Symplectic geometry and Hamiltonian Dynamics

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Lecture plan:

1. Symplectic spaces and manifolds
   • Definition of symplectic spaces [3, Sec. 1.1]
   • Orthogonal completion, symplectic, isotropic, co-isotropic and Lagrangian subspaces
   • Symplectic spaces decomposition; symplectic spaces are of even dimension [3, Prop. 1, Sec 1.1]
   • The group of symplectic linear maps $Sp(n)$
   • Definition of symplectic manifolds
   • Examples of symplectic manifolds: $\mathbb{R}^{2n}$, surfaces, $T^*Q$ and tautological 1-form [1, Sec. 2],

2. Lagrangian submanifolds and symplectomorphisms
   • Definition of symplectic and Lagrangian submanifolds
   • Lagrangian submanifolds of cotangent bundles [1, Sec. 3.2], [4, Prop. 3.26]
   • The group of symplectomorphisms $Symp(M, \omega)$
   • Symplectic and Hamiltonian vector fields [1, Def 18.2]
   • Hamiltonian symplectomorphisms and $Symp_0(M, \omega)$ as subgroups of $Symp(M, \omega)$ [4, Prop. 3.2]
   • Product bundles regular and twisted
   • Lagrangian graphs in twisted product bundles and symplectomorphisms[1, Thm. 3.8]
   • Symplectomorphisms are volume preserving

3. Symplectic rigidity and flexibility
   • Darboux theorem [3, Thm. 1, Sec. 1.3]
   • Equivalent definition of symplectic manifold via atlases
• Symplectic manifolds are locally indistinguishable
• Diffeomorphic surfaces of equal volume are symplectomorphic [3, Thm. 2, Sec. 1.3]
• Linear non-squeezing theorem (Sec. 2.4, [4])

4. Hamiltonian dynamics [3, Sec. 1.4]
• Definition of a Hamiltonian vector field
• Definition of a defining Hamiltonian of a hypersurface
• Defining Hamiltonians of a hypersurface generate the same dynamics up to reparametrization
• Periodic orbits as invariants of symplectomorphisms
• Arnold conjecture
• Existence of a Hamiltonian flow invariant measure
• Poincaré’s recurrence theorem [3, Thm. 3, Sec. 1.4]

5. Lagrangian dynamics and Legendre transform
• Classical mechanics
• Euler-Lagrange equations
• Fiber derivative diffeomorphism
• Energy function
• Legendre transform
• Conjugacy of the Euler-Lagrange flow and the Hamiltonian flow via the fiber derivative

6. Mechanical and magnetic Hamiltonian systems
• Jacobi metric for an energy level of a mechanical Hamiltonian
• Geodesics of the Jacobi metric [3, Sec. 4.4]
• Levi-Civita connection [5, Def. 31.8]
• Canonical splitting of $T(T^*Q)$ [5, Lem. 31.3]
• Twisted cotangent bundles
• Mechanical and magnetic Hamiltonians

7. Periodic orbits on a hypersurface
• Definition of a characteristic line bundle and closed characteristics [3, Sec. 4.2]
• Hamiltonian vector field spanning the characteristic line bundle of their regular level sets
• Orientable hypersurface $S$ admits a Hamiltonian $H$, such that $S$ is a regular level set of $H$ [3, Prop. 3, Sec. 4.2]
8. Contact type property

- Examples of contact type hypersurfaces - star-shaped hypersurfaces [3, Sec. 4.3] and regular level sets of mechanical Hamiltonians [3, Sec. 4.4];
- Dented sphere as example of a non-contact type hypersurface [3, Sec. 4.3];
- Contact type property and isomorphism of characteristic line bundles [3, Sec. 4.3];
- Weinstein conjecture

9. Contact manifolds

- A distribution of codimension 1 is locally defined as a kernel of a 1-form and it is co-orientable if and only if it is globally a kernel of a 1-form [2, Lem. 1.1.1];
- Recall definition of integrability and Frobenius integrability condition [5, Def. 11.11, Thm. 11.18];
- Definition of a contact manifold, contact structure, contact form [2, Def. 1.1.3];
- Remark: contact structure does not depend on the contact form and admits many contact forms;
- Examples of contact manifolds:
  - contact type hypersurfaces;
  - standard contact structure on $\mathbb{R}^{2n+1}$ [2, Ex. 1.1.5];
  - overtwisted contact structure in $\mathbb{R}^3$ [2, Ex. 2.1.6];
- Definition of a contactomorphism;
- Submanifolds of a contact manifold: contact, isotropic, Legendrian [2, Def. 1.5.11, Def. 1.5.13];
- Remark: dimension of a isotropic manifold is $\leq n$ [2, Prop. 1.5.12];
- Symplectic fillings [2, Def. 1.7.14];
- Symplectization of a contact manifold and symplectization of a strong symplectic filling [2, Ex. 1.4.7];
- Definition of the Reeb vector field, proof of uniqueness [2, Def. 1.1.9];
- Remark: Reeb vector field is associated to a contact form, not to a contact structure.
- Characteristic foliation [2, Def. 2.5.18] and examples of thereof:
  - on a sphere [2, Ex. 2.5.19];
on standard overtwisted disks [2, Sect. 4.5].

- Definition of an overtwisted disk [2, Def. 4.5.1], tight and overtwisted contact structures [2, Def. 4.5.2];
- Theorem (Eliashberg, Gromov): Overtwistedness as an obstruction to fillability [2, Thm. 6.5.6].

10. Symplectic neighborhood theorems and applications

- Moser theorem of global symplectomorphisms [1, Thm. 7.2];
- Moser theorem of symplectomorphic neighborhoods [1, Thm. 7.4];
- Whitney extension theorem [1, Thm. 8.5];
- Weinstein tubular neighbourhood theorem [1, Thm. 9.2];
- Tangent space to the group of symplectomorphisms [1, Sec. 9.3];
- Fixed points of symplectomorphisms [1, Thm. 9.6].

11. Complex structures on vector spaces

- Definition of a complex structures on a vector space [1, Def. 12.1];
- Every complex space is isomorphic to \((\mathbb{R}^{2n}, J_0)\) [4, Prop. 2.47];
- Definition of an \(\omega\)-compatible complex structure on a symplectic vector space [1, Def. 12.2];
- An \(\omega\)-compatible complex structure defines a positive inner product on a symplectic vector space;
- Existence of \(\omega\)-compatible complex structures on a symplectic vector space [1, Prop. 12.3].

12. Almost complex manifolds

- Definition of an almost complex manifold [1, Def. 12.4] and a complex manifold [1, Def. 15.1];
- Definition of an \(\omega\)-compatible almost complex structure on a symplectic manifold [1, Def. 12.5];
- Symplectic manifolds are almost complex [1, Prop. 12.6];
- Set of \(\omega\)-compatible almost complex structures is path-connected [1, Prop. 12.8].

References


