9 Axioms of quantum mechanics

9.1 Projections

Exercise 9.1. Show that \( P \) is an orthogonal projection if and only if there exists a closed subspace \( X \) be a closed subspace of \( \mathcal{H} \), such that

\[
P(x, y) = x, \quad (x, y) \in X \oplus X^\perp = \mathcal{H},
\]

where \( X^\perp \) denotes \( X \)’s orthogonal complement.

Recall that the space of orthogonal projections \( \mathcal{B}_{op}(\mathcal{H}) \) is a subset of \( \mathcal{B}(\mathcal{H}) \). Moreover, every \( P \in \mathcal{B}_{op}(\mathcal{H}) \) is a positive operator. Naturally, for every \( x \in \mathcal{H}, \|x\| = 1 \) the map

\[
y \mapsto \langle y, x \rangle x,
\]

is an orthogonal projection on the space \( \mathbb{C}x \). This way we have \( \mathcal{H} \subseteq \mathcal{B}_{op}(\mathcal{H}) \).

Exercise 9.2. Let \( P_1, P_2 \) be orthogonal projections onto subspaces \( X_1, X_2 \) of \( \mathcal{H} \), respectively. Prove the following:

1. \( P_1 + P_2 \) is an orthogonal projection if and only if \( X_1 \perp X_2 \). In that case \( P_1 + P_2 \) is an orthogonal projection onto the subspace \( X_1 \oplus X_2 \);
2. \( P_1 P_2 \) is an orthogonal projection if and only if \( P_1 P_2 = P_2 P_1 \). In that case \( P_1 P_2 \) is an orthogonal projection on \( X_1 \cap X_2 \);
3. \( X_1 \subseteq X_2 \) if and only if \( P_1 P_2 = P_1 = P_2 P_1 \).

Exercise 9.3. Let \( \{X_m\}_{m \in \mathbb{N}} \) be a sequence of closed subspaces of \( \mathcal{H} \), such that \( \bigoplus_{m=1}^{\infty} X_m \) is a closed subspace of \( \mathcal{H} \). Let \( \{P_m\}_{m \in \mathbb{N}} \) be a sequence of orthogonal projections on the subspaces \( X_m \), such that \( P_m P_n = 0, \ m \neq n \). Let \( P \) be an orthogonal projection on \( \bigoplus_{m=1}^{\infty} X_m \). Show that for every \( x \in \mathcal{H} \) we have

\[
P x = \lim_{n \to \infty} \sum_{m=1}^{n} P_m x.
\]

Definition 9.1. A resolution of the identity on a Hilbert space \( \mathcal{H} \) is a family of orthogonal projections \( \{P_s\}_{s \in \mathbb{R}} \), such that:

a) if \( s \leq t \) then \( P_s P_t = P_s \);

b) for all \( x \in \mathcal{H} \) one has \( \lim_{s \downarrow s_0} P_s x = P_{s_0} x \);

c) for all \( x \in \mathcal{H} \) one has \( \lim_{s \to -\infty} P_s x = 0 \) and \( \lim_{s \to -\infty} P_s x = x \).
9.2 Trace-class operators

Definition 9.2. An operator $A \in \mathcal{B}(\mathcal{H})$ is of trace class if for every orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of $\mathcal{H}$

$$\sum_{n=1}^{\infty} |\langle Ae_n, e_n \rangle| < +\infty.$$ 

We denote the set of trace class operators by $\mathcal{B}_1(\mathcal{H})$. The trace of a trace class operator is defined by

$$\text{Tr} A := \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle.$$ 

On the space $\mathcal{B}_1(\mathcal{H})$ we define a norm

$$\|A\|_1 := \text{Tr} |A|.$$

Exercise 9.4. Show that the function $\text{Tr} : \mathcal{B}_1 \to \mathbb{C}$ is a continuous, linear functional. Moreover, for every $A \in \mathcal{B}_1(\mathcal{H})$, $\text{Tr} A^* = \overline{\text{Tr} A}$.

Proposition 9.3. We have that $\mathcal{B}_1(\mathcal{H}) = (\mathcal{B}_\infty(\mathcal{H}))^*$ and the map

$$A \mapsto \text{Tr}(A \cdot),$$

is the isomorphism between $\mathcal{B}_1(\mathcal{H})$ onto the dual of $\mathcal{B}_\infty(\mathcal{H})$.

Exercise 9.5. Show that $(\mathcal{B}_1(\mathcal{H}), \|\cdot\|_1)$ is a Banach algebra, i.e. it is a Banach space with the property that for all $A, B \in \mathcal{B}_1(\mathcal{H})$, $\|AB\|_1 \leq \|A\|_1 \|B\|_1$.

Exercise 9.6. Show that $\mathcal{B}_1(\mathcal{H})$ is a two-sided ideal in the $\mathbb{C}^*$-algebra $\mathcal{B}(\mathcal{H})$.

Exercise 9.7. Show that for every $A \in \mathcal{B}_1(\mathcal{H})$ and every $B \in \mathcal{B}(\mathcal{H})$ the trace satisfies

$$\text{Tr} AB = \text{Tr} BA.$$ 

Exercise 9.8. Show that

$$\mathcal{B}_1(\mathcal{H}) \cap \mathcal{B}_{ad}(\mathcal{H}) \subseteq \mathcal{B}_\infty(\mathcal{H}) \cap \mathcal{B}_{ad}(\mathcal{H}).$$

In particular, for every $A \in \mathcal{B}_1(\mathcal{H}) \cap \mathcal{B}_{ad}(\mathcal{H})$, then there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ in $\mathbb{C}$ and an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of $\mathcal{H}$, such that

$$A = \sum_{n=1}^{\infty} a_n (\cdot, e_n), \quad \|A\|_1 = \sum_{n=1}^{\infty} |a_n|, \quad \text{Tr} A = \sum_{n=1}^{\infty} a_n.$$

Hint: for the second part use Hilbert-Schmidt theorem.
Exercise 9.9. For $A, B \in \mathcal{B}_{ad}(\mathcal{H})$ we define their commutator to be

$$[A, B] := AB - BA.$$  

Show that for every $\varepsilon > 0$ the space $\mathcal{B}_{ad}(\mathcal{H})$ with the Lie bracket $\frac{1}{\varepsilon} [\cdot, \cdot]$ is a Lie algebra.

Hint: You first need to show that $\frac{1}{\varepsilon} [A, B] \in \mathcal{B}_{ad}(\mathcal{H})$.

Exercise 9.10. Let $P$ be an orthogonal projection on a space $C_x$ for some $x \in \mathcal{H}$. Suppose that $A, B \in \mathcal{B}_1(\mathcal{H})$ be two positive operators of trace 1, such that there exists $0 < t < 1$ satisfying

$$P = tA + (1 - t)B.$$  

Show that then either $A = P$ or $B = P$.

9.3 Hilbert-Schmidt operators

Definition 9.4. An operator $A \in \mathcal{B}(\mathcal{H})$ is a Hilbert-Schmidt operator if for every orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of $\mathcal{H}$

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 < +\infty.$$  

We denote the set of Hilbert-Schmidt operators by $\mathcal{B}_2(\mathcal{H})$. On the space $\mathcal{B}_2(\mathcal{H})$ we define a norm

$$\|A\|_2 := \sqrt{\sum_{n=1}^{\infty} \|Ae_n\|^2}.$$  

Exercise 9.11. Show that $\mathcal{B}_2(\mathcal{H}) \subseteq \mathcal{B}_1(\mathcal{H})$.

Exercise 9.12. Show that $\mathcal{B}_2(\mathcal{H})$ is a Hilbert space with the inner product defined

$$\langle A, B \rangle := \text{Tr} AB^*.$$  

Exercise 9.13. Show that if $A \in \mathcal{B}_2(\mathcal{H})$ is normal then the sequence of its eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ is in $\ell_2$.

Exercise 9.14. Show that $\mathcal{B}_2(\mathcal{H})$ is a two-sided ideal in the $\mathbb{C}^*$-algebra $\mathcal{B}(\mathcal{H})$.

Exercise 9.15. Show that

$$\mathcal{B}_2(\mathcal{H}) = \left\{ A \in \mathcal{B}(\mathcal{H}) \mid AA^* \in \mathcal{B}_1(\mathcal{H}) \right\}.$$  

9.4 Axioms of quantum mechanics

A quantum system is described in the following way:
• To a quantum system we associate a complex, infinite-dimensional, separable Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle)\);

• The set of **observables** on \(\mathcal{H}\) is the set of linear, continuous, self-adjointed operators \(\mathcal{A} := \mathcal{L}_{ad}(\mathcal{H})\). The set of bounded observables is denoted \(\mathcal{A}_0 := \mathcal{B}_{ad}(\mathcal{H})\);

• The set of **states** is defined as

\[
\mathcal{S} := \{ A \in \mathcal{B}_1(\mathcal{H}) \mid \text{Tr} A = 1, \ A \geq 0 \}.
\]

• A **measurement** is a map

\[
\mathcal{A} \times \mathcal{S} \ni (A, \psi) \mapsto \mu_A \in \mathcal{P}(\mathbb{R}),
\]

which to every observable and every state assigns a probability measure on \(\mathbb{R}\). For every Borel subset \(E \subseteq \mathbb{R}\) the value \(\mu_A(E)\) is the probability that the measurement of an observable \(A\) on a system in state \(\psi\) will belong to \(E\).

Note that to every \(x \in \mathcal{H}\) the corresponding orthogonal projection \(P_x\) on \(\mathbb{C}x\) is a state \(P_x \in \mathcal{S}\). Moreover, by Exercise 9.10 \(P_x\) is not a convex combination of any other two states. Therefore for every \(x \in \mathcal{H}\) the corresponding \(P_x\) is called a **pure state**. Thus \(\mathcal{H} \subseteq \mathcal{S}\).

On the other hand, \(\mathcal{S} \subseteq \mathcal{B}_1(\mathcal{H}) \cap \mathcal{B}_{ad}(\mathcal{H})\), so by the Hilbert-Schmidt Theorem for every \(\psi \in \mathcal{S}\) there exists an orthonormal set \(e_n \in \mathcal{H}\) and a sequence \(a_n \in \mathbb{C}\), such that

\[
\psi = \sum_n a_n P_n, \quad \text{Tr} \psi = \sum_n a_n,
\]

where \(P_n\) are projections onto the spaces \(\mathbb{C}e_n\).