2 Lie algebras and the Poisson bracket

Definition 2.1. A Lie algebra is a vector space \( g \) over some field \( \mathbb{F} \) together with a binary operator \([\cdot, \cdot] : g \times g \rightarrow g\) called the Lie bracket, satisfying the following axioms:

- bilinearity;
- alternativity \([x, x] = 0\) for all \( x \in g \);
- the Jacobi identity
  \[ [x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0, \]
  for all \( x, y, z \in g \).

Exercise 2.1. Show that if \((g, [\cdot, \cdot])\) is a Lie algebra, then \((g, -[\cdot, \cdot])\) is also a Lie algebra.

Exercise 2.2. Let \( e_1, e_2, e_3 \) be the standard basis of \( \mathbb{R}^3 \). Show that if we set
\[
e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_2 \times e_3 = e_1,
\]
then we can extend it to a bilinear and anti-symmetric \( \times : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \). We call this map the cross product on \( \mathbb{R}^3 \). Show that \( \mathbb{R}^3 \) with the cross product is a Lie algebra.

Exercise 2.3. Let \( M \) be a smooth manifold, \( \mathfrak{X}(M) \) the set of smooth vector fields on \( M \). We define a Lie bracket \([\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)\) in the following way:
\[
[X, Y](f) := Y(X(f)) - X(Y(f)), \quad \forall X, Y \in \mathfrak{X}(M), \quad \forall f \in C^\infty(M).
\]
Show that \((\mathfrak{X}(M), [\cdot, \cdot])\) is a Lie algebra.

Exercise 2.4. Let \( X, Y \in \mathfrak{X}(M) \) be complete vector fields on a manifold \( M \) and let \( \phi_X, \phi_Y \) be the flows of \( X \) and \( Y \) respectively. Show that \([X, Y] \equiv 0\) if and only if the flows of \( X \) and \( Y \) commute, i.e., \( \phi_X \circ \phi_Y = \phi_Y \circ \phi_X \).

Exercise 2.5. Let \( M \) be a smooth manifold, \( \mathfrak{X}(M) \) the set of smooth vector fields on \( M \), \( \Omega(M) \) be the set of differential forms on \( M \). Show that for \( X, Y \in \mathfrak{X}(M) \) and \( \alpha \in \Omega(M) \) we have
\[
\iota_{[X, Y]} \alpha = \iota_Y \mathcal{L}_X \alpha - \mathcal{L}_X \iota_Y \alpha, \quad (2.1)
\]
where \([\cdot, \cdot]\) is the Lie bracket and \( \mathcal{L}_X \) is the Lie derivative.

Definition 2.2. Let \((M, \omega)\) be a symplectic manifold. We define the Poisson bracket on the set of smooth functions \( C^\infty(M) \) in the following way:
\[
\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M),
\]
\[
\{f, g\} := \omega(X_f, X_g),
\]
where \( X_f, X_g \) are the Hamiltonian vector fields corresponding to \( f \) and \( g \) respectively.
Theorem 2.3. On a symplectic manifold \((M, \omega)\) the set of smooth functions \(C^\infty(M)\) equipped with the Poisson bracket form a Lie algebra.

Before proving the theorem, we first will prove the following lemma:

Lemma 2.4. Let \((M, \omega)\) be a symplectic manifold. Then for any two smooth functions \(f, g \in C^\infty(M)\) one has

\[
X_{\{f, g\}} = -[X_f, X_g]. \tag{2.2}
\]

Proof. Let \(X, Y \in \mathfrak{X}(M)\) be two symplectic vector fields. Then by (2.1) we have

\[
i_{[X, Y]} \omega = i_Y \mathcal{L}_X \omega - \mathcal{L}_X i_Y \omega = -d i_X i_Y \omega - i_X d i_Y \omega = d(\omega(X, Y)).
\]

Consequently, for \(f, g \in C^\infty(M)\) we have

\[
i_{X_{\{f, g\}}} \omega = -d(\{f, g\}) = -d(\omega(X_f, X_g)) = -i_{[X_f, X_g]} \omega,
\]

which gives (2.2). \(\square\)

Corollary 2.5. Hamiltonian vector fields on a symplectic manifold \((M, \omega)\) form a sub-algebra of \((\mathfrak{X}(M), [\cdot, \cdot])\).

Proof of Theorem 2.3: The bilinearity and alternativity of the Poisson bracket are straightforward. What is left to prove is the Jacobi identity. Let \(f, g, h \in C^\infty(M)\), then

\[
\{h, \{f, g\}\} = -\{\{f, g\}, h\} = -dh(X_{\{f, g\}}) = dh([X_f, X_g]) = X_g(h(X_f)) - X_f(h(X_g))
\]

\[
= \{g, \{f, h\}\} - \{f, \{g, h\}\} = -\{g, \{h, f\}\} - \{f, \{g, h\}\}.
\]

\(\square\)

Corollary 2.6. For a symplectic manifold \((M, \omega)\) the map

\[
C^\infty(M) \to \mathfrak{X}(M),
\]

\[
H \mapsto X_H,
\]

is a Lie algebra anti-homomorphism.

Let \(((M, \omega), H)\) be a Hamiltonian system with \(\varphi^t\) being the Hamiltonian flow. Then for any \(f \in C^\infty(M)\)

\[
\frac{d}{dt} f \circ \varphi^t = \{H, f\}.
\]

This defines a dynamics on the set of observables \(C^\infty(M)\).

Definition 2.7. A function \(f \in C^\infty(M)\) satisfying \(\{H, f\} = 0\) is called an integral of motion for the Hamiltonian system \(((M, \omega), H)\).

Remark 2.8. The solutions to Hamilton’s equations of a given Hamiltonian system are contained in the level sets of integrals of motion.

Exercise 2.6. Show that integrals of motion of a Hamiltonian system \(((M, \omega), H)\) form a sub-algebra of \((C^\infty(M), \{\cdot, \cdot\})\).
Question: How to find integrals of motion?