Connected Components of spaces of Type-Preserving Representations

Inyoung Ryu

joint w/ Tian Yang

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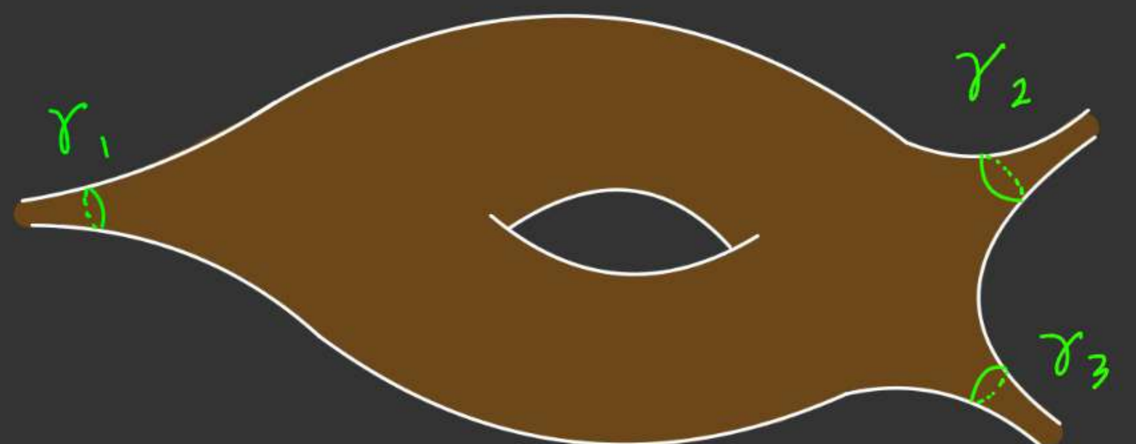
Def. Type-Preserving Representations

$\phi : \pi_1(S) \to PSL(2,\mathbb{R})$, $\phi(c_i)$: Parabolic for $i = 1, 2, \cdots, p$.



$c_i = [\gamma_i] \in \pi_1(S)$.

$R(S)$: The space of type-preserving representations.

Q. How many connected components does $R(S)$ have?

1. Relative Euler Class

Theorem (Goldman, Milnor-Wood Inequality)

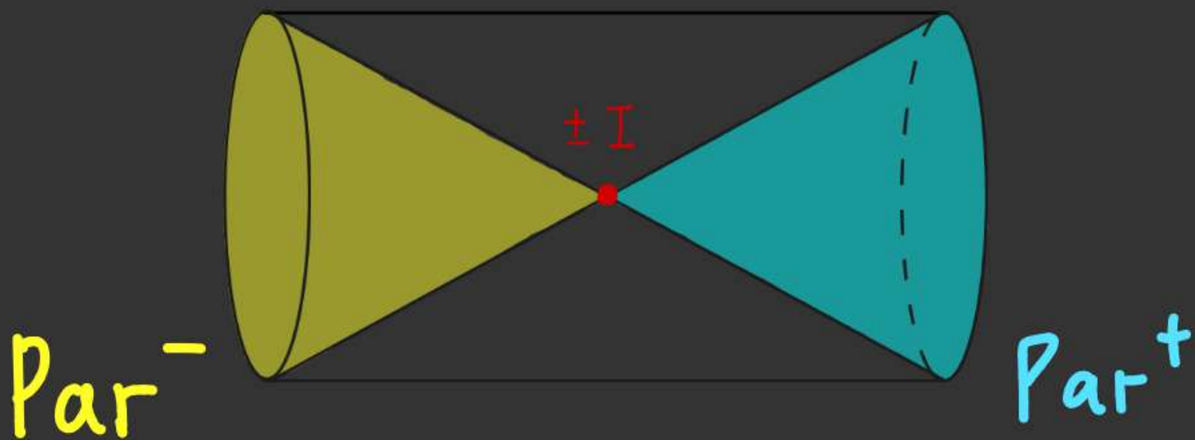
Let $S = S_g$ $(g > 1)$ and $n \in \mathbb{Z}$. Denote by $\mathrm{H}_n(S)$ the space of representations in $\mathrm{Hom}\big(\pi_1(S), \mathrm{PSL}(2,\mathbb{R})\big)$ with Euler class n. Then $\mathrm{H}_n(S)$ is nonempty if and only if

$$\chi(S) \leqslant n \leqslant -\chi(S).$$

Moreover, each nonempty $\mathrm{H}_n(S)$ is connected.

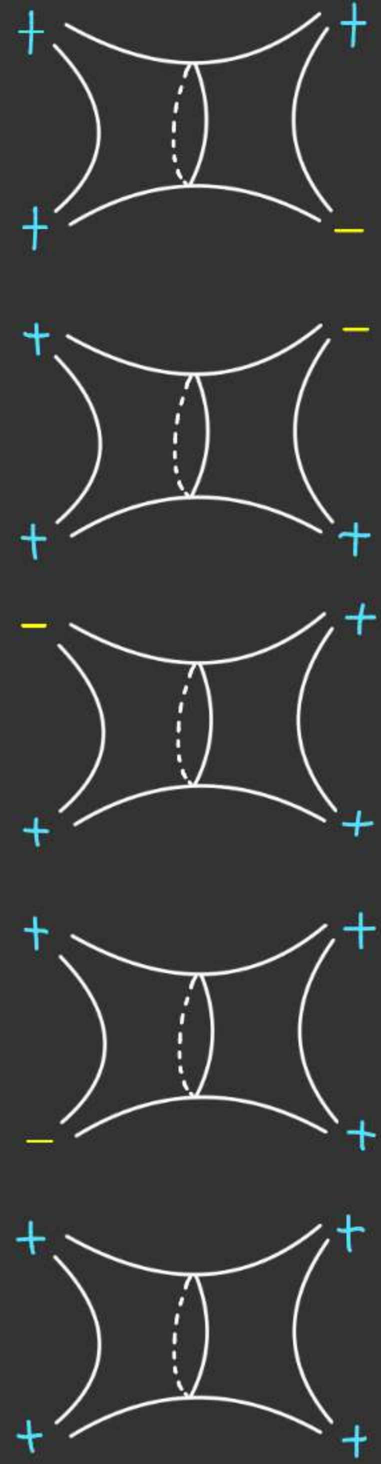
★ Euler Class is constant on each connected component!

2. Sign



$$s(\phi) = (s_1, s_2, \cdots, s_p)$$

$$s_i = \begin{cases} + & \phi(c_i) \in Par^+ \\ - & \phi(c_i) \in Par^- \end{cases}$$



$$R_n^s(S) = \{\phi \in R(S) \mid e(\phi) = n, s(\phi) = s\}$$

$\boxed{\begin{array}{l}\text{Q1. When is } R_n^s(S) \neq \phi\,? \\ \text{Q2. Is } R_n^s(S) \text{ connected?}\end{array}}$

Main Result.

Theorem (R. - Yang)

Let $S = S_{g,p}$ *for* $g > 0$. *Let* $p_\pm(s)$ *respectively be the number of* $+$ *and* $-$ *in* $s \in \{+,-\}^p$. *Then* $\mathcal{R}_n^s(S)$ *is nonempty if and only if*

$$\chi(S) + p_+(s) \leqslant n \leqslant -\chi(S) - p_-(s).$$

Moreover, each nonempty $\mathcal{R}_n^s(S)$ *is connected.*

Corollary

Let $S = S_{g,p}$ *with* $g > 0$. *Then the number of connected components of* $\mathcal{R}_n(S)$ *equals*

$$\sum_{k=\max\{n+p+\chi(S),0\}}^{\min\{n-\chi(S),p\}} \binom{p}{k}.$$

Hence $\mathcal{R}(S)$ *has* $\displaystyle\sum_{n=\chi(S)}^{-\chi(S)} \sum_{k=\max\{n+p+\chi(S),0\}}^{\min\{n-\chi(S),p\}} \binom{p}{k}$ *components.*

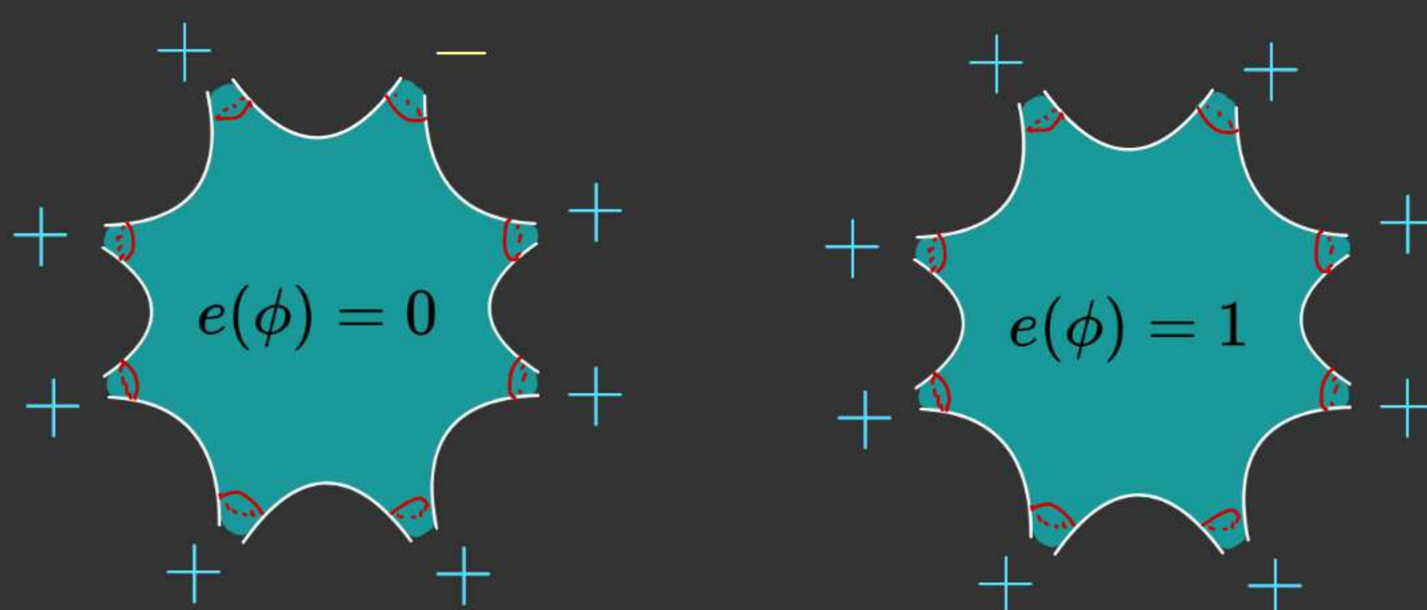
Theorem (R. - Yang)

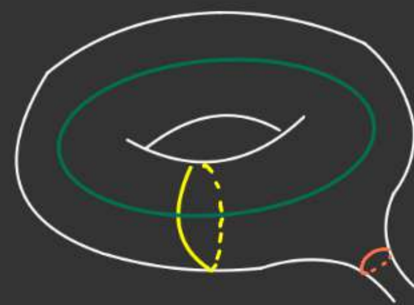
When $S = S_{0,p}$, there are extra components that do not satisfy the inequality:

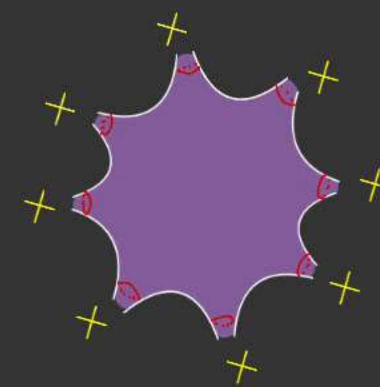
1. *$2p$ components $\mathcal{R}_n^s(S)$ with $n = 0$, $p_+(s) = 1$ or $p_-(s) = 1$ consisting of abelian representations.*
2. *2 components $\mathcal{R}_n^s(S)$ with $(n, s) = (1, +, \cdots, +)$ and $(-1, -, \cdots, -)$.*

Hence the number of components of $\mathcal{R}(S)$ is

$$\sum_{n=\chi(S)}^{-\chi(S)} \sum_{k=\max\{n+p+\chi(S),0\}}^{\min\{n-\chi(S),p\}} \binom{p}{k} + 2p + 2.$$







Thank you!