Minimal volume entropy of mapping tori of 3-manifolds

Giuseppe Bargagnati
joint work with Alberto Casali, Francesco Milizia, and Marco Moraschini

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Minimal volume entropy

Setting: (M, g) closed Riemannian manifold.

Definition

The *volume entropy* of (M, g) is

$$\mathrm{Ent}(M, g) := \lim_{R \to +\infty} \frac{1}{R} \cdot \log\big(\mathrm{Vol}_{\tilde{g}}(B(x, R))\big).$$

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The *minimal volume entropy* of M is

$$\mathrm{MinEnt}(M) := \inf_{g\,\in\,\mathrm{Riem}(M)} \mathrm{Ent}(M,g)\,\mathrm{Vol}(M,g)^{\frac{1}{\dim(M)}}.$$

Facts and questions about MinEnt

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Question/Conjecture

Does the following implication hold?

$$\|M\| = 0 \Rightarrow \mathrm{MinEnt}(M) = 0$$

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Naive approach: try to compute MinEnt of manifolds whose simplicial volume is known to vanish.

Mapping tori

Definition

Let M be a smooth manifold and $f\colon M \to M$ a diffeomorphism. The mapping torus of f of M is

$$M_f := M \times [0,1]/_{(x,0)\sim(f(x),1)}.$$

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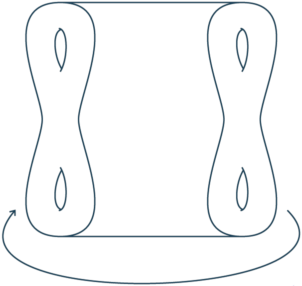

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Theorem (BN20)

Let M be a connected closed oriented 3-manifold and let $f\colon M \to M$ be a self-diffeomorphism. Then

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$$\operatorname{MinEnt}(M_f) = 0.$$

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Thank you for the attention!