Rigidity and stationary actions of arithmetic groups

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Calculus I

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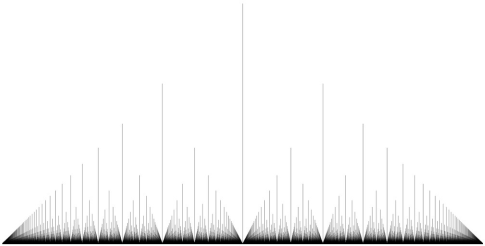
Answer: yes.

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$



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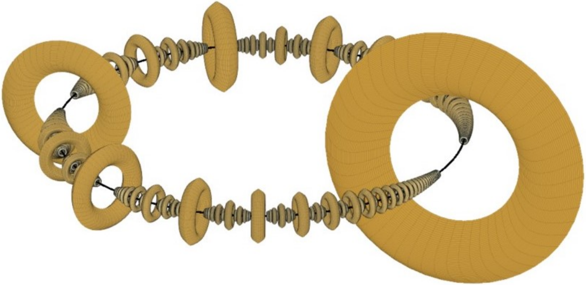
$X :=$ 

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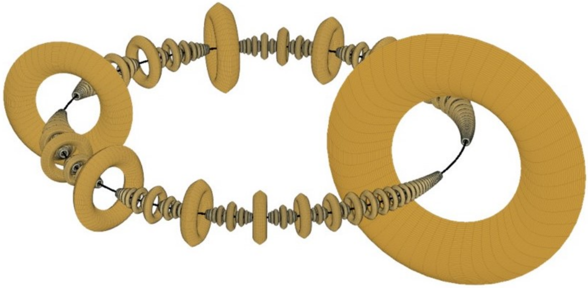
$X :=$ 

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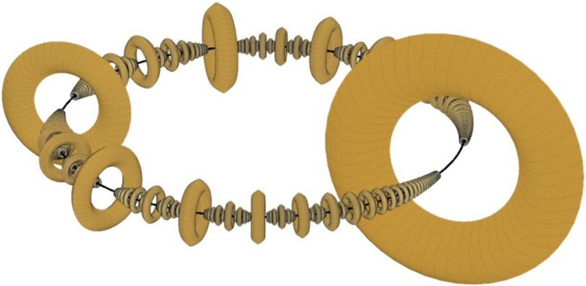
$X :=$ 

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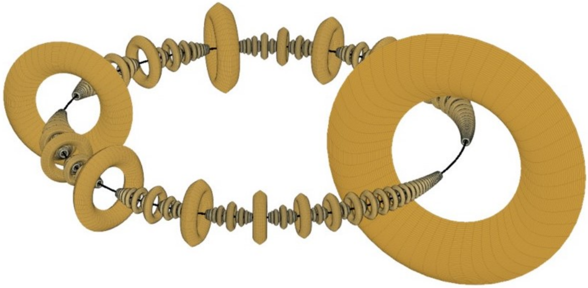
$X :=$ 

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Proposition 1.

The action of Γ *on* X *is* μ-*stiff* *(for many* $\mu \in \mathrm{Prob}(\Gamma)$*), that is, for any* $\nu \in \mathrm{Prob}(X)$*:*

$$\sum_{\gamma \in \Gamma} \mu(\gamma)\gamma.\nu = \nu \qquad \Rightarrow \qquad \forall \gamma \in \Gamma : \gamma.\nu = \nu$$

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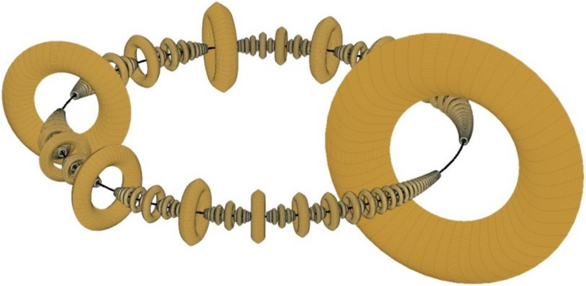
$\mathrm{Ch}(\mathrm{H}(\mathbb{Z})) =$ 

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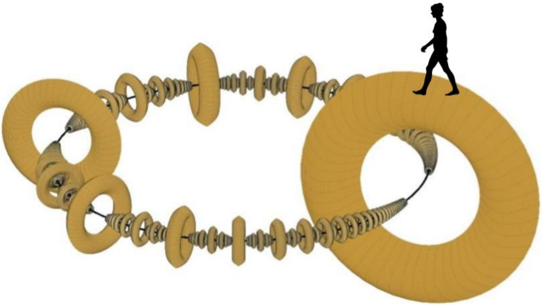

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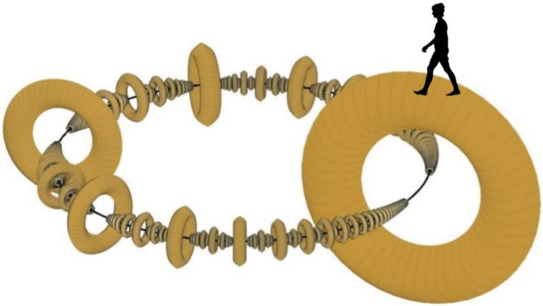

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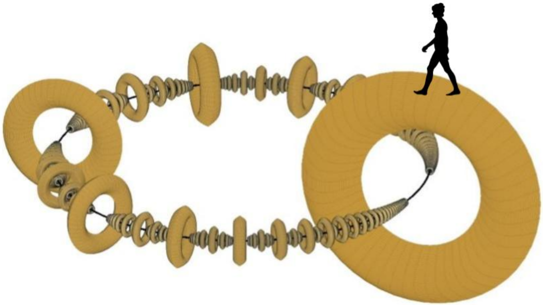

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All of the above can be generalized as follows:

Results [Bader-V 23']

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Let Γ be any arithmetic group (e.g $\mathrm{SL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$, $\mathrm{Sp}_{2n}(\mathbb{Z}) \ltimes \mathrm{H}_n(\mathbb{Z})$,...). Then the action of $\Gamma \curvearrowright \mathrm{Ch}(\mathrm{Rad}(\Gamma))$ is stiff.

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If the semisimple part of the arithmetic group Γ is of higher rank, then Γ is "charmenable". In particular:

1. *Every normal subgroup is either amenable or co-amenable.*
2. *Every trace is either amenable or supported on $\mathrm{Rad}(\Gamma)$.*
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6. *Something about C^*-algs and von Neumann algs.*

Thank you

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