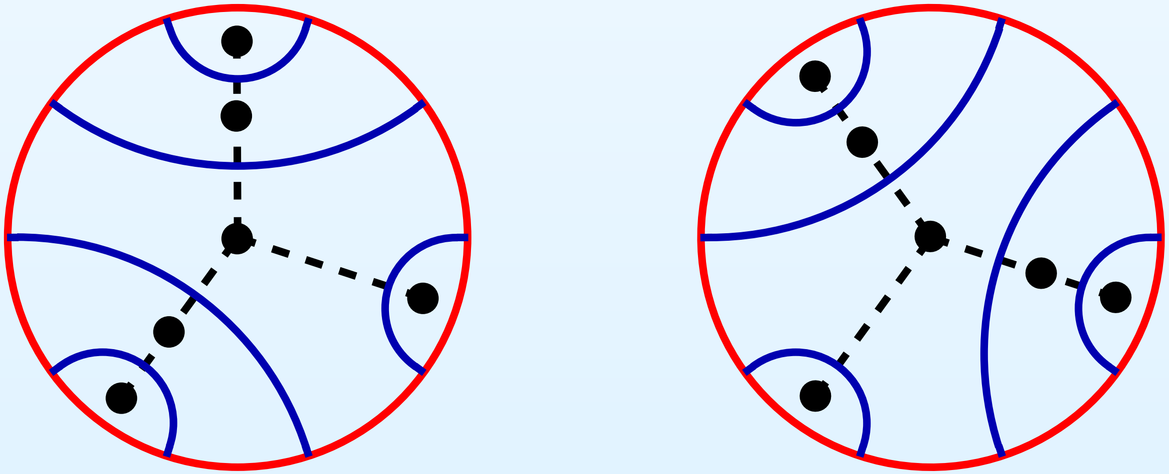

Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

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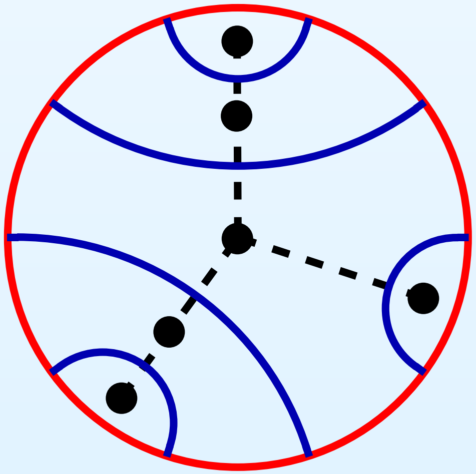

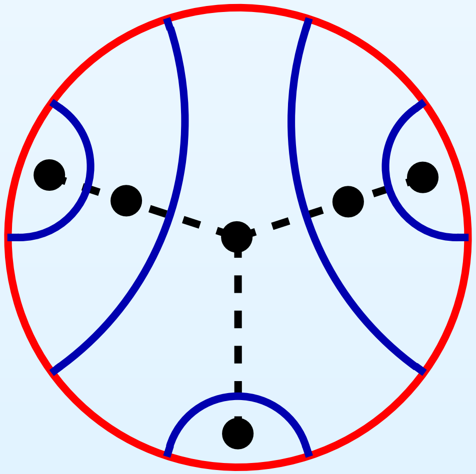

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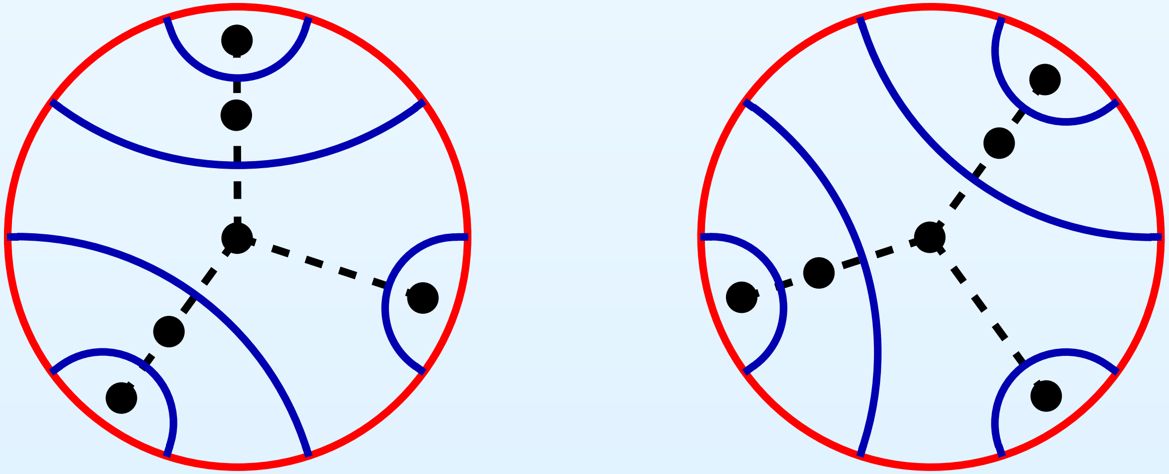

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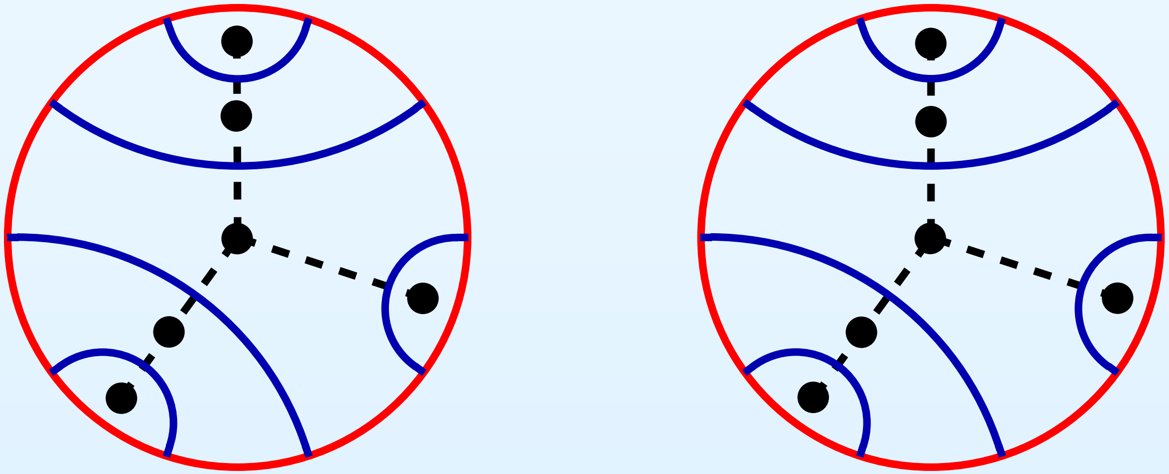

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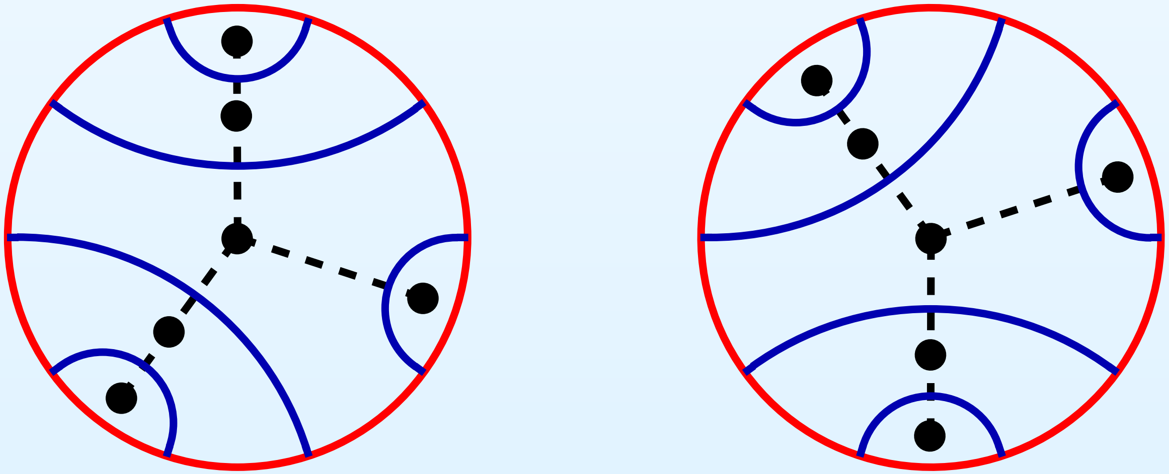

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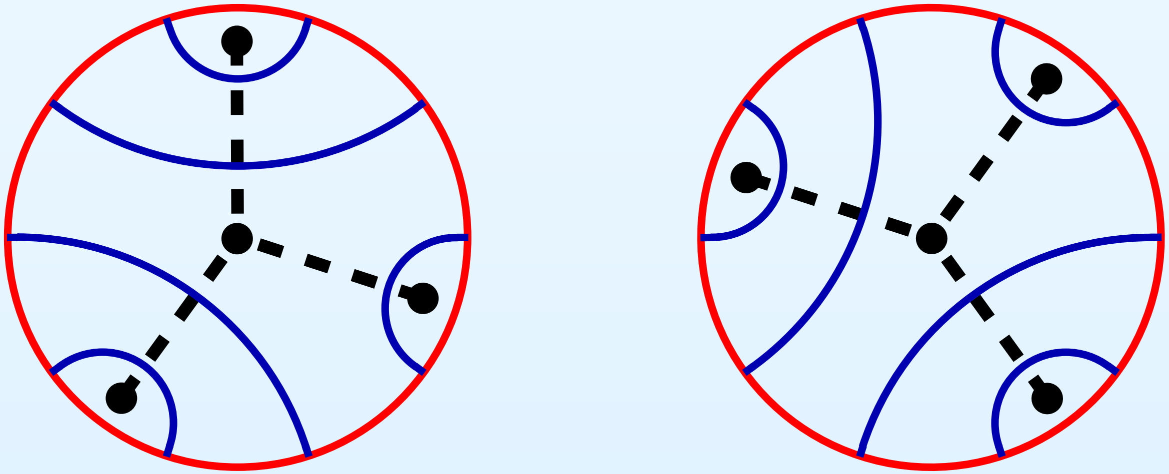

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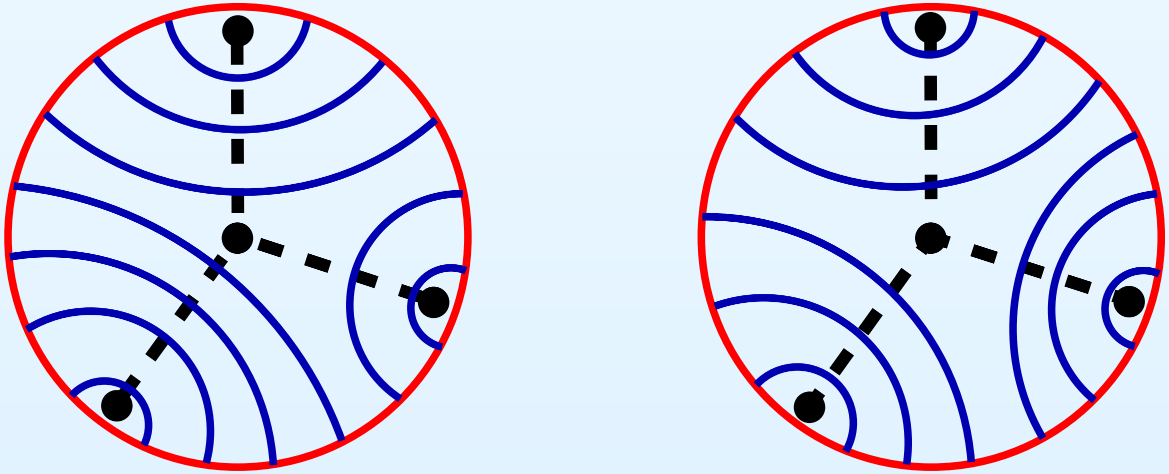

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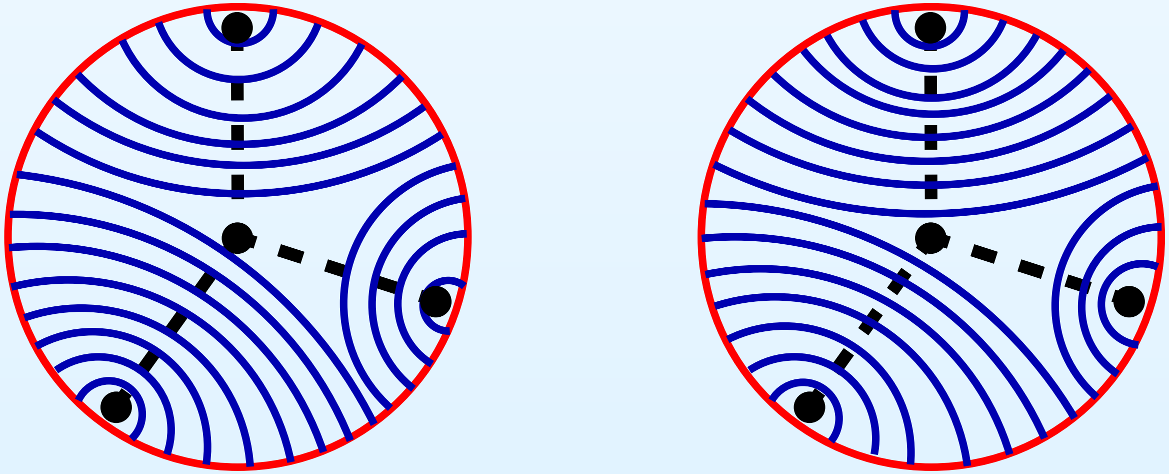

Asymptotic frequency of meanders

Question. *What is the asymptotic probability*

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) \sim ? \quad \text{as } N \to +\infty$$

to get a meander (i.e. a connected curve) by a random gluing of a random pair of arc systems as above with $n \leq N$ arcs?
Does it behave like $N^{-\alpha}$? Like $\exp(-\beta N)$? If so, describe how α (respectively β) depend on $\mathcal{T}_{North}, \mathcal{T}_{South}$.

Asymptotic frequency of meanders

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Does it behave like $N^{-\alpha}$? Like $\exp(-\beta N)$? If so, describe how α (respectively β) depend on $\mathcal{T}_{North}, \mathcal{T}_{South}$.

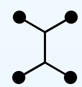
Theorem. *For any pair of trees $\mathcal{T}_{North}, \mathcal{T}_{South}$ the quantity $\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N)$ admits a strictly positive limit as $N \to +\infty$.*
We have an explicit formula for this limit in terms of the total number of vertices of valence $1, 3, 4, \dots$ of the two trees.

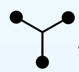
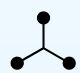
I have to confess that the fact that this asymptotic frequency is nonzero was unexpected to me.

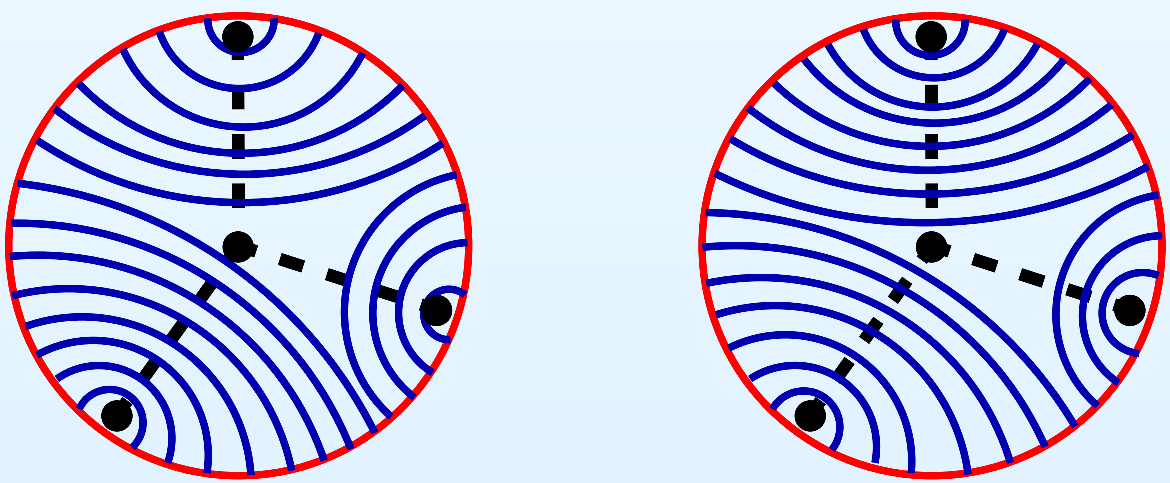
Asymptotic frequency of meanders

Theorem. *Let* $p_{North}, p_{South} \geq 2$*. Let* $p = p_{North} + p_{South}$*. The frequency* $\mathrm{P}_{connected}(p_{North}, p_{South}; N)$ *of meanders obtained by all possible identifications of all arc systems with at most* N *arcs represented by all possible pairs of plane trees having* p_{North}, p_{South} *of leaves (vertices of valence one) has the following limit:*

$$\lim_{N \to +\infty} \mathrm{P}_{connected}(p_{North}, p_{South}; N) = \frac{1}{2}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2}.$$

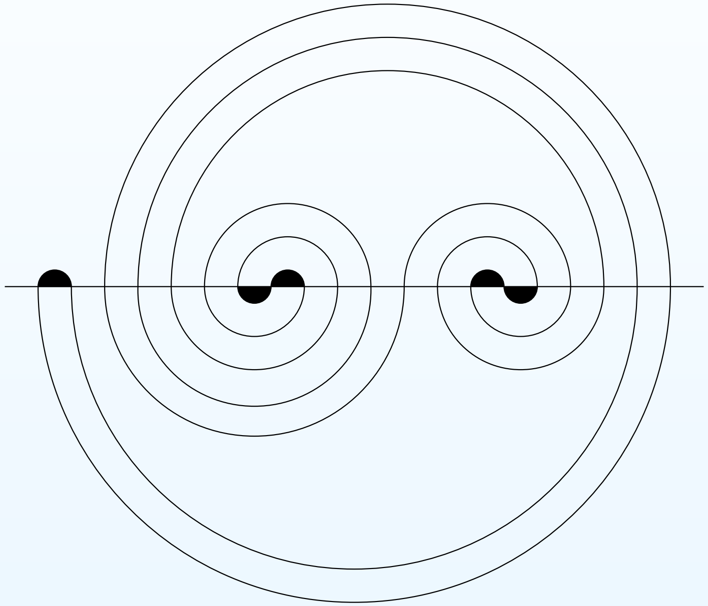
Example. $\lim_{N \to +\infty} \mathrm{P}_{connected}($  ,  $, N) =$

$= \lim_{N \to +\infty} \mathrm{P}_{connected}($  ,  $, N) = \frac{280}{\pi^6} \approx 0.291245\,.$

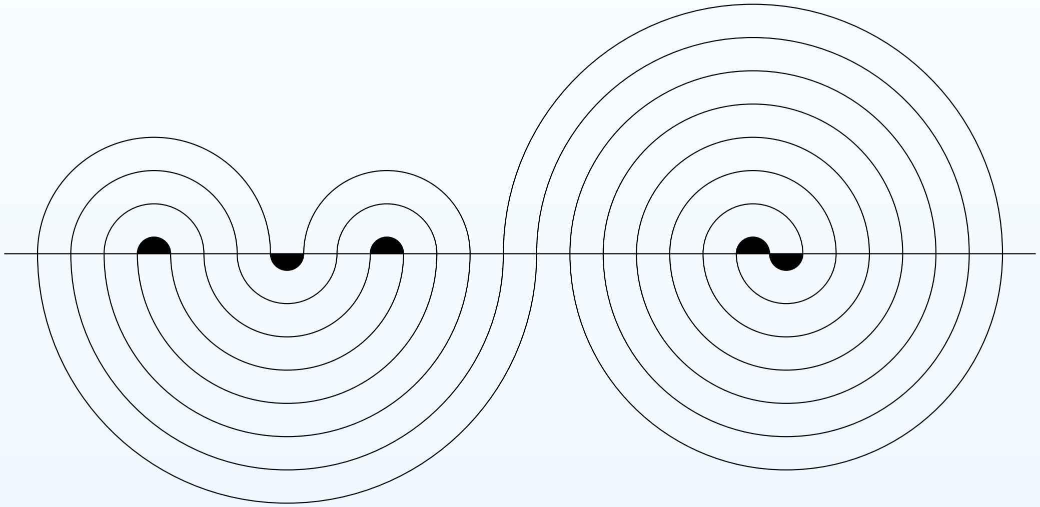


Meanders with and without maximal arc

These two meanders have 5 minimal arcs ("pimples") each.



Meander with a maximal arc ("rainbow") contributes to $\mathcal{M}_5^+(N)$



Meander without maximal arc contributes to $\mathcal{M}_5^-(N)$

Let $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ be the numbers of closed meanders respectively with and without maximal arc ("rainbow") and having at most $2N$ crossings with the horizontal line and exactly p minimal arcs ("pimples"). We consider p as a parameter and we study the leading terms of the asymptotics of $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ as $N \to +\infty$.

Counting formulae for meanders

Theorem. *For any fixed p the numbers $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ of closed meanders with p minimal arcs (pimples) and with at most $2N$ crossings have the following asymtotics as $N \to +\infty$:*

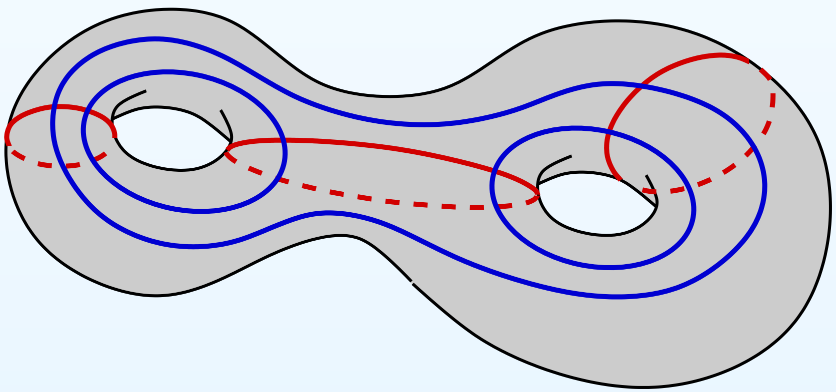
$$\mathcal{M}_p^+(N) = \frac{2}{p!\,(p-3)!}\left(\frac{2}{\pi^2}\right)^{p-2} \cdot \binom{2p-2}{p-1}^2 \cdot \frac{N^{2p-4}}{4p-8} + o(N^{2p-4})\,.$$

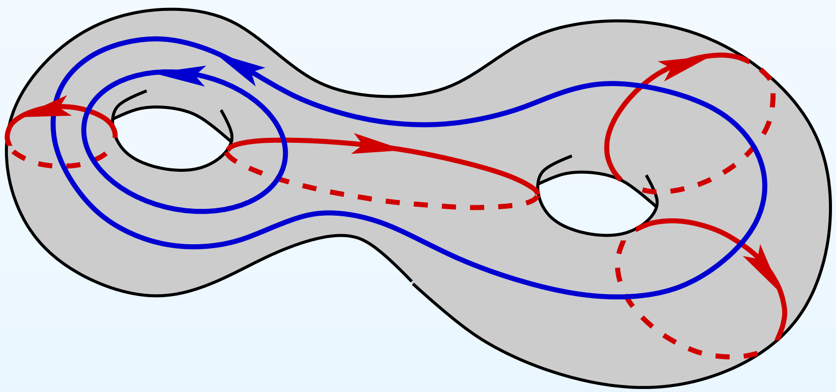
$$\mathcal{M}_p^-(N) = \frac{4}{p!\,(p-4)!}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2}^2 \cdot \frac{N^{2p-5}}{4p-10} + o(N^{2p-5})\,.$$

Note that $\mathcal{M}_p^+(N)$ grows as N^{2p-4} while $\mathcal{M}_p^-(N)$ grows as N^{2p-5}.

Meanders in higher genera

A pair of smooth simple closed transverse oriented multicurves is called *positively intersecting* if each connected component of each multicurve is oriented in such way that all intersections match the orientation of the surface. A pair of transverse multicurves is called *orientable* if it admits an orientation with positive intersection and *non-orientable* otherwise.





Exercise. Verify that the pair of transverse multicurves on the right is positively intersecting and that the pair of multicurves on the left is non-orientable.

Definition. A *meander* on a surface of genus g is a pair of smooth transverse simple closed curves considered up to a diffeomorphism of the surface. A *positive meander* is a positively intersecting pair of simple closed curves.

Results: general (non-orientable) case

Fix the genus g of the surface. Fix a nonnegative integer p denoting the number of bigons produced by intersections of pairs of multicurves.

Observation. *The following quantities have polynomial asymptotics:*

- *Number of pairs of transverse multicurves with at most N intersections and with exactly p bigons $= c(g,p) \cdot N^{6g-6+2p} + o(N^{6g-6+2p})$.*
- *Number of pairs (simple closed curve, transverse multicurve) with at most N intersections and p bigons $= c_1(g,p) \cdot N^{6g-6+2p} + o(N^{6g-6+2p})$.*
- *Number of meanders with at most N intersections and with exactly p bigons $= c_{1,1}(g,p) \cdot N^{6g-6+2p} + o(N^{6g-6+2p})$.*

Theorem. *The coefficients $c(g,p), c_1(g,p), c_{1,1}(g,p)$ satisfy the following relation:*

$$\frac{c_1(g,p)}{c(g,p)} = \frac{c_{1,1}(g,p)}{c_1(g,p)}\,.$$

Results: general (non-orientable) case

The coefficients in these asymptotics have the following arithmetic nature:

$$c(g,p) = r(g,p)\cdot \pi^{6g-6+2n}, \quad c_1(g,p) = r_1(g,p); \quad r,\, r_1 \in \mathbb{Q}\,.$$

For small p and for g, say, up to 500, we can compute the rational numbers $r(g,p)$ explicitly (based on results of D. Chen–M. Möller–A. Sauvaget combined with results of M. Kazarian or D. Yang, D. Zagier, and Y. Zhang).

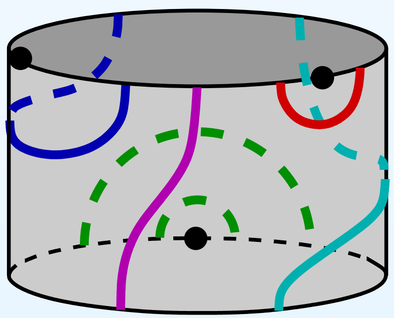
For the same range of g and p we can compute the rational numbers $r_1(g,p)$ explicitly (based on our own results).

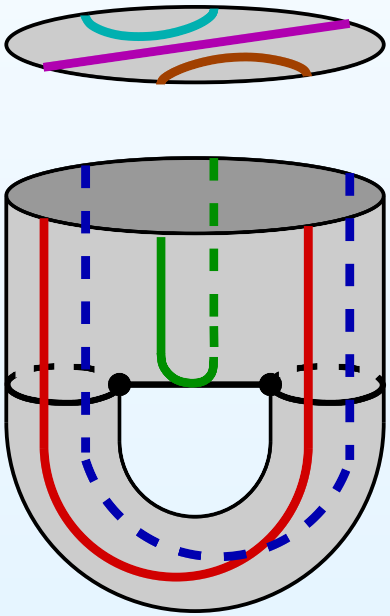
For any fixed p we also have a simple asymptotic formulae for $r(g,p)$ and $r_1(g,p)$ as $g \to +\infty$ (our results combined with results of A. Aggarwal).

Since $c_{1,1}(g,p) = \dfrac{c_1^2(g,p)}{c(g,p)}$, in all these cases we get absolutely explicit asymptotic formulae for the count of meanders in genus g.

Asymptotic frequency of meanders

As in genus 0, we can construct multicurves from systems of arcs on a surface of genus $g-1$ with two boundary components or on a pair of surfaces of genera g_1 and g_2, where $g_1+g_2=g$, each with a single boundary component. As before we fix the total number p of bigons. We assume that there are exactly n arcs landing to each of the two boundary components, and that $n \le N$.





Theorem. *The asymptotic probability to get a meander after a random gluing of a random system of arcs as above is* $\frac{c_1(g,p)}{c(g,p)}$.

Results on positively intersecting pairs of multicurves

The case of positively intersecting pairs of multicurves is analogous. However, the power of N and all the coefficients in the polynomial asymptotics do change. Fix the genus g of the surface.

Observation. *The following quantities have polynomial asymptotics:*

- *Number of pairs of transverse positively intersecting multicurves with at most N intersections $= c^+(g) \cdot N^{4g-3} + o(N^{4g-3})$.*
- *Number of positively intersecting pairs (simple closed curve, transverse multicurve) with at most N intersections $= c_1^+(g) \cdot N^{4g-3} + o(N^{4g-3})$.*
- *Number of positive meanders with at most N intersections $= c_{1,1}^+(g) \cdot N^{4g-3} + o(N^{4g-3})$.*

Theorem. *The coefficients $c^+(g), c_1^+(g), c_{1,1}^+(g)$ satisfy the relation:*

$$\frac{c_1^+(g)}{c^+(g)} = \frac{c_{1,1}^+(g)}{c_1^+(g)}\,.$$

Results on positively intersecting pairs of multicurves

The coefficients in these asymptotics have the following arithmetic nature:

$$c^+(g) = r^+(g) \cdot \pi^{2g}, \quad c_1^+(g) = r_1^+(g) \text{ where } r^+(g),\, r_1^+(g) \in \mathbb{Q}\,.$$

(the first result was proved by A. Eskin and A. Okounkov; the second one is simple.) For g, say, up to 2000, we can compute the rational numbers $r^+(g)$ explicitly (small g — due to A. Eskin and A. Okounkov, 2003; larger g — D. Chen–M. Möller–A. Sauvaget–D. Zagier, 2020).

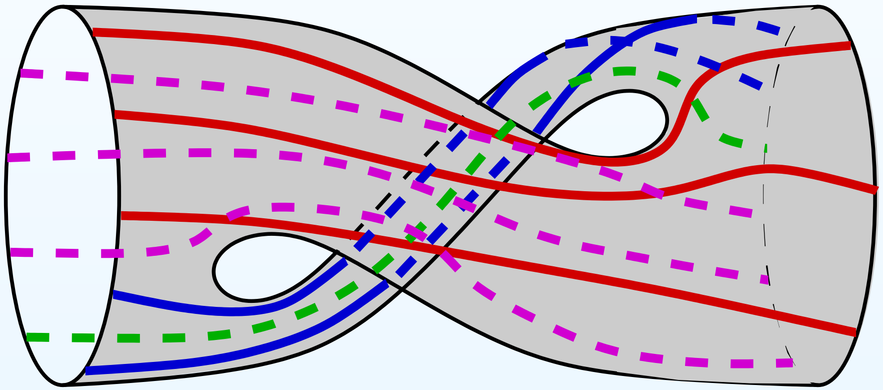
For any g we have a simple formula for $r_1^+(g)$ (our own results, 2020).

We also have simple asymptotic formulae for $r^+(g)$ and $r_1^+(g)$ as $g \to +\infty$ (results of D. Chen–M. Möller–D. Zagier, 2018, for $r^+(g)$; independent result of A. Aggarwal, 2019; result of D. Zagier of 1995 for $r_1^+(g)$).

Since $c_{1,1}^+(g) = \dfrac{\big(c_1^+(g)\big)^2}{c^+(g)}$, we can count positive meanders in genus g.

Asymptotic frequency of positive meanders

As before we can glue systems of arcs on a surface of genus $g-1$ with two boundary components. This time we assume that each of n arc goes from one boundary component to the other, and that $n \leq N$.



Theorem. *The asymptotic probability to get a positive meander after a random gluing of a system of arcs as above is* $\frac{c_1^+(g)}{c_g^+}$*. We have*

$$\frac{c_1^+(g)}{c_g^+} = \frac{1}{4g} + o\left(\frac{1}{g}\right) \text{ as } g \to +\infty\,.$$

Meanders count: summary

Translation to the language of square-tiled surfaces

Every square-tiled surface defines a pair of transverse simple closed multicurves. The number of squares is the number of intersections of the two multicurves.

Reciprocal is not always true since in genera higher than 0 a pair of transverse multicurves might chop the surface into components more complicated than topological discs. However, it happens rarely in terms of the asymptotic count, so for the purposes of the count we can pretend that we have a bijection.

Bigons arising from intersection of transverse multicurves correspond to simple poles of the associated quadratic differentials. Thus, the count of pairs of transverse multicurves on a surface of genus g with at most N intersections and with p bigons corresponds to the count of square-tiled surfaces of genus g with p poles tiled by at most N squares, i.e. to evaluation of the Masur–Veech volume of the moduli space $\mathcal{Q}_{g,p}$. In this way we get the asymptotics $c(g,p) \cdot N^{6g-6+2p}$ for the number of multicurves and the constant $c(g,p)$.

How we count meanders

A pair of transverse multicurves associated to a square-tiled surface is orientable if and only if the square-tiled surface is Abelian. Thus, the count of positively intersecting pairs of transverse multicurves in genus g corresponds to the count of Abelian square-tiled surfaces in genus g, i.e. to the evaluation of the Masur–Veech volumes of the corresponding moduli space of Abelian differentials. In this way we get the asymptotics $c^+(g) \cdot N^{4g-3}$ and the constant $c_1^+(g)$ for the count of positively intersecting multicurves.

Pairs (simple closed curve, transverse multicurve) correspond to square-tiled surfaces having single horizontal band of squares. We found a way to count such square-tiled surfaces both in the Abelian and in the quadratic case and to evaluate the constants $c_1(g,p)$ and $c_1^+(g)$ in the corresponding asymptotics $c_1(g,p) \cdot N^{6g-6+2p}$ and $c_1^+(g) \cdot N^{4g-3}$ respectively.

Meanders correspond to square-tiled surfaces having single horizontal and single vertical band of squares. We apply our non-correlation theorem to get

$$c_{1,1}(g,p) = \frac{c_1^2(g,p)}{c(g,p)} \quad \text{and} \quad c_{1,1}^+(g) = \frac{\left(c_1^+(g)\right)^2}{c^+(g)}\,.$$