Boundary Representations of Locally Compact Hyperbolic Groups

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Boundary Representations

Definition

A locally compact group G is hyperbolic if it admits a proper cocompact isometric action on a proper geodesic Gromov hyperbolic metric space.

Examples

$\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{Q}_p)$, Discrete hyperbolic groups.

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Given a left invariant metric d on G (with some mild assumptions), one constructs a Patterson-Sullivan measure μ_d on the Gromov boundary ∂G. One can then consider the Koopman representation $\pi_d : G \to U(L^2(\mu_d))$.

Classification of Boundary Representations

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Theorem (G. 2023)

Let G be locally compact, second countable, unimodular and non-elementary hyperbolic then:

- *The representations π_d are irreducible.*
- *Two such representations π_{d_1} and π_{d_2} are unitarily equivalent if and only if there exist $L, C > 0$ such that:* $L \cdot d_2(g,h) - C \leq d_2(g,h) \leq L \cdot d_2(g,h) + C.$

This generalizes theorems by Garncarek [3] and by Bader, Muchnik [1].

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Proof.

π_{d_1} and π_{d_2} are weakly equivalent $\implies$ π_{d_1} and π_{d_2} are unitarily equivalent $\implies$ d_1 and d_2 are roughly similar. □

Thanks for Coming!

- Uri Bader and Roman Muchnik, *Boundary unitary representations—irreducibility and rigidity*, J. Mod. Dyn. **5** (2011), no. 1, 49–69. MR 2787597
- Pierre-Emmanuel Caprace, Mehrdad Kalantar, and Nicolas Monod, *A type i conjecture and boundary representations of hyperbolic groups*, Proceedings of the London Mathematical Society **127** (2023), no. 2, 447–486.
- Łukasz Garncarek, *Boundary representations of hyperbolic groups*, arXive preprint (2014).