First-order theory of torsion-free Tarski monsters

Ventotene VII – Lightning talk

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G is uniformly perfect iff there exists some n such that G satisfies the positive sentence

$$\forall g \exists x_1, y_1, \dots, x_n, y_n : [x_1, y_1] \cdots [x_n, y_n] = g.$$

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- Non-elementary hyperbolic groups (Sela 2009, Heil 2018);
- Acylindrically hyperbolic groups (André, Fruchter 2022);
- Many groups acting on trees (Casals-Ruiz, Garreta, de la Nuez González 2021).

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In each of these respects, this is drastically different from previous examples.

The construction

G arises as the limit of a sequence of small cancellation quotients

$$\Gamma_0 \twoheadrightarrow \Gamma_1 \twoheadrightarrow \Gamma_2 \twoheadrightarrow \cdots \twoheadrightarrow G$$

where Γ_0 is an arbitrary torsion-free non-elementary hyperbolic group - with property (T).

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The triviality of the first order theory then follows from the result on hyperbolic groups (Sela, Heil) and quantifier reduction for positive sentences (Casals-Ruiz, Garreta, de la Nuez González).

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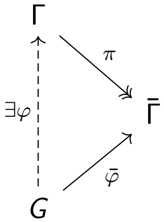
Theorem (Coulon, FF, Ho 2025)

Let Γ be a torsion-free non-elementary hyperbolic group. Let G be a finitely generated group, .
If $\pi\colon \Gamma \to \bar{\Gamma}$ is a nice enough small cancellation quotient, then every morphism $\bar{\varphi}\colon G \to \bar{\Gamma}$ lifts to Γ.



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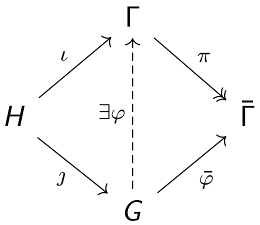
...is to relate the theory of a torsion-free hyperbolic group Γ to that of its small cancellation quotient $\bar{\Gamma}$.

Theorem (Coulon, FF, Ho 2025)

Let Γ be a torsion-free non-elementary hyperbolic group. Let G be a finitely generated group, H an arbitrary group with morphisms $\jmath\colon H \to G$ and $\iota\colon H \to \Gamma$.
If $\pi : \Gamma \to \bar{\Gamma}$ is a nice enough small cancellation quotient, then every morphism $\bar{\varphi}\colon G \to \bar{\Gamma}$ compatible with $H, \jmath$ and ι lifts to Γ.



Thank you for the attention!