Deformations of hyperbolic 3-manifolds in $\mathbb{H}^5$

A combinatorial approach featuring quaternions

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Ventotene 2025

Introduction

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How? With a combinatorial approach based on an ideal triangulation of M, generalizing Thurston's classic framework.

Why dimension 5?

Let $\mathcal{H}$ be the algebra of quaternions.The algebraic motivation is the inclusion $\mathbb{C} \hookrightarrow \mathcal{H}$.

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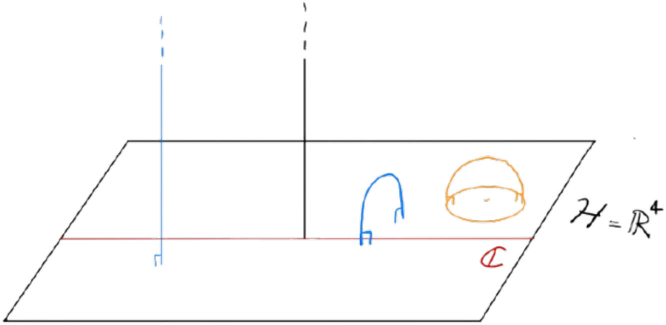

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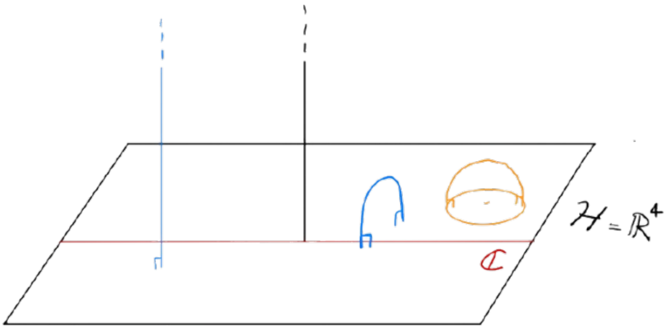
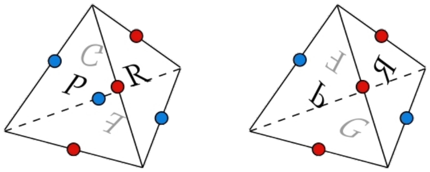


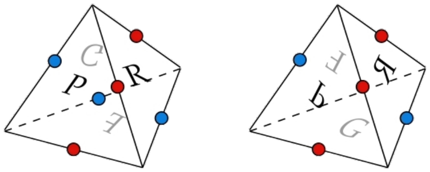
With the upper-half space model, we have $\mathbb{H}^5 \cong \mathcal{H} \times \mathbb{R}_+$ and

$$\partial_\infty \mathbb{H}^5 = \hat{\mathcal{H}}, \quad \mathrm{Isom}^+(\mathbb{H}^5) \cong \mathrm{PSL}(2, \mathcal{H}).$$



The figure-8 knot complement

- $\mathcal{T}$ = ideal triangulation of M. We assign algebraic data to the combinatorial pieces of $\mathcal{T}$ in a way that is globally consistent.



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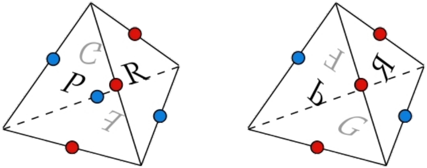

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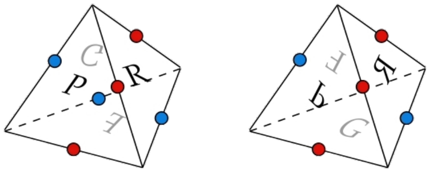

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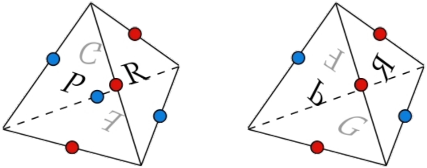

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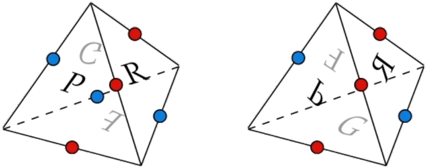
Parameters	A **complex shape parameter** ($z \in \mathbb{C}$) per edge.	
Conditions		



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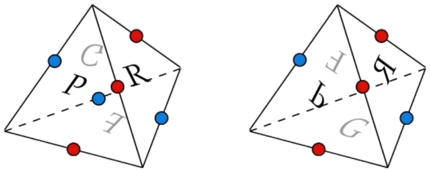

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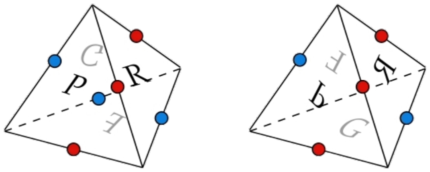

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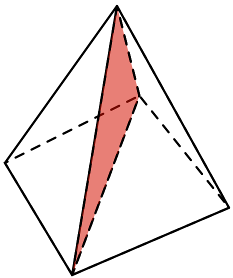

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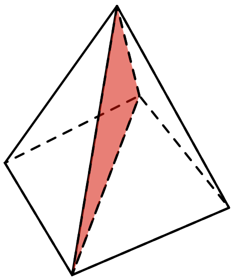

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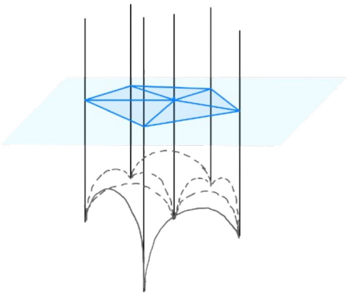

The Cusp Condition

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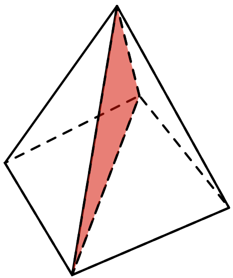

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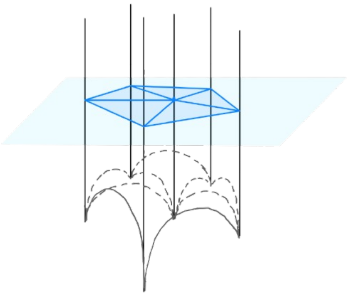

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Conjecture (Almost Theorem!)

An assignment of quaternionic parameters to the ideal triangulation $\mathcal{T}$ induces a represention $\pi_1(M) \to \mathrm{PSL}(2, \mathcal{H})$ if and only if it satisfies two main axioms:

1. Cusp condition
2. Face condition

Thank you for your attention!