Stiefel-Whitney and Chern Classes
Part 3
Antonia Giannopoulou
May 4, 2018

**Goal** Last week Shengxuan Liu provided the intuition behind the definition of characteristic classes which are cohomological invariants that describe the obstructions to constructing everywhere independent sets of sections of a vector bundle. Characteristic classes measure the extent to which the bundle is twisted. Muze Ren defined axiomatically the Stiefel-Whitney classes for real vector bundles and the Chern classes for complex vector bundles and proved the existence and uniqueness of those classes. The evident similarities in the definition of those classes leads to the question of whether they are somehow related. Today’s goal is to understand how these classes are related.

But first let’s explore the Stiefel-Whitney and Chern classes a bit more by working on some examples!

**Example 1.** Since the product bundle $E = B \times \mathbb{R}^n$ is the pullback of a $n$-dimensional bundle over a point and the cohomology of a point is zero in positive dimensions, the naturality of Stiefel-Whitney classes implies that the product bundle has $w_i(E) = 0$ for $i > 0$.

**Example 2** (Stability). The Whitney sum formula implies that taking the direct sum of a bundle with a product bundle does not change its Stiefel-Whitney classes. For example, the tangent bundle $\tau(S^n)$ to $S^n$ is stably trivial since its direct sum with the normal bundle $\nu(S^n)$ to $S^n$ in $\mathbb{R}^{n+1}$, which is a trivial line bundle, produces a trivial line bundle. Hence, $w_i(\tau(S^n)) = 0$ for $i > 0$. This implies that the Stiefel-Whitney classes cannot tell us if $S^n$ is
parallelizable and hence they are not complete invariants (at least in the case of $n$-dimensional vector bundles with $n > 1$).

The last observation generalizes to the case of an arbitrary submanifold $M^n$ of $\mathbb{R}^n$ since $\tau(M) \oplus \nu(M) \cong M \times \mathbb{R}^n$ and we get that $w(\nu(E)) = w^{-1}(\tau(M))$, where $w^{-1}(\tau(M))$ is the multiplicative inverse of $w(\tau(M))$. (Note that the collection of all $w = 1 + w_1 + w_2 + \ldots \in H^*(B; \mathbb{Z}_2)$ with leading term 1 forms a commutative group under multiplication.)

**Remark.** The identity

$$(1 + w_1 + w_2 + \ldots)(1 + w_1' + w_2' + \ldots) = 1 + (w_1 + w_1') + (w_2 + w_1w_1' + w_2') + \ldots$$

tells us that $w(E_1)$ and $w(E_1 \oplus E_2)$ completely determine $w(E_2)$.

**Example 3.** Proposition 1.4 in Hatcher’s book tells us that for any vector bundle $E \to B$ over a compact Hausdorff space $B$ there exists a vector bundle $E' \to B$ such that $E \oplus E'$ is the trivial bundle. The following argument that uses the Stiefel-Whitney classes tells us that the theorem fails when the base space is not compact. Consider the canonical line bundle $E_1(\mathbb{R}^\infty) \to \mathbb{R}P^\infty$ and assume there is a vector bundle $E \to \mathbb{R}P^\infty$ whose sum with the canonical line bundle is trivial. Using the Whitney sum formula together with the fact that $w(E_1(\mathbb{R}^\infty)) = 1 + \omega$, where $\omega$ is a generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$, we get that $w(E)$ should be $(1 + \omega)^{-1} = 1 + \omega + \omega^2 + \ldots$ since we are using $\mathbb{Z}_2$ coefficients. Thus, we get $w_1(E) = \omega^i$ which is nonzero in $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2)$ and this leads to a contradiction since $w_i(E) = 0$ for $i > \dim E$.

The restriction of the canonical bundle to the subspace $\mathbb{R}P^n \subset \mathbb{R}P^\infty$, namely $E_1(\mathbb{R}^{n+1})$ satisfies the hypotheses of the theorem and has an “inverse” bundle. We have already seen that the orthogonal complement of this bundle is the $n$-dimensional vector bundle with total space

$$E_1^\perp(\mathbb{R}^{n+1}) = \{(l, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1}| v \perp l\}$$

and projection $p : E_1^\perp(\mathbb{R}^{n+1}) \to \mathbb{R}P^n$ given by $p(l, v) = l$. Using the Whitney sum formula, we have that $w(E_1^\perp(\mathbb{R}^{n+1})) = 1 + \omega + \omega^2 + \ldots + \omega^n$. Here $\omega$ denotes a generator of $H^1(\mathbb{R}P^n; \mathbb{Z}_2)$.

**Example 4.** Consider the $n$-fold Cartesian product $(E_1)^n \to (G_1)^n$ of the canonical line bundle over $G_1 = \mathbb{R}P^\infty = \bigcup_k \mathbb{R}P^k$ with itself. Observe that this vector bundle is the direct sum $\pi_1^*(E_1) \oplus \ldots \oplus \pi_n^*(E_1)$ where $\pi_i : (G_1)^n \to G_1$.
is the projection onto the $i$-th factor. Using the Whitney sum formula, we compute

$$w((E_1)^n) = w(\pi_1^*(E_1)) \cdots w(\pi_n^*(E_1)) = \prod_i (1 + \alpha_i) \in \mathbb{Z}_2 [\alpha_1, ..., \alpha_n] \cong H^*((\mathbb{R}P^n)^n; \mathbb{Z}_2),$$

where $\alpha_i = \pi_i^* w_1(E_1)$. Hence, $w_i((E_1)^n)$ is nonzero for $0 < i \leq n$ and is the $i$-th elementary symmetric polynomial in the $\alpha_j$'s, i.e. the sum of all the products of $i$ different $\alpha_j$'s.

The same argument applies to the n-fold Cartesian product of the canonical line bundle over $\mathbb{C}P^\infty$ and we get that the first $n$ Chern classes are nonzero and are in bijection with the elementary symmetric polynomials in $n$ variables.

The identification of the $w_i$'s and the $c_i$'s with the elementary symmetric functions can be done in a more general setting. The splitting principle associates to every $n$-dimensional vector bundle $E \to B$ a space $F(E)$ and a map $p : F(E) \to B$ such that the pullback of $E$ over this map is a direct sum of $n$ line bundles, namely $L_1 \oplus ... \oplus L_n \to F(E)$. We see that $w(E)$ pulls back to $w(L_1 \oplus ... \oplus L_n) = (1 + \alpha_1)...(1 + \alpha_n) = 1 + \sigma_1 + ... + \sigma_n$, where $\alpha_i = w_1(L_i)$.

Thus, we have embedded $H^*(B; \mathbb{Z}_2)$ in a larger ring $H^*(F(E); \mathbb{Z}_2)$ such that $w_i(E)$ becomes the $i$-th elementary symmetric polynomial in the elements $\alpha_1, ..., \alpha_n$ of $H^*(F(E); \mathbb{Z}_2)$. (Note that $p^* : H^*(B; \mathbb{Z}_2) \to H^*(F(E); \mathbb{Z}_2)$ is injective.)

## Relation between Stiefel-Whitney and Chern classes

### Definition 1.
Let $\xi = (E, B, \pi)$ be a $n$-dimensional complex vector bundle. Then, the underlying real vector bundle, $\xi_\mathbb{R}$, of $\xi$ is the $2n$-dimensional real vector bundle with the same underlying structure $(E, B, \pi)$ but where in each fiber $\pi^{-1}(b)$ we consider the real vector space structure. This procedure is called restriction to scalars.

### Remark.
The underlying real vector bundle $\xi_\mathbb{R}$ of a complex vector bundle $\xi$ has a canonical preferred orientation.

To see this, fix a homeomorphism $\phi : \mathbb{C}^n \to \mathbb{R}^{2n}$ and choose an open cover
\{U_\lambda\} \) of \( B \) and trivializations \( h_\lambda : \pi^{-1}(U_\lambda) \to U_\lambda \times \mathbb{C}^n \). Then, we observe that \( k_\lambda = (id_{U_\lambda} \times \phi) \circ h_\lambda : \pi^{-1}(U_\lambda) \to U_\lambda \times \mathbb{R}^{2n} \) are trivializations of the underlying real vector bundle.

This is an orienting atlas of the vector bundle \( \mathbb{R}^{2n} \to E \to B \). Indeed, if \( x \in U_i \cap U_j \), then we have that the maps \( h_i|_{E_x}, h_j|_{E_x} : E_x \to \mathbb{C}^n \) are homotopic since \( GL_n(\mathbb{C}) \) is path connected. Hence,

\[
\langle \gamma_x \rangle = H^n(\phi \circ h_i|_{E_x}) \langle \gamma_i \rangle = H^n(h_i|_{E_x}) \circ H^n(\phi) \langle \gamma_i \rangle \\
= H^n(h_j|_{E_x}) \circ H^n(\phi) \langle \gamma_i \rangle \\
= H^n(\phi \circ h_j|_{E_x}) \langle \gamma_i \rangle \\
= \langle \gamma_x' \rangle
\]

The two trivializations are compatible and hence, the vector bundle is orientable.

(Recall that \( \langle \gamma_i \rangle \) is the canonical generator, see definition 35.11 in the lecture notes of AT II.)

**Remark.** The Stiefel-Whitney classes of the underlying real vector bundle of a given complex vector bundle vanish in odd degree. Indeed, first note that the Stiefel-Whitney classes of the underlying real vector bundle of the canonical bundle over the infinite complex Grassmannian, \( E_n \to G_n(\mathbb{C}^\infty) \), satisfy \( w_{2i+1}(E_n) = 0 \) (since the cell structure of \( G_n(\mathbb{C}^\infty) \) has no cells in odd dimensions and hence \( H^{2i+1}(G_n(\mathbb{C}^\infty); \mathbb{Z}_2) = 0 \)). Furthermore, we know that any complex vector bundle is obtained as a pullback of the canonical \( n \)-dimensional vector bundle over \( G_n(\mathbb{C}^\infty) \); that is there exists \( f \in [B, G_n] \) such that \( f^*(E_n) = E \). Regarding all bundles as real vector bundles and using the naturality property of the Stiefel-Whitney class, we see that \( w_{2i+1}(E) = f^*w_{2i+1}(E_n) = 0 \).

**Proposition 1.** Regarding a \( n \)-dimensional complex vector bundle \( \xi : E \xrightarrow{\pi} B \) as a \( 2n \)-dimensional real vector bundle \( \xi_{\mathbb{R}} \), then \( w_{2i+1}(E) = 0 \) and \( w_{2i}(E) \) is the image of \( c_i(E) \) under the coefficient homomorphism \( H^{2i}(B; \mathbb{Z}) \to H^{2i}(B; \mathbb{Z}_2) \).

Idea behind the proof.

Let’s start with the defining relation for Chern classes

\[
x_C(E)^n - c_1(E)x_C(E)^{n-1} + \ldots + (-1)^n c_n(E) \cdot 1 = 0.
\]

This is a relation in \( H^*(\mathbb{C}P(E); \mathbb{Z}) \). By \( \mathbb{Z}_2 \)-reduction we get a relation in \( H^*(\mathbb{C}P(E); \mathbb{Z}_2) \), namely \( \bar{x}_C(E)^n + \bar{c}_1(E)\bar{x}_C(E)^{n-1} + \ldots + \bar{c}_n(E) \cdot 1 = 0 \). How
The bundle $E$ has two projectivizations $\mathbb{R}P^{2n-1} \to \mathbb{R}P(E) \xrightarrow{\mathbb{R}P(g)} \mathbb{R}P^\infty$ and $\mathbb{C}P^{n-1} \to \mathbb{C}P(E) \xrightarrow{\mathbb{C}P(g)} \mathbb{C}P^\infty$, consisting of all the real and all the complex lines in the fibers of $E$, respectively.

Since a real line is all the real scalar multiples of any nonzero vector and a complex line is all the complex scalar multiples, there is a natural projection $p : \mathbb{R}P(E) \to \mathbb{C}P(E)$ sending each real line to the complex line containing it. If $v \in E_x \setminus 0_x$, then $p$ maps $\mathbb{R}P(E) \ni v \to Cv \in \mathbb{C}P(E)$.

This projection $p$ fits into a commuting diagram

\[
\begin{array}{ccc}
\mathbb{R}P^{2n-1} & \xrightarrow{\pi} & \mathbb{R}P(E) \\
\downarrow & & \downarrow p \\
\mathbb{C}P^{n-1} & \xrightarrow{\pi'} & \mathbb{C}P(E) \\
\end{array}
\]

where the left column is the restriction of $p$ to a fiber of $E$ and the maps $\mathbb{R}P(g)$ and $\mathbb{C}P(g)$ are obtained by projectivizing, over $\mathbb{R}$ and $\mathbb{C}$ a map $g : E \to \mathbb{C}P^\infty$ which is a $\mathbb{C}$-linear injection on fibers. (We have seen the construction of such a map $g$ in the proof of Theorem 1.16 in Hatcher’s book.)

Note that all three vertical maps in this diagram are fiber bundles with fiber $\mathbb{R}P^1$, the real lines in a complex line.

The fiber bundle $\mathbb{R}P^\infty \to \mathbb{C}P^\infty$ satisfies the hypothesis of the Leray-Hirsch theorem with $\mathbb{Z}_2$ coefficients. First, note that $H^*(\mathbb{R}P^1; \mathbb{Z}_2) \cong \mathbb{Z}_2[\chi]/(\chi^2)$ is a finitely generated free $\mathbb{Z}_2$-module. Also note that a generator 1 of $H^0(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2$ pulls back to a generator of $H^0(\mathbb{R}P^1; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and a generator $\alpha$ of $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2$ (as groups) pulls back to a generator of $H^1(\mathbb{R}P^1; \mathbb{Z}_2)$ via the inclusion $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^\infty$.\(^1\)

Hence,

\[
H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong H^*(\mathbb{C}P^\infty; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(\mathbb{R}P^1; \mathbb{Z}_2)
\]

\(^1\) $\mathbb{R}P^1$ is the 1-skeleton of $\mathbb{R}P^\infty$ and thus, the inclusion $\iota : \mathbb{R}P^1 \to \mathbb{R}P^\infty$ induces isomorphisms $\iota^* : H^k(\mathbb{R}P^\infty; \mathbb{Z}_2) \to H^k(\mathbb{R}P^1; \mathbb{Z}_2)$ for $k \leq 1$. (see page 262, Proposition 12.1.8 in [3])

\(^2\) The existence of classes $c_j \in H^k(E; R)$ whose restrictions $i^*(c_j)$ form a basis for $H^*(F; R)$ in each fiber $F$, where $i : F \to E$ is the inclusion, is equivalent to the existence of a cohomology extension of the fibre.
and $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2)$ is a free $H^*(\mathbb{C}P^\infty; \mathbb{Z}_2)$-module with basis 1 and $\alpha$.

The Leary-Hirsch theorem tells us that if $\beta$ is the standard generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$, the $\mathbb{Z}_2$-reduction $\bar{\beta} \in H^2(\mathbb{C}P^\infty; \mathbb{Z}_2)$ pulls back to a generator of $H^2(\mathbb{R}P^\infty; \mathbb{Z}_2)$, namely the square $\alpha^2$ of the generator $\alpha \in H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$.

Hence, the $\mathbb{Z}_2$-reduction $\bar{x}_C(E) = \mathbb{C}P(g)^\ast(\bar{\beta}) \in H^2(\mathbb{C}P(E); \mathbb{Z}_2)$ of the basic class $x_C(E) = \mathbb{C}P(g)^\ast(\beta)$ pulls back to the square of the basic class $x_{\mathbb{R}}(E) = \mathbb{R}P(g)^\ast(\alpha) \in H^1(\mathbb{R}P(E); \mathbb{Z}_2)$. Indeed,

$$p^*\mathbb{C}P(g)^\ast(\bar{\beta}) = (\mathbb{C}P(g) \circ p)^\ast(\bar{\beta})$$
$$= (p' \circ \mathbb{R}P(g))^\ast(\bar{\beta})$$
$$= \mathbb{R}P(g)^\ast(p')^\ast(\bar{\beta})$$
$$= \mathbb{R}P(g)\alpha^2$$
$$= x_{\mathbb{R}}(E)^2$$

We have that the $\mathbb{Z}_2$-reduction of the defining relation for the Chern classes of $E$, $x_C(E)^n + \bar{c}_1(E)x_C(E)^{n-1} + ... + \bar{c}_n(E) \cdot 1 = 0$, pulls back to the relation $x_{\mathbb{R}}(E)^{2n} + \bar{c}_1(E)x_{\mathbb{R}}(E)^{2n-2} + ... + \bar{c}_n(E) \cdot 1 = 0$, which is the defining relation for the Stiefel-Whitney classes of $E$. Thus, $w_{2i+1}(E) = 0$ and $w_{2i}(E) = \bar{c}_i(E)$.

\[ \square \]

References