# A new method for geometric bordism

**Leonardo Ferrari**

Institute of Mathematics of the Polish Academy of Sciences

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Every smooth 3-manifold bounds smoothly, but not all hyperbolic 3-manifolds bound geometrically.

**Proposition**

If $N$ embeds totally geodesically and admits an orientation-reversing, fixed-point free (nor-fpf) involution, then it bounds geometrically.

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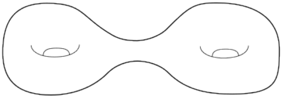
There are hyperbolic surfaces that bound geometrically without such maps.

## Theorem (F.)

*It is (sometimes) possible to construct a geometric boundary, for surfaces that tessellate into regular polygons, without using a nor-fpf involution.*

# An idea of the construction



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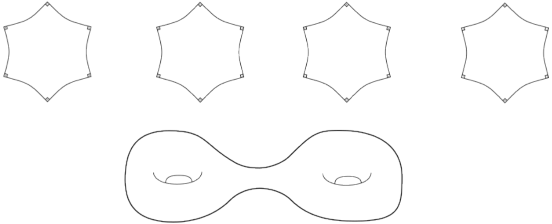

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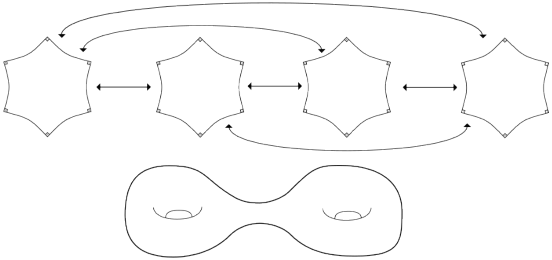

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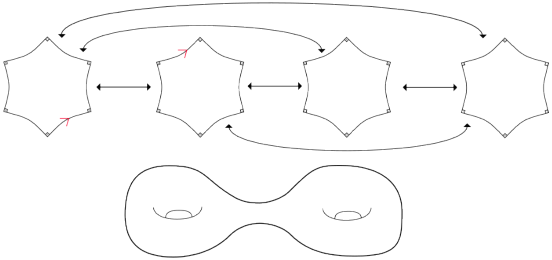

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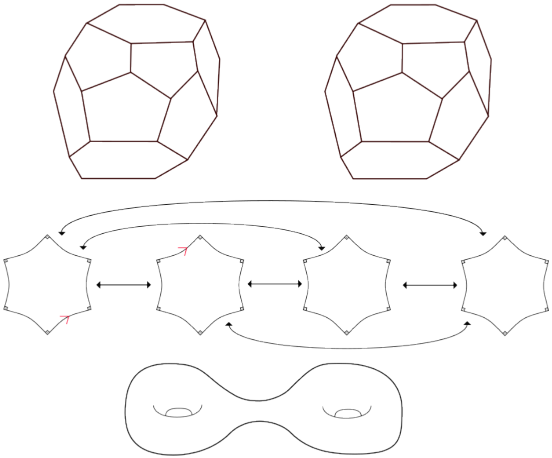

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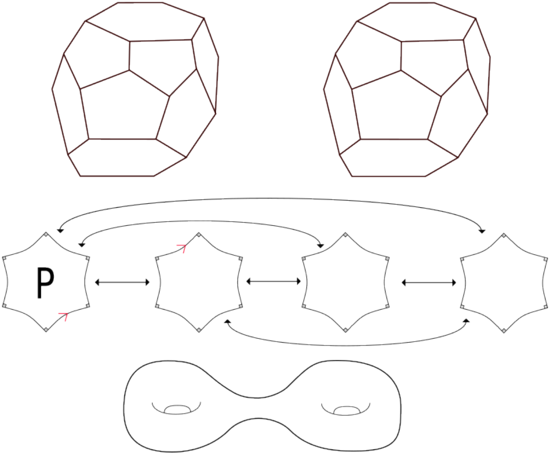

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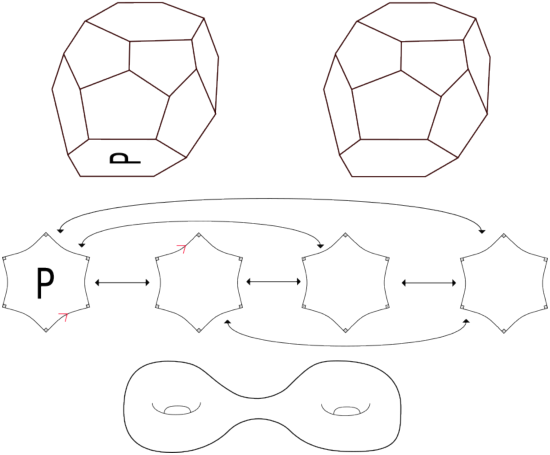

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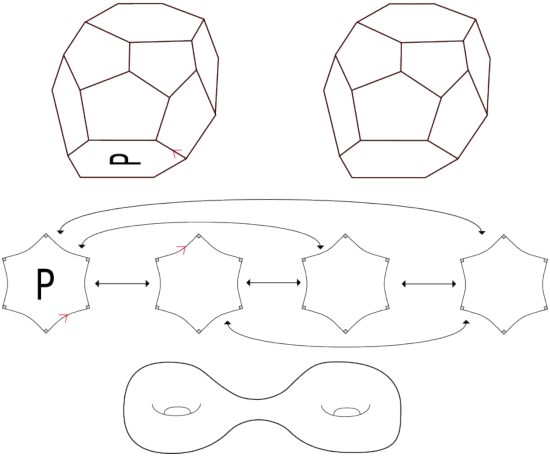

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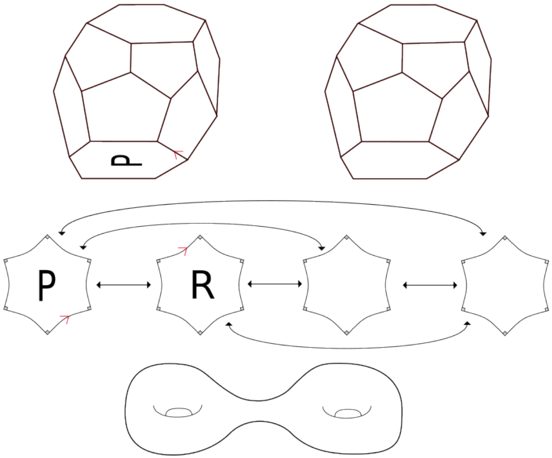

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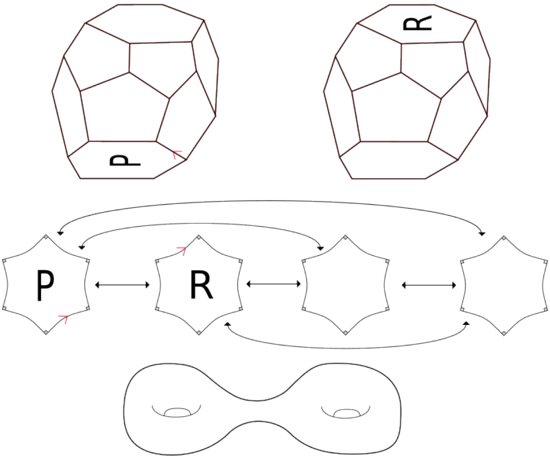

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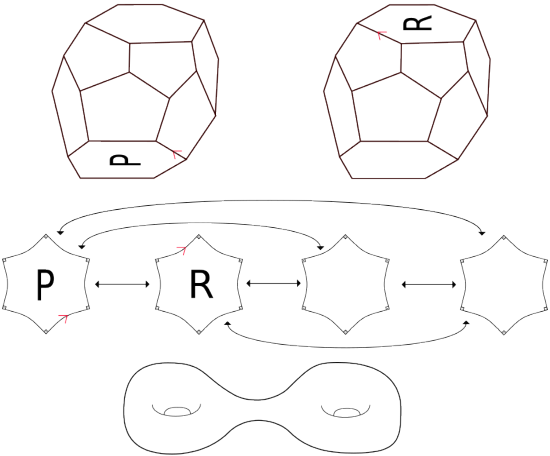

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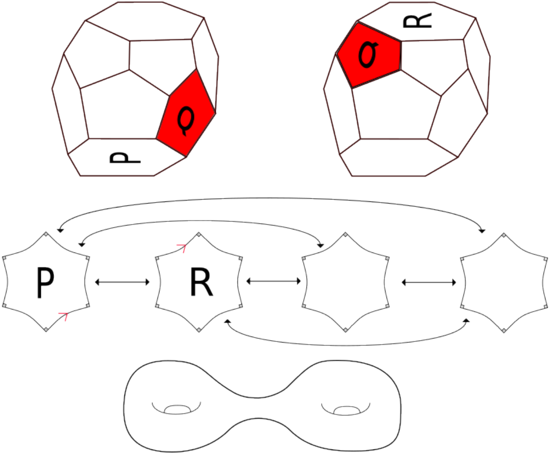

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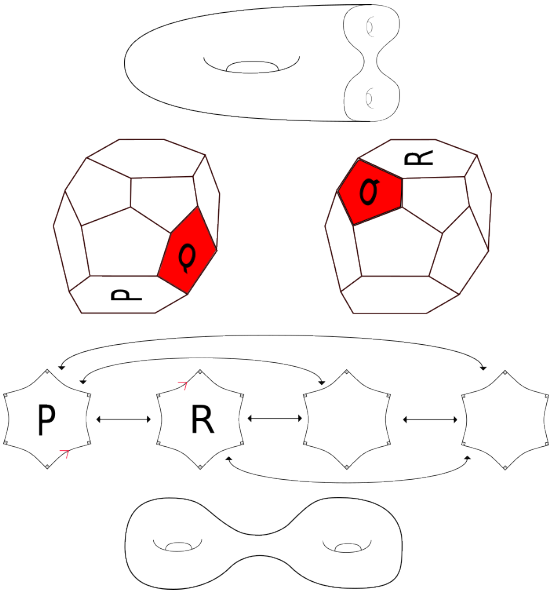

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