Cocycle rigidity and bounded cohomology of groupoids

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Cocycles rigidity

Let $\mathbf{G}, \mathbf{H}$ be lcsc groups and $\mathbf{G} \curvearrowright (X, \mu_X)$ be a p.m.p action. A measurable map $\sigma : \mathbf{G} \times X \to \mathbf{H}$ is a **measurable cocycle** if

$$\sigma(g_1 g_2, x) = \sigma(g_1, g_2 x)\sigma(g_2, x)\,.$$

for all $g_1, g_2 \in \mathbf{G}$, $x \in X$

A cocycle is **rigid** if (up to *equivalence*) it does not depend on the parameter.

HIGHER RANK

Zimmer, '80. Orbit equivalent p.m.p actions of higher rank groups comes from isomorphisms of the groups. (Consequence of cocycle superrigidity)

RANK 1

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Cocycles and bounded cohomology

Following Burger-Iozzi, **continuous bounded cohomology** can be used to introduce **numerical invariants** of

- **representations** of real/complex hyperbolic lattices inside Hermitian groups ([Burger–Iozzi], [Burger–Iozzi–Wienhard],[Pozzetti]).
- **cocycles** of real/complex hyperbolic lattices in Hermitian groups ([Savini–Moraschini], [S.–Savini]).

Numerical invariants are real numbers attached to a **representation**/**measurable cocycle** invariant up to conjugation/cohomology.
They are obtained through the pullback of preferred cohomology classes along the **representation**/**measurable cocycle**.

$$\rho : \mathbf{G} \to \mathbf{H} \qquad \rightsquigarrow \qquad \rho^* : \mathrm{H}^\bullet_\mathrm{b}(\mathbf{H}, \mathbb{R}) \to \mathrm{H}^\bullet_\mathrm{b}(\mathbf{G}, \mathbb{R}) .$$

Problem: How we pull back cohomology classes along a cocycle?

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Problem: How we pull back cohomology classes along a cocycle?

A better perspective

A cocycle is a **groupoid homomorphism** of a p.m.p action $G \curvearrowright X$ to a group H.

Question: Can bounded cohomology be defined for (measured) groupoids?

YES! [S. –Savini, 23]

A few words:

- The construction is based on **Burger-Monod** theory of continuous bounded cohomology and bounded cohomology of groupoids extends the one of groups.
- Bounded cohomology of groupoids is invariant under orbit equivalence.
- For a group action $G \curvearrowright X$ we obtain

$$H_b^{\bullet}(G \curvearrowright X, \mathbb{R}) \cong H_b^{\bullet}(G, L^{\infty}(X, \mathbb{R})).$$

- For amenable groupoids (such as amenable actions) bounded cohomology vanishes.

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Thanks for your attention!