Stiefel-Whitney classes and Chern classes
Part I: Introduction and Motivation

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We come back to the problem of classification of vector bundles over a given basis space. In any branches of mathematics, there is always a guiding problem. Interestingly, classification is always a guiding problem in geometry. An easy way to solve such a problem is to find a very good invariant that can distinguish everything up to some condition. Examples are cardinality in set theory. However, other invariants, like homology, cohomology, homotopy group in topology, Gromov-Witten invariants, Donaldson-Thomas invariants in algebraic geometry, can only partially solve the problem. As we are working on algebraic topology, a natural idea comes out of head is using homology/cohomology. One step further, we would like to consider some particular cohomological classes associated to the vector bundle. Historically, the first characteristic classes was proposed by Hopf, Stiefel and Whitney (and if you are born early enough, you might meet the first two professors somewhere in ETH...), but the idea of characteristic class can date back to Schubert Calculus and Italian School of Algebraic Geometry.

Loosely speaking, characteristic class is just to measure how a vector bundle is twisted. To classify a vector bundle, let’s first recall theorem 1.16 in [2].

Theorem 0.1. Let $X$ be a topological space, $G_n$ is the grassmanian manifold for $\mathbb{R}^\infty$, $E_n$ be the universal bundle on $G_n$. For paracompact $X$, the map $[X,G_n] \to \text{Vect}^n(X)$, $[f] \mapsto f^*(E_n)$, is a bijection.

\[ \begin{array}{ccc}
 X & \xrightarrow{f} & G_n \\
 \downarrow \pi & & \downarrow \pi' \\
 f^*(E_n) & \xrightarrow{=} & E_n \\
 \end{array} \]

\[ \begin{array}{ccc}
 X & \xrightarrow{f_E} & G_n \\
 \downarrow \pi & & \downarrow \pi' \\
 f_E^*(E_n) & \xrightarrow{=} & E_n \\
 \end{array} \]

By this theorem, to classify the vector bundles on a vector space is equivalent to find out what $[X,G_n]$ is. But this is not as simple as one think. So we turn around to see whether $f$ induces a trivial map in homology or cohomology. This is the idea of characteristic class.
1 Characteristic Classes

**Definition 1.1** (Characteristic class). Let $R$ be a ring. A characteristic class $u_q$ of degree $q$ for a vector bundle of dimension $n$ is a natural transformation from the a contravariant functor $\text{Vect}^n$, from $\text{Top}$ to $\text{Vect}^n$, category of set of isomorphism classes of $n$-vector bundles, to another contravariant functor $H^q(\ast; R)$.[10]

For a natural transformation, we mean that for each object $X$ in $\text{Top}$, we have a morphism $u_q(X): \text{Vect}^n(X) \to H^q(X; R)$ such that $u_q(X) \circ \text{Vect}^n(g) = H^q(g; R) \circ u_q(Y)$, where $g: X \to Y$ is a continuous map for topological spaces.

![Diagram]

Well, this definition is really dry..., so let’s proceed with some more geometric ideas. If one want to call an obvious invariant of a vector bundle, the first one should be orientability. This is the idea of first Stiefel-Whitney class. There is a higher degree analogue which we will elaborate in details later, called $q$-th Stiefel-Whitney class. We use $w_i$ to denote $i$-th Stiefel-Whitney class.

**Theorem 1.1.** $H^*(G_n, \mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[w_1, ..., w_n]$ on the Stiefel-Whitney classes of universal bundle.[2]

Thus $f$ induces trivial map on $\mathbb{Z}_2$ cohomology if and only if the Stiefel-Whitney classes are trivial.

One may think whether characteristic class can classify vector bundles up to isomorphism. Well, this is not true. For Stiefel-Whitney class and Chern class, they are stable, which means after taking direct sum of trivial vector bundle, we still have same characteristic class. Thus for vector bundles that is stably trivial, for example, tangent bundle of sphere, the characteristic class is trivial.

2 Historical Remarks

(Please find [1] as a reference for this section.)

Although we have seen a bit topological idea of characteristic class, one of the importance is the connection of topology and geometry it provides, both in a differential and a algebraic way.

Although the Stiefel-Whitney classes is the first characteristic class in history, the simplest characteristic class should be Euler characteristic.

**Definition 2.1** (Euler characteristic). If $M$ be a finite cell complex, the Euler characteristic class is define as

$$\chi(E) = \sum_k (-1)^k \alpha_k = \sum_k (-1)^k b_k,$$

where $\alpha_k$ is number of $k$-cells and $b_k$ is the $k$-dimensional Betti number, i.e., rank of $H_k(X)$. 

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The connection of geometry and topology falls into next theorem.

**Theorem 2.1** (Hopf(1927)).

\[ \chi(M) = \sum_i \text{index}_{x_i} \text{ of } v, \]

where \( v \) is a vector field with isolated zeros on \( M \) and the sum is taking over all zeros of \( v \).

The index of an isolated zero \( x \) is defined to be the degree of the map \( f : S_\epsilon \to S_\epsilon \), where we take first \( S_\epsilon \) to be a small enough sphere around \( x \), and the second one is around \( v(x) \) and the map is \( \frac{v(y)}{|v(y)|} \). This connects \( \chi(M) \) with geometrical meaning. (Well, one may also consider Gauss-Bonnet theorem, but this will be the story of Chern classes....) If we make one step further, we would like to take \( k \) vector fields that the exterior product is 0, i.e. the linear dependent part of vector fields is a \( k - 1 \) manifold. Depending on the parity, this gives us a \( (k - 1) \)-dimensional cycle, whose homology class is independent of the choice of the vector fields. (If we consider \( \mathbb{Z}_2 \) coefficients, then it is always the case.) However, it is more proper to work in cohomology. Then this leads to the Stiefel-Whitney cohomology classes (just for manifolds so far). Whitney proposed that such a construction can be made in the vector bundle to arbitrary topological spaces. He noticed that \( \pi_i(V_k(\mathbb{R}^n)) = 0 \) for all \( 0 \leq i < n \). Thus universal bundle can be viewed somehow as the total space has a string of vanishing homotopy groups while the base space has lots of homological properties. The importance of universal bundle is encoding in theorem 0.1.

Let \( u \in H^i(G_m(\mathbb{R}^n); \mathbb{R}) \). This is called a universal characteristic class of degree \( i \). Then it follows from the theorem that \( f_\ast^u \) depends only on vector bundle \( E \) on \( X \). It is called a characteristic class corresponding to the universal class \( u \).

Let us use some examples to describe the construction of characteristic classes in a more geometric way.

**Example 2.1** (Stiefel-Whitney Class). Consider all \( m \)-dimensional linear subspaces \( S \) in \( \mathbb{R}^{m+n} \) with

\[ \dim(S \cap \mathbb{R}^{i+n-1}) \geq i, \]

where \( 1 \leq i \leq m \) and \( \mathbb{R}^{i+n-1} \) is a fixed subspace of dimension \( i + n - 1 \). (Sometimes this condition is called Schubert condition.) Then they form a \((m + n - i)\)-cycle mod 2 in \( G_m(\mathbb{R}^n) \). Then the dual of this homology class is an element in \( H^i(G_m(\mathbb{R}^n); \mathbb{Z}_2) \). This is called the \( i \)th universal Stiefel-Whitney class \( w_i \). Its image of the corresponding pullback map is called the Stiefel-Whitney class of the bundle \( E \), denoted as \( w_i(E) := f_\ast^u(w_i) \).

**Example 2.2** (Potryagin Class). If we replace the above process by considering

\[ \dim(S \cap \mathbb{R}^{2i+2n-2}) \geq 2i, \]

the resulting class in coefficient \( \mathbb{Z} \) is called the universal Potryagin class. The corresponding class is called the Pontryagin class.

**Example 2.3** (Chern Class). Here we meet the first good property of Chern class. It is known that \( Gr_n(\mathbb{C}^N) \) has a simpler topological properties than real
ones. It is connected, no torsion, and all odd-dimensional homology classes are trivial. We repeat the same process, and the resulting classes are universal Chern class and Chern class.

One notice that if we apply the above process to the tangent bundle of a manifold, the Stiefel-Whitney class and Pontryagin class are invariants of differentiable structure. Similarly, the Chern class is an invariant of complex structure.

3 What do Characteristic Classes measure?

Well, this question is too ... hard, but at least let me show some understanding. What can you think first to distinguish one vector bundle from trivial bundle? Orientability! Let \( E \rightarrow B \) be a vector bundle with \( B \) connected. Then orientability is detected by \( \pi_1(B) \rightarrow \mathbb{Z}_2 \), by giving 0 if after one loop the orientation of fibre is preserved, or 1 otherwise. As \( \mathbb{Z}_2 \) is abelian, by universal property of abelianization, we can use \( H_1 \) to replace \( \pi_1 \). Then such a homomorphism is a cocycle in \( H^1(B;\mathbb{Z}_2) \). This is exactly the first Stiefel-Whitney class. Now if we are considering a cell complex, we can easily forget cell structures higher than 1 and still have the same orientability. Furthermore, a vector bundle on \( S^1 \) is trivial if and only if it is orientable. This can be deduced to 1-skeleton. Thus we see that \( w_1(E) \) measures the triviality of \( E \). Thus we have following theorem:

**Theorem 3.1.** A vector bundle \( E \rightarrow B \) is orientable if and only if \( w_1(B) = 0 \), assuming that \( X \) is homotopy equivalent to a CW complex.\[2\]

Now let’s move one dimension higher. Then if \( E \) is trivial over \( B \), then we will have a \( n \) orthonomal sections over \( B^1 \), where fibres is of dimension \( n \). Like the process of dimension 1, we ask whether such a frame can be extended to 2-cells. Let’s consider the pullback bundle to \( D^2 \) via characteristic map. As the pullback bundle is trivial, we have a map as \( \partial D^2 \rightarrow O(n) \) by choosing a trivialization on \( B^1 \). If such a choice gives a same orientation, then we can sharpen it to \( SO(n) \). As we are working on \( B^1 \), we actually get one element in \( \pi_1(SO(n)) \) for each one cell. Obviously if the sections can extend to orthonomal basis to \( B^2 \) if and only if all the image is 0 for each two cell. \( SO(1) = 1 \), \( SO(2) = S^1 \), and for \( n > 3 \), we use \( \mathbb{Z}_2 \) coefficients. But by the obvious inclusion \( SO(2) \rightarrow SO(n) \), we get a surjection on \( \pi_1 \). So a generator is a rotation of a plane through \( 2\pi \) and leave the orthogonal complement fixed. So we can see such a map as an obstruction to extending the sections over the 2-cell as a 2-dimensional cellular cochain with coefficients in \( \pi_1(SO(n)) \). Thus it is a cellular cocyle, so defines as an element in \( H^2(B;\pi_1(SO(n))) \). When \( n = 1 \), it is trivial. When \( n = 2 \), we are working on coefficient \( \mathbb{Z} \) and such a class is called Euler class. For \( n > 2 \), we are working in \( \mathbb{Z}_2 \) coefficient and they are Stiefel-Whitney class. We can mimic this process for higher skeleton, but let me stop here.

This process is not practical for higher skeleton. One reason is that the higher homotopy group of \( SO(n) \) is very difficulty to compute. Another disadvantage is that if during one step we are stuck, then we cannot proceed further.

One way to solve this problem is that we consider sections fewer than \( n \). Suppose we have a section on \( B^{k-1} \). We pullback via the characteristic map of
the $k$-cell, so we have a section over $S^{n-1}$. The section extends if and only if the map is nullhomotopic. For $k < n$ we can always extend, so the first obstruction is on $k = n$. So we get an obstruction $w(E)$. However, this only works if we have a good orientation. If we only working on $\mathbb{Z}_2$ coefficient, we can forget the orientation, so we have a well-defined Stiefel-Whitney class. Such a process can be generalized to $k$ sections. To do such a generalization, we replace the unit ball by Stiefel manifold. We have following theorem:

**Theorem 3.2.** The first nonvanishing homotopy group of $V_k(\mathbb{R}^n)$ is $\pi_{n-k}$, which is $\mathbb{Z}$ if $n-k$ is even or $k = 1$, and $\mathbb{Z}_2$ otherwise.[2]

This means that there is no obstruction in finding up to $k$ sections over $n-k$ skeleton by applying a similar argument. The first obstruction lies in $H^{n-k+1}(B; \pi_{n-k}(V_k(\mathbb{R}^n)))$. We choose the $\mathbb{Z}_2$ coefficient to avoid orientation problem. Thus we get the Stiefel-Whitney class. For complex vector bundles, we work in a same fashion, except we only consider even case, and the resulting class is called Chern class. Although it seems that Chern class loses half of the information as we only consider even dimensional case, Chern class is widely use in algebraic geometry. The algebro-geometric analogue of homology (or cohomology) is called Chow group. (Both homologial and cohomological analogue are called Chow group.) (Loosely speaking, the $i$-th Chow group is the group of all dimension (codimension) $i$ irreducible subvarieties up to rational equivalence.) And if it happens that if the variety is smooth over complex number, there is a natural map from Chow ring (in this case Chow group is a ring) to $H^{2j}$. Thus Chern class encodes sufficient information one need in algebraic geometry. Moreover, computation of Chern class in algebraic geometry is much simpler, as for a locally free sheaf, the Chern class can be computed directly as product of Chern polynomial in short exact sequence, and the divisor case can be computed directly by considering twisted sheaf.

**Definition 3.1 (Chern Polynomial).** The Chern polynomial is defined to be

$$c_t(E) = c_0(E) + c_1(E)t + ... + c_r(E)t^r.$$ 

**Theorem 3.3.** 1. If $E \cong L(D)$, where $D$ is a divisor (loosely speaking, a codimension one cycle), then $c_t(E) = 1 + Dt$. 2. If $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ is an exact sequence of locally free sheaves (vector bundles) on $X$, then

$$c_t(E) = c_t(E')c_t(E'').$$

(For a reference of locally free sheaves, please refer to [3]. For a reference for Chow group and Chow ring, please refer to [3][5]. For a reference for Chern classes in algebraic geometry, please refer to [3][5][6][7].)

4 **Reference**