Classifying Vector Bundles

Notes for the Seminar Vector Bundles in Algebraic Topology

Initial remarks

The content of these notes follows closely the one of [1] from page 6 to page 21. We highlighted some of the key objects and results, and presented them with a slightly changed structure. Sometimes we wrote things in more detail, some of the material got cut. Whenever it was possible, the notation and the formulation of the statements were not changed radically to facilitate the comparison with the content of the above mentioned reference textbook.

As in [1] a map denotes a continuous function.

A VECTOR WHAAAAAT? A Quick recap

DEFINITION 1. Let $B$ and $E$ be topological spaces (we will call the first the base space and the second the total space).
An $n$-dimensional vector bundle is a map $p : E \to B$ such that:
1) $p^{-1}(b)$ is a real vector space for each $b \in B$
2) We can locally trivialize the vector bundle, i.e. there is a cover with open sets $U_\alpha$ such that for every such open set there exists a homeomorphism $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ taking $p^{-1}(b)$ to $B \times \mathbb{R}^n$ by a vector space isomorphism for every $b \in U_\alpha$. we call each $h_\alpha$ a local trivialization.

A complex vector bundle is defined similarly, simply taking $\mathbb{C}$ as a field instead of $\mathbb{R}$.

DEFINITION 2. An isomorphism between vector bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ is a homeomorphism $h : E_1 \to E_2$ taking each fiber $p_1^{-1}(b)$ to the fiber $p_2^{-1}(b)$ by a linear isomorphism.

DEFINITION 3. A section of a vector bundle is a map $s : B \to E$ assigning each $b \in B$ a vector $s(b)$ in the fiber $p^{-1}(b)$.

Hey, you look like a trivial bundle!

The goal of this section is to find a criterion to identify vector bundles isomorphic to the trivial one. In particular, we will find a strong correlation between the existence of linearly independent sections and existence of an isomorphism with the trivial bundle.

We start by invoking the following lemma:

LEMMA 4. (lemma 1.1 in [1]) Let $h : E_1 \to E_2$ be a map between vector bundles over the same base space $B$. If the restriction of $h$ to each fiber $p_1^{-1}(b)$ is a vector space isomorphism, then $h$ is a vector bundles isomorphism.

Proof. The function $h$ maps each fiber of $E_1$ to one fiber of $E_2$ by linear isomorphism, so injectivity follows by assumption. Every point in $E_2$ belongs to a fiber so also surjectivity comes for free. It remains only to check that the inverse $h^{-1}$ is continuous. This is a local question so we can restrict to an open set $U \subset B$ over which $E_1$ and $E_2$ are trivial (by taking the refinement of the two covers). Because the trivializations are homeomorphism we can study the equivalent problem of the continuity of the map $h := h_1 \circ h^{-1} \circ h_2^{-1} : U \times \mathbb{R}^n \to U \times \mathbb{R}^n$ obtained by composition with local trivializations. The map has the form $h(\chi, \nu) = (\chi, g_\chi(\nu))$. In particular $g_\chi \in GL_n(\mathbb{R})$ and depends continuously on $\chi$, i.e. each entry of the matrix depends continuously on $\chi$. The entries of the inverse matrix $g_\chi^{-1}$ can be expressed continuously as functions of the entries of $g_\chi$ so they also depend continuously on $\chi$. Therefore $h^{-1}(\chi, \nu) = (\chi, g_\chi^{-1}(\nu))$ is continuous. ∎
Now we are ready to state a criterion to identify $n$-dimensional vector bundles isomorphic to the trivial bundle:

**PROPOSITION 5.** (stated before lemma 1.1 in [1]) An $n$-dimensional bundle $p : E \to B$ is isomorphic to the trivial bundle if it has $n$ sections $s_1, \ldots, s_n$ such that the vectors $s_1(b), \ldots, s_n(b)$ are linearly independent in each fiber $p^{-1}(b)$

**Proof.** ($\Rightarrow$) The trivial bundle has such sections and an isomorphism of vector bundles preserves the linear independence of the fibers.

($\Leftarrow$) If one has $n$ linearly independent sections $s_i$ then the map $h : B \times \mathbb{R}^n \to E$ given by $h(b, t_1, \ldots, t_n) = \sum_i t_i s_i(b)$ is a linear isomorphism in each fiber, and is continuous since the its composition with a local trivialization $p^{-1}(U) \to U \times \mathbb{R}^n$ is continuous. By the previous lemma we have that $h$ is an isomorphism. \qed

Let’s add some $\oplus$, et voilà: everything becomes trivial!

Given two vector bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ over the same base space $B$, we want to create a third vector bundle over $B$ whose fiber over each point of $B$ is the direct sum of the original fibers.

**DEFINITION 6.** We define the direct sum of $E_1$ and $E_2$ as the space $E_1 \oplus E_2 = \{(\nu_1, \nu_2) \in E_1 \times E_2 \mid p_1(\nu_1) = p_2(\nu_2)\}$

We can define a projection $E_1 \oplus E_2 \to B$ sending $(\nu_1, \nu_2)$ to the point $p_1(\nu_1) = p_2(\nu_2)$

The fibers are then the direct sum of the fibers of $E_1$ and $E_2$ as desired.

Now we want to show that we are indeed dealing with a vector bundle. In order to prove that we invoke the two following facts:

(a) Given a vector bundle $p : E \to B$ and a subspace $A \subset B$, then $p|_A$ is a vector bundle over $A$. We call this the restriction of $E$ over $A$.

(b) Given vector bundles $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$, then $p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$ is also a vector bundle.

Then if $E_1$ and $E_2$ have the same base space $B$, the restriction of the product $E_1 \times E_2$ over the diagonal $D = \{(b, b) \in B \times B\}$ is exactly $E_1 \oplus E_2$.

So by first using fact (b) and then fact (a) we can verify the local triviality condition, proving that we indeed constructed a vector bundle.

**PROPOSITION 7.** The direct sum of vector bundles is an associative operation.

**Proof.** We start with $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ and $p_3 : E_3 \to B$

We have that:

$E_2 \oplus E_3 = \{(e_2, e_3) \in E_2 \times E_3 \mid p_2(e_2) = p_3(e_3)\}$ and $p_{2,3} : E_2 \oplus E_3 \to B, p_{2,3}(e_2, e_3) = p_2(e_2) = p_3(e_3)$

Then $E_1 \oplus (E_2 \oplus E_3) = \{(e_1, e_2, e_3) \in E_1 \times E_2 \times E_3 \mid p_1(e_1) = p_{2,3}(e_2, e_3) = p_2(e_2) = p_3(e_3)\}$ and by "going back" we can see that constructed sets are the same. Of course also the projections onto the base space $B$ are the same. \qed

**DEFINITION 8.** A vector bundle is said to be **stably trivial** if it becomes trivial after taking the direct sum with a vector bundle isomorphic to the trivial bundle.

**REMARK 9.** An example of vector bundle which is not trivial but nevertheless stably trivial is $TS^2$

**DEFINITION 10.** A vector subbundle of a vector bundle $p : E \to B$ is defined as a subspace $E_0 \subset E$ intersecting each fiber of $E$ in a vector subspace, such that the restriction $p|_{E_0}$ is a vector bundle.

We now define a condition weaker than compactness for Hausdorff spaces. Many results that we will state later assuming compactness of a certain space, actually work under the paracompactness assumption.
DEFINITION 11. A Hausdorff space $X$ is paracompact if for each open cover $U_{\alpha}$ of $X$ there is a partition of the unity $\phi_{\beta}$ subordinate to the cover. This means that the $\phi$’s are maps $X \to I$ such that each $\phi_{\beta}$ has support contained in some $U_{\alpha}$ and each $\chi \in X$ has a neighborhood in which only finitely many $\phi_{\beta}$’s are nonzero, and $\sum_{\beta} \phi_{\beta} = 1$.

The main result of this section is that, under certain conditions on the base space $B$, we can trivialize every vector bundle through direct sum with an appropriate complement. We will use the following result without proving it. The proof can be found in [1]

PROPOSITION 12. (Proposition 1.4 in [1]) If $p : E \to B$ is a vector bundle over a compact base space $B$ (or more in general paracompact) and $E_0 \subset E$ is a vector subbundle, then there is a vector subbundle $E_0' \subset E$ such that $E_0 \oplus E_0'$ is isomorphic to $E$.

PROPOSITION 13. (Proposition 1.4 in [1]) For each vector bundle $E \to B$ with a compact Hausdorff base space $B$ there exists a vector bundle $E' \to B$ such that $E \oplus E'$ is isomorphic to a trivial bundle.

Proof. Each point $\chi \in B$ is contained in an open set $U_{\chi}$ over which the vector bundle is trivial. By Urisohn’s Lemma there is a map $\phi_{\chi} : B \to [0, 1]$ that is 0 outside $U_{\chi}$ and nonzero at $\chi$. Letting $\chi$ vary, we obtain an open cover of $B$ with sets $\phi_{\chi}^{-1}(0, 1]$.

By compactness we find a finite subcover and we relabel the corresponding $U_{\chi}$’s and $\phi_{\chi}$’s with $U_i$ and $\phi_i$.

We now define $g_i : E \to \mathbb{R}^n$ by $g_i(\nu) = \phi_i(p(\nu))((\pi_i h_i(\nu))$ where $p : E \to B$ and $\pi_i h_i$ is the composition of the local trivialization $h_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^n$ with the projection $\pi_i : U_i \times \mathbb{R}^n \to \mathbb{R}^n$.

Then $g_i$ is a linear injection on every fiber over $\phi_i^{-1}(0, 1]$. By setting every $g_i$ as a coordinate of a map $g : E \to \mathbb{R}^N$ (where $\mathbb{R}^N$ is a product of copies of $\mathbb{R}^n$) we obtain a linear injection from each fiber onto a $n$–dimensional subspace. One could think that this is not true for fibers contained in more than one $U_i$. They have indeed more not trivial components $g_i$, but these are related to each other and depend on only one of them, so the image is still $n$–dimensional.

We now set $g$ as second coordinate of a map $f : E \to B \times \mathbb{R}^N$ with first coordinate $p$. The image of $f$ is a subbundle of the product $B \times \mathbb{N}$ since projection of $\mathbb{R}^N$ onto the $i$th $\mathbb{R}^n$ factor gives the second coordinate of a local trivialization over $\phi_i^{-1}(0, 1]$.

Thus we have $E$ isomorphic to a subbundle of $B \times \mathbb{R}^n$ and by the previous proposition there is a complementary subbundle $E'$ with $E \oplus E'$ isomorphic to $B \times \mathbb{R}^N$. \hfill \Box

It’s time for some tensor product

In this part we will briefly introduce the tensor product of vector bundles and some of its properties. They will be useful in the following parts of the seminar.

DEFINITION 14. Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$. We define $E_1 \otimes E_2$ to be the disjoint union of the vector spaces $p_1^{-1}(\chi) \otimes p_2^{-1}(\chi)$ for $\chi \in B$ with the following topology.

We work with an open cover of $B$ over which both initial bundles are trivial. Choose isomorphisms $h_i : p_i^{-1}(U) \to U \times \mathbb{R}^{n_i}$ for each open set $U$ in such a cover.

The topology $\mathcal{T}_U$ on the set $p_1^{-1}(U) \otimes p_2^{-1}(U)$ is defined by letting the tensor product map $h_1 \otimes h_2 : p_1^{-1}(U) \otimes p_2^{-1}(U) \to U \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$ be a homeomorphism.

REMARK 15. Up to isomorphism, the tensor product $\otimes$ is associative, commutative and has an identity element.

Just few short remarks about this proposition before moving on:

(i) Associativity follows from associativity of the tensor product between vector spaces

(ii) Commutativity is not in general true as equality, but only up to isomorphism

(iii) The identity element is the trivial line bundle

REMARK 16. Up to isomorphisms, the tensor product $\otimes$ is distributive with respect to the direct sum $\oplus$
Do you wanna pull back?

**DEFINITION 17.** We define \( \text{Vect}^n(B) \) to be the set of isomorphism classes of \( n \)-dimensional real vector bundles over \( B \).

We define the complex counterpart as \( \text{Vect}^n(B) \)

We will use the notation \( E_1 \approx E_2 \) referring to two vector bundles over the same base space that are isomorphic and hence belong to the same class in \( \text{Vect}^n \) (respectively in \( \text{Vect}^n \)).

Now the goal is to construct a vector bundle \( p' : E' \to A \) given a map \( f : A \to B \) and a vector bundle \( p : E \to B \). But first, let’s show how the map \( f : B \to B \) induces a function \( f^* : \text{Vect}^n(B) \to \text{Vect}^n(A) \) for every \( n \). In particular \( f^*(E) \) is the vector bundle that we are trying to construct. We name it pullback of \( E \) by \( f \).

**PROPOSITION 18** (Universal property of the pullback). (Proposition 1.5 in [1]) Given a map \( f : A \to B \) and a vector bundle \( p : E \to B \), then there exists a vector bundle \( p' : E' \to A \) with a map \( f' : E' \to E \) taking the fiber over each point \( a \in A \) isomorphically onto the fiber of \( E \) over \( f(a) \), and such a vector bundle \( E' \) is unique up to isomorphism.

**Proof.** Let’s start by constructing explicitly a pullback

\[
E' := \{(a, \nu) \in A \times E | f(a) = p(\nu)\}
\]

with \( p' : \)

\[
\begin{align*}
E' \xrightarrow{f'} E \\
(a, \nu) \mapsto (a, \nu) \mapsto \nu
\end{align*}
\]

We now have the following commutative diagram:

\[
\begin{array}{ccc}
E' & \xrightarrow{f'} & E \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
\]

Let \( \Gamma \) be the graph of \( f \), then we can represent \( p' \) as the composition

\[
E' \to \Gamma \to A, \quad (a, \nu) \mapsto (a, p(\nu)) = (a, f(a)) \to a.
\]

The first of the two maps is the restriction of the vector bundle \( Id \times p : A \times E \to A \times B \) over the graph \( \Gamma \), so it is a vector bundle, and the second map is a homeomorphism, so their composition \( p' : E' \to A \) is a vector bundle. The map \( f' \) takes the fiber of \( E' \) over \( a \) isomorphically onto the fiber of \( E \) over \( f(a) \).

For the uniqueness statement, we can construct an isomorphism from an arbitrary \( E' \) satisfying the conditions in the proposition to the particular one just constructed by sending \( \nu' \in E' \) to the pair \((p'(\nu'), f'(\nu'))\). This map takes each fiber of \( E' \) to the corresponding fiber of \( f^*(E) \) by a vector space isomorphism, so by Lemma 1.4 it is an isomorphism of vector bundles.

**PROPOSITION 19** (Elementary properties of pullbacks).

(i) \((gf)^*(E) \approx f^*(g^*(E))\)

(ii) \(Id^*(E) \approx E\)

(iii) \(f^*(E_1 \oplus E_2) \approx f^*(E_1) \oplus f^*(E_2)\)

(iv) \(f^*(E_1 \oplus E_2) \approx f^*(E_1) \oplus f^*(E_2)\)

**Proof.** The various proofs will consist in verifying that the vector bundle on the right side satisfy the universal property of the pullback and by uniqueness up to isomorphism we have the claims.

(i) We have the following diagrams representing the situation. We want to show that the second one respects the properties listed in Proposition 1.12.

\[
\begin{array}{ccc}
(gf)^*(E) & \xrightarrow{(gf)} & E \\
\downarrow & & \downarrow \\
A & \xrightarrow{g} & B
\end{array}
\quad
\begin{array}{ccc}
f^*(g^*(E)) & \xrightarrow{f'} & g^*(E) & \xrightarrow{g'} & E \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C
\end{array}
\]

By universal property \( f' \) maps the fiber over each \( a \in A \) isomorphically onto the fiber of \( g^*(E) \) over \( f(a) \).

Then always by universal property, \( g' \) maps every fiber over \( f(a) \in B \) isomorphically onto the fiber of \( E \), over \( gf(a) \). The composition of isomorphisms is still an isomorphism, so \( f^*(g^*(E)) \) satisfies the universal property of pullback bundles and must be isomorphic to \((gf)^*\).
(ii) The claim follows from the following diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{Id} & E \\
\downarrow & & \downarrow \\
A & \xrightarrow{Id} & A
\end{array}
\]

(iii) The claim follows from looking at these diagrams:

\[
\begin{array}{ccc}
f^*(E_1) & \xrightarrow{f_1} & E_1 \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
\quad
\begin{array}{ccc}
f^*(E_2) & \xrightarrow{f_2} & E_2 \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
\]

given

\[
\begin{array}{ccc}
f^*(E_1) \oplus f^*(E_2) & \xrightarrow{(f_1; f_2)} & E_1 \oplus E_2 \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & A
\end{array}
\]

we construct the following diagram

\[
\begin{array}{ccc}
f^*(E_1) \otimes f^*(E_2) & \xrightarrow{(f_1 \otimes f_2)} & E_1 \otimes E_2 \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & A
\end{array}
\]

And we prove that \( f^*(E_1) \oplus f^*(E_2) \) respects the universal property by recalling that, by definition, the fibers of \( E_1 \oplus E_2 \) are the direct sum of the fibers of \( E_1 \) and \( E_2 \), which are isomorphic to fibers of \( f^*(E_1) \) and \( f^*(E_2) \) respectively by universal property applied to the first two diagrams.

(iv) Similar to (iii): we have

\[
\begin{array}{ccc}
f^*(E_1) \otimes f^*(E_2) & \xrightarrow{(f_1 \otimes f_2)} & E_1 \otimes E_2 \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & A
\end{array}
\]

and by definition the fibers of \( E_1 \otimes E_2 \) are the tensor product of the fibers of \( E_1 \) and \( E_2 \), which are isomorphic to fibers of \( f^*(E_1) \) and \( f^*(E_2) \) respectively by universal property.

The following technical Lemma will allow us to prove the main result about pullbacks. We will omit the proof but it can be found directly in [1]

**Lemma 20.** (Proposition 1.7 in [1]) Let \( X \) be compact (or more in general paracompact). Then the restrictions of a vector bundle \( E \to X \times I \) over \( X \times 0 \) and \( X \times 1 \) are isomorphic.

And now we have

**Theorem 21.** (Theorem 1.6 in [1]) Given a vector bundle \( p: E \to B \) and homotopic maps \( f_0, f_1: A \to B \) over a compact Hausdorff (or more generally paracompact Hausdorff) space \( A \).

Then the pullbacks \( f_0^*(E) \) and \( f_1^*(E) \) are isomorphic.

**Proof.** Let \( F: A \times I \to B \) be a homotopy between \( f_0 \) and \( f_1 \). The restrictions of \( F^*(E) \) over \( A \times \{0\} \) and \( A \times \{1\} \) are then \( f_0^*(E) \) and \( f_1^*(E) \). The claim follows then from the previous Lemma.

A direct consequence of this theorem is:

**Corollary 22.** (Corollary 1.8 in [1]) A homotopy equivalence \( f: A \to B \) of compact spaces (or more generally paracompact) induces a bijection \( f^*: \text{Vect}^n(B) \to \text{Vect}^n(A) \).

As a consequence every vector bundle over a contractible compact space (or more generally paracompact) is trivial.

**Proof.** Let \( g: B \to A \) such that \( gf \approx Id_A \) and \( fg \approx Id_B \). Then by the properties of the pullback we have \( f^*g^* = Id^* = Id \) and \( g^*f^* = Id^* = Id \)

**References**