Embedding problems between L^p and ℓ^p spaces.

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Groups and Rigidity Around the Zimmer Program
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Definitions

1. $\ell_p = \{(x_n)_{n\in\mathbb{N}} : \sum_n |x_n|^p < \infty\}$.
2. $L_p = \{f : \mathbb{R} \to \mathbb{R}$ Lebesgue measurable $: \int_{\mathbb{R}} |f|^p \; d\mu < +\infty\}/ \sim$

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Linear embeddings

A linear map $T\colon (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ between normed vector spaces is a linear embedding if there exists a constant $M > 0$ s.t. for all $x \in X$:

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Bi-Lipschitz embeddings

A map $f\colon (X, d_X) \to (Y, d_Y)$ is called a bi-Lipschitz embedding if there exist constants $s > 0$ and $D \geq 1$ such that for any $x_1, x_2 \in X$ the following inequalities hold.

$$sd_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq sDd_X(x_1, x_2).$$

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Heinrich and Mankiewicz using Rademacher's differentiability theorem

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Coarse embeddings

A map $f\colon (X, d_X) \to (Y, d_Y)$ is a a coarse embedding if there exist non decreasing functions $\rho_-, \rho_+\colon [0,\infty) \to [0,\infty)$ such that $\lim_{t\to\infty} \rho_-(t) = \infty$ and for any $x_1, x_2 \in X$ the following inequalities hold.

$$\rho_-(d_X(x_1, x_2)) \leq d_Y(f(x_1), f(x_2)) \leq \rho_+(d_X(x_1, x_2)).$$

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Proposition

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Equivariant coarse embeddings

Because of the above one can view proper, affine, isometric actions on V as a special version of a coarse embedding. Admitting such an action can be called an equivariant coarse embedding into V.

Theorem (Ś)

If a normed vector space $(V, +)$ admits an equivariant coarse embedding into ℓ_p then it also embeds in a bi-Lipschitz way into ℓ_p.

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Theorem (Cornulier, Tessera, Valette)

Let G be a locally compact, compactly generated, amenable group. If G coarsely embeds into the Hilbert space, then there exists an equivariant coarse embedding of G into Hilbert space

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Question

Can we use similar techniques to reduce the coarse embedding problem of L_p into ℓ_p to the equivariant coarse embedding case that we've already solved?

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