Aut-invariant quasimorphisms

Ventotene VI

Francesco Fournier-Facio
joint with Ric Wade

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Quasimorphisms

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Quasimorphisms are important in:

- Bounded cohomology;
- Stable commutator length;
- Geometric group theory;
- Knot theory;
- Symplectic geometry;
- Dynamics...

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All acylindrically hyperbolic groups have many homogeneous quasimorphisms.

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Theorem (Brooks, Epstein–Fujiwara, Fujiwara, Bestvina–Fujiwara)

All acylindrically hyperbolic groups have many homogeneous quasimorphisms.

(Here *many* means: infinite-dimensional, even modulo homomorphisms.)

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Lemma

Homogeneous quasimorphisms are conjugacy-invariant, i.e.

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Question (Miklós Abért, 2009)

Do non-abelian free groups admit Aut-*invariant homogeneous quasimorphisms?*

Theorem (FF–Wade)

Let G be a finitely generated group that is not virtually abelian. Suppose that either:

- *G is hyperbolic;*
- *G is a Right Angled Artin or Coxeter group;*
- *G has infinitely many ends...*

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Answers Abért's question, generalizes previous partial results of Brandenbursky–Marcinkowski and Karlhofer.
Recovers unboundedness results for Aut-invariant norms of Bardakov–Shpilrain–Tolstykh, Brandenbursky–Marcinkowski, Marcinkowski.

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At the opposite end of the spectrum:

Question

Does there exist a finitely generated group with many homogeneous quasimorphisms, but no Aut*-invariant ones?*

Thank you for your attention!