HIGHER COARSE MEDIANS IN HIGHER RANK

j/w Mitul Islam

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Grazia Rago

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Goal: define a coarse n-median on $SL_n\mathbb{R}/SO(n)$

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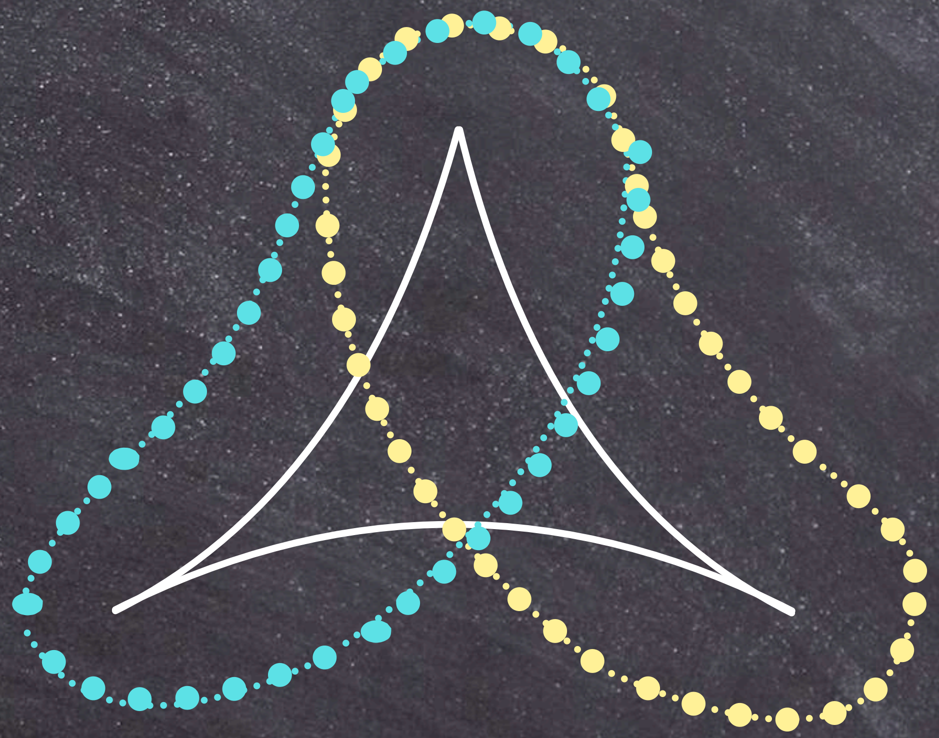
Def. Given $v^0, \dots, v^k \in \overline{\Omega}$, the straight simplex $S = CH(v^0, \dots, v^k)$ is

- δ-slim if $F^i \subseteq \bigcup_{j \neq i} N_\delta(F^j)$ for all $i = 0, \dots, k$

  w/ $F^i = CH(v^0, \dots, \hat{v}^i, \dots, v^k)$



$N_\delta(\ )$ = δ-neighbourhood for the Hilbert metric

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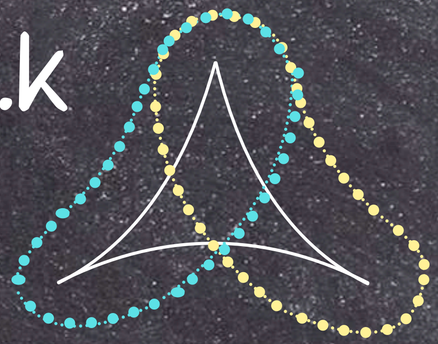

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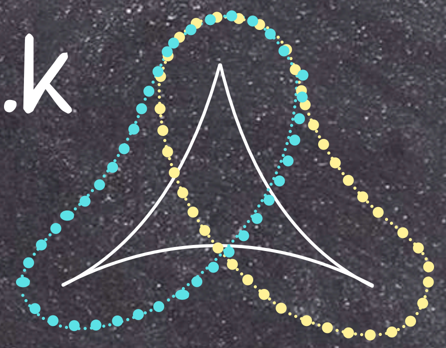

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Proposition [Islam-R.] : i.e. Ω has a cocompact action by isometries

Let Ω be a divisible p.c.d. Fix $r \in \mathbb{N}$. TFAE:

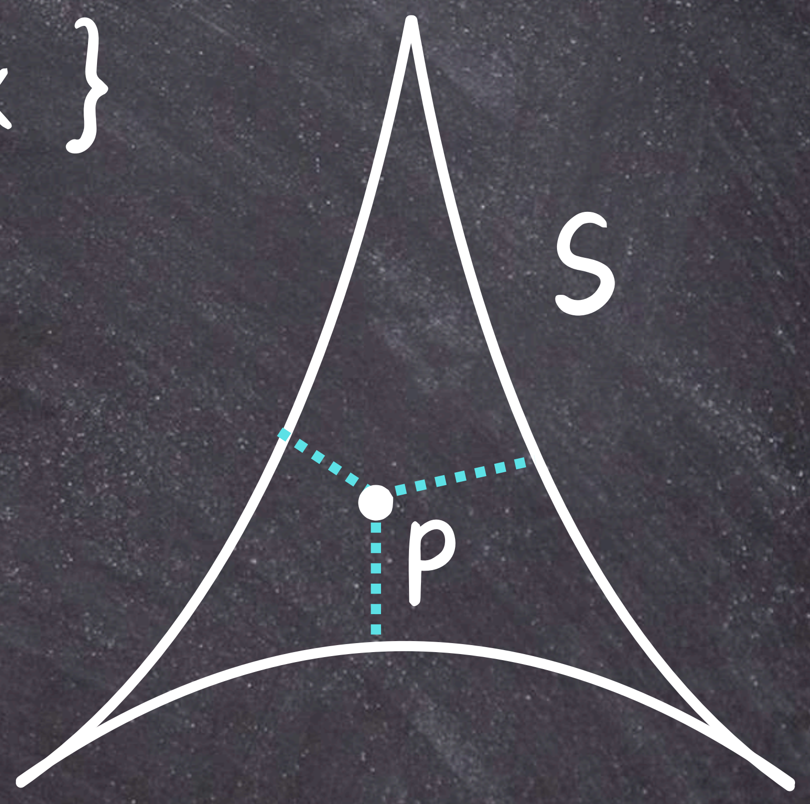
- The maximum dimension of a PES in Ω is less then or equal to r
- There exists $\delta > 0$ such that all $(r+1)$-simplices are δ-slim

Def. [δ-centroid]

Let $\Omega \subset \mathbb{R}\mathbb{P}^d$ be a p.c.d. and $S \subset \Omega$ be a simplex. Fix $\delta > 0$.

$$C_\delta(S) := \{ p \in S \mid d_\Omega(p , F^i) < \delta \text{ for all } i=0,\dots,k \}$$

Hilbert metric (↑ d_Ω)



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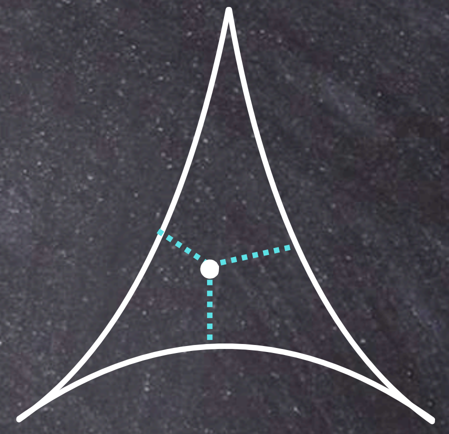

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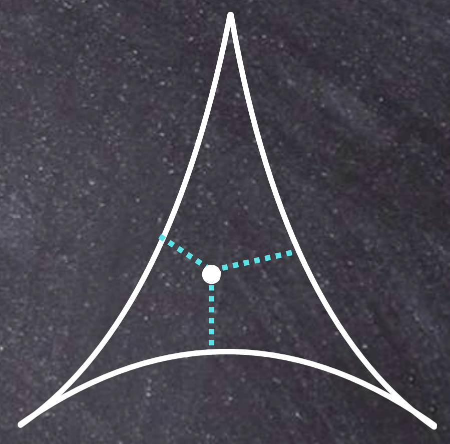

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Let Ω be a <u>divisible</u> p.c.d. and $1 \leq r \leq d$ be the max dim of a PES. Fix $\delta > 0$ s.t. any $(r+1)$-simplex is δ-slim. There is a constant $K(\delta) > 0$ s.t. if $v^0, \dots, v^{r+1} \in \Omega$ are generic, then

$$\operatorname{diam}_\Omega(C_\delta(CH(v^0, \dots, v^{r+1}))) < K(\delta).$$

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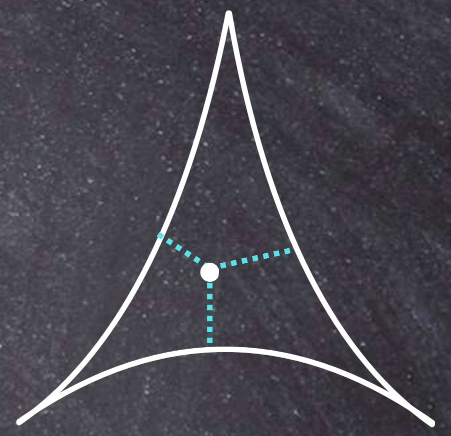

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Theorem [Islam-R.]: (Ω, d_Ω) admits a coarse r-median.

Thanks for your attention !