Convergence and Riemannian bounds on Lagrangian submanifolds

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We consider collections of Lagrangian submanifolds of a given symplectic manifold which respect uniform bounds of curvature type coming from an auxiliary Riemannian metric. We prove that, for a large class of metrics on these collections, convergence to an embedded Lagrangian submanifold implies convergence to it in the Hausdorff metric. This class of metrics includes well-known metrics such as the Lagrangian Hofer metric, the spectral metric and the shadow metrics introduced by Biran, Cornea and Shelukhin [5]. The proof relies on a version of the monotonicity lemma, applied on a carefully-chosen metric ball.

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1. Introduction and main result

Spaces of Lagrangian submanifolds are generally analyzed in a metric sense at large scale. For example, there has been a great amount of work on whether the space of Lagrangian submanifolds, subject to some topological constraints, has infinite diameter, or whether the metric admits upper bounds in terms of intersection numbers (see for example [5, 20, 22, 33, 38]). However, a less studied problem is the local behavior of those metrics, even in the case of the well-known Lagrangian Hofer metric. This is probably due to the fact that, without additional constraints, converging sequences can be quite wild from a set-theoretic standpoint. Furthermore, before the advent of shadow metrics, one only looked at Hamiltonian isotopic Lagrangian submanifolds — or at least, conjecturaly Hamiltonian isotopic ones.

The purpose of this paper is to show that, when we only look at Lagrangian submanifolds behaving well with respect to some auxiliary Riemannian metric, converging sequences cannot be wild. More precisely, we will show that they also converge to the same Lagrangian submanifold in the Hausdorff metric associated to the auxiliary Riemannian metric. Note however that we only look at sequences converging to an embedded Lagrangian submanifold. This restriction is necessary since sequences converging in certain metrics could theoretically converge to an im-

mersed Lagrangian submanifold. As noted above, this result is of particular interest when applied to the weighted fragmentation metrics of Biran, Cornea, Shelukhin and Zhang [5, 6]. Indeed, these metrics exist on spaces of Lagrangian submanifolds which are not necessarily of the same homotopy type, let alone Hamiltonian isotopic. But, as we shall see below, the presence of bounds coming from a Riemannian metric forces neighboring Lagrangian submanifolds to be homeomorphic.

1.1. *Some notation and definitions*

Before writing down the main result in a more precise form, we need to fix some notation and definitions.

1.1.1. *Riemannian bounds*

We begin by introducing some Riemannian quantities that will serve to restrict the classes of the Lagrangian submanifolds that we will consider.

Let L be a submanifold of a Riemannian manifold (M, g). We can see its second fundamental form B_L as a section of the bundle $(TL \otimes TL \otimes TL^\perp)^* \to L$, where $\perp$ denotes the orthogonal complement with respect to g. We thus define the norm of the second fundamental form to be

$$||B_L|| := \sup_{x \in L} |B_L(x)|_{\tilde{g}_x},$$

where $\tilde{g}_x$ is the scalar product induced by g on $(T_xL \otimes T_xL \otimes T_xL^\perp)^*$. When $\dim L = 1$ and $\dim M = 2$ — a case that will be of particular interest to us later on — this is just the supremum of the geodesic curvature of L.

In general, uniformly bounding the norm of the second fundamental form will not be enough for our purposes. We will thus make use of another quantity, which gives a better control on the way L is embedded in M.

Definition 1.1. Let (M, g) be a Riemannian manifold, and L be a submanifold. Let $\epsilon \in (0, 1]$. We say that L is ϵ-*tame* if

$$\frac{d_M(x, y)}{\min\{1, d_L(x, y)\}} \geq \epsilon \qquad \forall x \neq y \in L,$$

where d_M is the distance function on M induced by g, and d_L is the distance function on L induced by $g|_L$.

Remark 1.1. This is a variation of the tame condition appearing in work of Sikorav [35]. More precisely, it is equivalent to the (T'1) condition. This condition also appeared under the name ϵ-Lipschitz in work of Groman and Solomon [17, 18].

1.1.2. *Collections of Lagrangian submanifolds*

In general, there is no hope of being able to meaningfully compare two arbitrary Lagrangian submanifolds of a given symplectic manifold. That is why, when defining

metrics on spaces of Lagrangian submanifolds, it is necessary to consider more restricted collections of Lagrangian submanifolds, e.g. Hamiltonian isotopic ones, exact ones, etc.

In the present paper, the symplectic properties of the Lagrangian submanifolds considered will not matter inasmuch as they allow for the definition of well-behaved metrics between them. However, to give an idea of the collections to which our result applies, we present some interesting choices of collection $\mathscr{L}^{\star}(M)$ of Lagrangian submanifolds of a given symplectic manifold (M, ω). Note that throughout this paper, we will assume that M is connected, and either closed or convex at infinity. We will also assume that the Lagrangian submanifolds are closed and connected.

- $(\star = L_0)$: Here, L_0 is some fixed Lagrangian submanifold. Then, $\mathscr{L}^{L_0}(M)$ is the Hamiltonian orbit of L_0, i.e. the set of Lagrangian submanifolds of the form $\phi(L_0)$ for some Hamiltonian diffeomorphism with compact support ϕ.
- $(\star = e)$: For this collection to make sense, we need to suppose that M is exact. Then, $\mathscr{L}^e(M)$ is the collection of exact Lagrangian submanifolds.
- $(\star = we)$: Here, $\mathscr{L}^{we}(M)$ is the collection of weakly exact Lagrangian submanifolds, i.e. Lagrangian submanifolds L such that the morphism $\omega : \pi_2(M, L) \to \mathbb{R}$ given by integration with respect to ω is trivial. Note that the existence of such a Lagrangian submanifold implies that M is symplectically aspherical.
- $(\star = m(\rho, d))$: Here, $\rho > 0$ and $d \in \mathbb{Z}_2$. Then, $\mathscr{L}^{m(\rho,d)}(M)$ is the collection of Lagrangian submanifolds L such that

  (a) the Maslov index $\mu : \pi_2(M, L) \to \mathbb{Z}$ satisfies $\omega = \rho\mu$;
  (b) the minimum Maslov number is $N_L \geq 2$;
  (c) the modulo 2 count of J-holomorphic disks with boundary along L of Maslov index 2 — for J generic — is equal to d.

As noted by Biran, Cornea and Zhang [6], when one is interested in fragmentation metrics, it might be necessary to restrict oneself to a subcollection of one of the above choices.

The result that we will present does not hold for all Lagrangian submanifolds in $\mathscr{L}^{\star}(M)$. Indeed, we will need to impose some uniform bounds coming from Riemannian geometry. Therefore, we fix a Riemannian metric g on M and constants $\Lambda \geq 0$, $\epsilon \in (0, 1]$. We then introduce two new types of subcollections:

$$\mathscr{L}^{\star}_{\Lambda}(M, g) := \{L \in \mathscr{L}^{\star}(M) \mid ||B_L|| \leq \Lambda\}$$
$$\mathscr{L}^{\star}_{\Lambda,\epsilon}(M, g) := \{L \in \mathscr{L}^{\star}_{\Lambda}(M, g) \mid L \text{ is } \epsilon\text{-tame}\}.$$

We recall that we always consider our Lagrangian submanifolds to be closed and connected. When it is evident from the context, we will omit g from the notation.

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1.1.3. *Chekanov-type metrics on collections of Lagrangian submanifolds*

We now describe the type of metrics that we will be putting on our collections $\mathscr{L}^{\star}(M)$. These will be metrics d that interact well with regards to J-holomorphic curves for a compatible almost complex structure J. In particular, they will include well-known metrics appearing in symplectic topology:

- $(d = d_H)$: This is the case of the Lagrangian Hofer metric, due to Chekanov [10]. It is then understood that $\mathscr{L}^{\star}(M) \subseteq \mathscr{L}^{L_0}(M)$ for some L_0.
- $(d = \gamma)$: This is the case of the spectral metric, originally due to Viterbo [37] on $\mathscr{L}^{L_0}(T^*L_0)$. In fact, the metric may be extended to the whole collection $\mathscr{L}^e(T^*L_0) \cap \mathscr{L}^{m(1,0)}(T^*L_0)$ — not just $\mathscr{L}^{L_0}(T^*L_0)$ — by work from Fukaya, Seidel, and Smith [14, 15], Abouzaid [1], and Kragh [25]. For any symplectic manifold M, it has also been defined on $\mathscr{L}^{L_0}(M)$ for L_0 weakly exact by Leclercq [26] and for L_0 monotone with nonvanishing quantum homology by Kislev and Shelukhin [23], following work of Leclercq and Zapolsky [38]. In this case, we always assume that $\mathscr{L}^{\star}(M)$ is included in one of the collection mentioned above.
- $(d = \gamma_{\text{ext}})$: This is a variant of the usual spectral metric, as defined by Kislev and Shelukhin [23]. We also have that $\mathscr{L}^{\star}(M) \subseteq \mathscr{L}^{L_0}(M)$ for $L_0 \in \mathscr{L}^{we}(M)$. However, M has then to be closed and monotone, i.e. the diagonal of $(M \times M, \omega \oplus -\omega)$ is in $\mathscr{L}^{m(\rho,d)}(M \times M)$.

These metrics have in common that they all respect a Chekanov-type theorem relating $d(L, L')$ to the Gromov width $w(L; L')$ of L in $M - L'$. In turn, this is because whenever $L, L' \in \mathscr{L}^{\star}(M)$ intersect transversally, $d(L, L')$ is small enough, and J is generic, there is a J-holomorphic strip with boundary along L and L' and with area bounded from above by $d(L, L')$ going through every point of $L \cup L'$ (cf. [3, 8, 23]). In fact, without the restrictions on $d(L, L')$ and J, we still get a J-holomorphic curve with the above properties by Gromov compactness. The curve may however be a disk or a sphere instead of a strip.

Our results are however most interesting when working with metrics which are finite between possibly-nondiffeomorphic Lagrangian submanifolds. In all known examples of such metrics, they are defined using auxiliary families $\mathscr{F}, \mathscr{F}' \subseteq \mathscr{L}^{\star}(M)$. They still respect a Chekanov-type theorem, but the J-holomorphic curve used in the proof of the theorem might be more general J-holomorphic polygons with boundary along L, L', and elements of $\mathscr{F}$ or $\mathscr{F}'$.

For this reason, we will consider in what we call Chekanov-type pseudometrics: pseudometrics $d^{\mathscr{F}}$ with the more general property that for any ω-compatible almost complex structure, any $L, L' \in \mathscr{L}^{\star}(M)$ intersecting transversally, and any point $x \in L \cup L'$, there exists a J-holomorphic polygon u passing through x such that

$$\omega(u) \leq 2d^{\mathscr{F}}(L, L').$$

Furthermore, u has to have boundary along L, L', and along Lagrangian submanifolds in $\mathscr{F}$ (see [32] for a more detailed exposition on J-holomorphic polygons and

on their role in symplectic topology — contrary to Seidel, we allow our polygons to have 0 or 1 side, i.e. to be spheres or disks). If the Lagrangian submanifolds above do not intersect transversally, then they should be replaced by arbitrarily small Hamiltonian perturbation. A more precise definition will be given in Subsec. 2.1. Note that we allow here our pseudometrics to take infinite values.

If $d^{\mathscr{F}}$ and $d^{\mathscr{F}'}$ are Chekanov-type pseudometrics, and if the intersection

$$\left(\overline{\bigcup_{F \in \mathscr{F}} F}\right) \bigcap \left(\overline{\bigcup_{F' \in \mathscr{F}'} F'}\right) \tag{1.1}$$

is discrete, then we will call $d = \hat{d}^{\mathscr{F},\mathscr{F}'} := \max\{d^{\mathscr{F}}, d^{\mathscr{F}'}\}$ a Chekanov-type metric. As we shall see below, the discreteness of (1.1) insures that the term "metric" is indeed here warranted.

Note that Chekanov-type metrics not only include the previously-mentionned metrics (under the case $d = d^{\mathscr{F}} = d^{\mathscr{F}'}$ and $\mathscr{F} = \mathscr{F}' = \emptyset$), but some more recent ones as well.

- $(\hat{d}^{\mathscr{F},\mathscr{F}'} = \hat{d}_{\mathcal{S}}^{\mathscr{F},\mathscr{F}'})$: These are the shadow metrics appearing in work of Biran, Cornea, and Shelukhin [5, 12]. It is then understood that $\mathscr{L}^\star(M) \subseteq \mathscr{L}^e(M)$, $\mathscr{L}^\star(M) \subseteq \mathscr{L}^{we}(M)$, or $\mathscr{L}^\star(M) \subseteq \mathscr{L}^{m(\rho,d)}(M)$ for some $\rho > 0$ and $d \in \mathbb{Z}_2$.
- $(\hat{d}^{\mathscr{F},\mathscr{F}'} = \hat{s}_{\mathrm{alg}}^{\mathscr{F},\mathscr{F}'})$: These are the so-called algebraic fragmentation metrics also appearing in work of Biran, Cornea, and Shelukhin [5]. As above, it is then understood that $\mathscr{L}^\star(M)$ is in $\mathscr{L}^e(M)$, $\mathscr{L}^{we}(M)$, or some $\mathscr{L}^{m(\rho,d)}(M)$.
- $(\hat{d}^{\mathscr{F},\mathscr{F}'} = \hat{D}^{\mathscr{F},\mathscr{F}'})$: There are possibly many other weighted fragmentation pseudometrics — as defined by Biran, Cornea, and Zhang [6] — that belong to this class.

1.2. *A conjecture on convergence in Lagrangian spaces*

We now introduce a conjecture due to Cornea and explain how we will show it holds under some additional assumptions in high dimensions and without them in dimension 2.

We fix a connected symplectic manifold with compatible almost complex structure (M, ω, J) which is closed or convex at infinity. We suppose that J is such that g_J is complete, has uniformly bounded sectional curvature, and has injectivity radius uniformly bounded away from zero. Note that a symplectic manifold that is convex at infinity always admits such a J. This has been proven for the case when M is a twisted cotangent bundle by Cieliebak, Ginzburg, and Kerman [11], following a suggestion of Sikorav. As noted there, the proof easily adapts to the case when M is instead convex at infinity.

We also fix a collection of Lagrangian submanifold $\mathscr{L}^\star(M)$ and a Chekanov-type metric $\hat{d}^{\mathscr{F},\mathscr{F}'}$ on it. By the examples in the previous subsection, we have multiple choices of appropriate collections $\mathscr{L}^\star(M)$ and metrics $\hat{d}^{\mathscr{F},\mathscr{F}'}$.

Conjecture 1.1 (Cornea, 2018). The topology induced by the Hausdorff metric

on $\mathscr{L}_\Lambda^\star(M)$ is coaser than the one induced by the Chekanov-type metric $\hat{d}^{\mathscr{F},\mathscr{F}'}$. In other words, if $L_n \to L_0$ in $\hat{d}^{\mathscr{F},\mathscr{F}'}$, then $L_n \to L_0$ in the Hausdorff metric δ_H induced by g_J.

Remark 1.2. The conjecture was originally stated for the weighted fragmentation metrics appearing in [5]. However, the proof lends itself naturally to a generalization to a larger class of metrics behaving well with respect to J-holomorphic curves.

The main purpose of this paper is to prove the conjecture under the slightly stronger hypothesis that the Lagrangian submanifolds are also ϵ-tame.

Theorem A. *Let $\{L_n\}_{n\geq 1} \subseteq \mathscr{L}_{\Lambda,\epsilon}^\star(M)$ be such that $L_n \to L_0 \in \mathscr{L}^\star(M)$ with respect to a Chekanov-type metric $\hat{d}^{\mathscr{F},\mathscr{F}'}$. Then, $L_n \to L_0$ in the Hausdorff metric δ_H induced by g_J.*

Moreover, when $\dim M = 2$, the statement where $\mathscr{L}_{\Lambda,\epsilon}^\star(M)$ is replaced by $\mathscr{L}_\Lambda^\star(M)$ holds, i.e. the conjecture holds as initially stated in dimension 2.

Remark 1.3.

- We have fixed an almost complex structure J in order to have a fixed Riemannian metric $g = g_J$ through which to impose Riemannian bounds on the L_n's, i.e. to have precise Λ and ϵ. However, convergence in the Hausdorff metric associated to a Riemannian metric g is independent of the precise g. Therefore, Theorem A implies that whenever $\{L_n\} \subseteq \mathscr{L}_{\Lambda,\epsilon}^\star(M, g_J)$ for *some* J, Λ, and ϵ, then convergence in any Chekanov-type metric implies convergence in any Hausdorff metric.
- There is however still the related question of whether, for any compatible almost complex structures J and J', $L \in \mathscr{L}_{\Lambda,\epsilon}^\star(M, g_J)$ implies $L \in \mathscr{L}_{\Lambda',\epsilon'}^\star(M, g_{J'})$ for some Λ' and ϵ' independent of L, i.e. the question of whether respecting *some* Riemannian bounds is independent of J. Finding such a $\Lambda' = \Lambda'(\Lambda, J - J')$ is a direct computation, but finding $\epsilon' = \epsilon'(\Lambda, \epsilon, J - J')$ has proven to be beyond reach. Indeed, this requires to compare geodesics between two arbitrary metrics, which is highly nontrivial. However, we expect such ϵ' to exist. At the very least, this implies that Theorem A is an entirely symplectic statement in dimension 2.

Note that this implies in the ϵ-tame case that, for n large, L_n is diffeomorphic to L_0. Indeed, although convergence in the Hausdorff metric only implies that $\{(L_n, d_M|_{L_n})\}$ converges to $(L_0, d_M|_{L_0})$ in the Gromov–Hausdorff metric — not necessarily $\{(L_n, d_{L_n})\}$ to (L_0, d_{L_0}) — and thus results such as Perelman's stability theorem [30] do not directly applies, Gromov's [19] and Katsuda's [21] proof of the existence of a diffeomorphism between Gromov–Hausdorff-close geometrically-bounded manifolds still goes through. This is because Gromov's construction is ultimately local, and in this regime, the ϵ-tame condition allows for the comparison of $d_M|_{L_n}$ and d_{L_n}.

Actually, we expect L_n to be Hamiltonian isotopic to L_0 for n large; we know that to be true in some cases. Indeed, since $\{L_n\}_{n\geq 1}$ converges to L_0 in the Haus-

dorff metric, L_n is eventually in a Weinstein neighborhood of L_0. Therefore, if the nearby Lagrangian conjecture holds for L_0, and if each L_n is exact in the Weinstein neighborhood, e.g. it is exact in M or simply connected, then L_n has to be Hamiltonian isotopic to L_0. In particular, this is the case when $\{L_n\} \subseteq \mathscr{L}^e_\Lambda(M)$ and $\dim M = 2$.

We now give the broad idea of the proof of Theorem A:

(i) Since $\hat{d}^{\mathscr{F},\mathscr{F}'}$ is a Chekanov-type metric, for any $x \in L_0 - (L_n \cup (\cup F))$ and $x' \in L_n - (L_0 \cup (\cup F))$, there exist J-holomorphic polygons u and u' passing through x and x', respectively — modulo arbitrarily small perturbations. Furthermore, their area is bounded from above by $2\hat{d}^{\mathscr{F},\mathscr{F}'}(L_n, L_0)$.

(ii) By finding appropriate metric balls in M and using a version of the monotonicity lemma, it is possible to find a lower bound for the area of these polygons for n large. This bound depends only on M, Λ, ϵ, and the distances $d_M(x, L_n \cup (\cup F))$, $d_M(x, L_n \cup (\cup F'))$, $d_M(x', L_0 \cup (\cup F))$ and $d_M(x', L_0 \cup (\cup F))$. By the previous step, this turns into a lower bound of $\hat{d}^{\mathscr{F},\mathscr{F}'}(L_n, L_0)$.

(iii) Using the fact that $(\overline{\cup F}) \cap (\overline{\cup F'})$ is discrete, it is possible to turn the dependence on the different distances onto one on the Hausdorff distance $\delta_H(L_n, L_0)$.

(iv) The fact that $\hat{d}^{\mathscr{F},\mathscr{F}'}(L_n, L_0) \to 0$ then forces that $\delta_H(L_n, L_0) \to 0$.

We can get rid of the dependence on ϵ when $\dim M = 2$ because, in that case, we can make a better choice of metric balls on which the monotonicity lemma is applied.

1.3. *Structure of the paper*

The remainder of the paper is almost entirely dedicated to the proof of Theorem A.

In Sec. 2, we give a proper definition of Chekanov-type metrics. We then give the proof of the main theorem without any restriction on the dimension of the symplectic manifold M. This is done in two steps: proving an appropriate version of the monotonicity lemma, and showing that the existence of an appropriate J-holomorphic polygon implies the result.

In Sec. 3, we explain how to modify the proof of Sec. 2 to get rid of ϵ when $\dim M = 2$. Essentially, it suffices to modify the ball on which we apply the monotonicity lemma. To find such a ball, we develop some combinatoric arguments for curves on surfaces. This section ends with some proofs of results from Riemannian geometry that we have not found explicited in the literature.

In Sec. 4, we end the paper with examples of sequences of Lagrangian submanifolds that do not respect uniform Riemannian bounds, some respecting the conclusion of Theorem A, while others not. This shows that a complete characterization of sequences respecting said conclusion lies beyond simple Riemannian bounds.

2. The general proof

In this section, we give the proof of Theorem A without restrictions on the dimension of M. Actually, we prove a more precise estimate on the relation between $\hat{d}^{\mathscr{F},\mathscr{F}'}$ and δ_H. In what follows, the almost complex structure J is fixed. We also fix constants $K_0, r_0 > 0$ such that the sectional curvature of M respects $|K| \leq K_0$, and its injectivity radius respects $r_{\mathrm{inj}}(M) \geq r_0$. We always assume that (M, g_J) is complete.

Theorem 2.1. *For any Chekanov-type metric $\hat{d}^{\mathscr{F},\mathscr{F}'}$ on $\mathscr{L}^\star_{\Lambda,\epsilon}(M, g_J)$, there exist constants $R = R\big(K_0, r_0, \Lambda, \epsilon, (\overline{\bigcup_{F \in \mathscr{F}} F}) \cap (\overline{\bigcup_{F' \in \mathscr{F}'} F'})\big) > 0$ and $C = C(K_0, r_0, \epsilon) > 0$ such that whenever $\hat{d}^{\mathscr{F},\mathscr{F}'}(L, L') < R$, then*

$$\hat{d}^{\mathscr{F},\mathscr{F}'}(L, L') \geq C\delta_H(L, L')^2.$$

Clearly, the first part of Theorem A follows directly from Theorem 2.1.

Remark 2.1. As explained above, the proof relies on a version of the monotonicity lemma. However, it was pointed out to us by Shelukhin that a modification of Groman–Solomon's reverse isoperimetric inequality [17] would also yield a proof of Theorem A. Such a modification of the inequality has turned out to be much more difficult to prove than what ended up being presented here. Furthermore, additional Riemannian bounds seem then to be required, e.g. C^1 bounds on B_L. On the other hand, this would have the advantage of giving a linear inequality in Theorem 2.1, instead of a quadratic one.

2.1. *Chekanov-type metrics*

Before going in the proof of Theorem 2.1, we need to give a precise definition of what we mean by Chekanov-type. We also explain how the metrics enumerated in the introduction fit in this definition.

We begin by clarifying what we mean by a *J*-holomorphic polygon, essentially following Seidel's book [32].

Definition 2.1. Let (M, ω) be a symplectic manifold equipped with a ω-compatible almost complex structure J. Let $L_0, \dots, L_k$ be pairwise transverse Lagrangian submanifolds. Denote by S_r the closed unit disk with $|r| \leq k + 1$ punctures at its boundary. We equip S_r with the standard complex structure j and area form σ — when $|r| = 0$, we take the convention that S_r is the sphere of area π. We label the components of ∂S_r counterclockwise from C_0 to $C_{|r|-1}$, and the puncture from ζ_0 to $\zeta_{|r|-1}$ accordingly. A *J-holomorphic polygon with boundary along* $L_0, \dots, L_k$ is a smooth map $u : S_r \to M$ such that

  (i) u is (j, J)-holomorphic;
 (ii) $E(u) := \int_{S_r} |du|^2_{g_J} \sigma < \infty$;
(iii) $u(C_i) \subseteq L_i$ for all $i \in \{0, \dots, |r| - 1\}$;
 (iv) near ζ_i, there are conformal coordinates $(s, t) \in [0, \infty) \times [0, 1]$ such that $\lim_{s \to \infty} u(s, t) =: p_i \in L_{i-1} \cap L_i$.

Furthermore, for some $x \in L_0$, we will say that u *passes through* x if $x \in \overline{u(C_0)}$.

This allows us to define precisely what are Chekanov-type metrics.

Definition 2.2. Let $\mathscr{L}^\star(M)$ be a collection of Lagrangian submanifolds on M. Let $\mathscr{F} \subseteq \mathscr{L}^\star(M)$. We say that a pseudometric $d^{\mathscr{F}}$ on $\mathscr{L}^\star(M)$ is of *Chekanov-type* if for all compatible almost complex structure J, all $\delta > 0$, and all $L, L' \in \mathscr{L}^\star(M)$, there exist Lagrangian submanifolds $F_1, \ldots, F_k \in \mathscr{F}$ with the following property.

For any C^0- and Hofer-small Hamiltonian perturbations $\widetilde{L}, \widetilde{L}', \widetilde{F}_1, \ldots, \widetilde{F}_k$ of the Lagrangian submanifolds above making them pairwise transverse, and for any $x \in \widetilde{L} \cup \widetilde{L}'$, there exists a nonconstant J-holomorphic polygon $u : S_r \to M$ such that

(i) has boundary along $\widetilde{L}$, $\widetilde{L}'$ and $\widetilde{F}_1, \ldots, \widetilde{F}_k$;
(ii) passes through x;
(iii) respects the bound

$$\omega(u) \leq d^{\mathscr{F}}(L, L') + \delta.$$

Let $\mathscr{F}' \subseteq \mathscr{L}^\star(M)$ be such that

$$\left(\overline{\bigcup_{F \in \mathscr{F}} F}\right) \bigcap \left(\overline{\bigcup_{F' \in \mathscr{F}'} F'}\right)$$

is discrete. We will call $\hat{d}^{\mathscr{F}, \mathscr{F}'} := \max\{d^{\mathscr{F}}, d^{\mathscr{F}'}\}$ a *Chekanov-type metric* if $d^{\mathscr{F}}$ and $d^{\mathscr{F}'}$ are both Chekanov-type pseudometrics.

We conclude this subsection by explaining how the metrics mentioned in the introduction are indeed Chekanov-type metrics. In the case when $\hat{d}^{\mathscr{F}, \mathscr{F}'}$ is either a shadow metric $\hat{d}_{\mathcal{S}}^{\mathscr{F}, \mathscr{F}'}$ or an algebraic one $\hat{s}_{\mathrm{alg}}^{\mathscr{F}, \mathscr{F}'}$, this is proven in the course of Theorem 5.0.2 of [5]. By the inequality $\hat{d}_{\mathcal{S}}^{\emptyset, \emptyset} \leq d_H$, this also implies the result for the Lagrangian Hofer metric.

In fact, in the case of the Lagrangian Hofer metric, when the Lagrangian submanifolds involved are weakly exact or monotone, it is possible to get a sharper result. In that case, d_H is not only of Chekanov-type on $\mathscr{L}^{L_0}(M)$, but we may also take the J-holomorphic polygon u appearing in the definition of a Chekanov-type metric to have at most 2 sides. Indeed, this appears as Corollary 3.9 in [3] for the weakly exact case and as Theorem 1.2 in [8] for the monotone one.

The same is true of the spectral metric γ and its variant γ_{ext} on $\mathscr{L}^{L_0}(M)$. This follows from the proof of Theorem E in [23]. When $M = T^*L_0$, and γ is extended to $\mathscr{L}^e(T^*L_0) \cap \mathscr{L}^{m(1,0)}(T^*L_0)$, the same stay true. Indeed, the spectral metric between L and L' is then taken to be the usual one, but using different decorations on L and L'. Therefore, the underlying manifolds are the same, and there is still a strip between them.

We remark that the polygons which are found in the proof of Theorem 5.0.2 of [5] and of Theorem E of [23] are J-holomorphic for a very specific J. This choice is made to allow for the use Lelong's inequality to get a Chekanov-type theorem.

However, the same argument works perfectly well for any J, even without any genericity assumption.

2.2. *A version of the monotonicity lemma*

We now prove a slightly improved version of Sikorav's monotonicity lemma for curves with Lagrangian boundary [35]. The constants appearing in it will only depend on bounds coming from the Riemannian metric g_J on M and its restriction on Lagrangian submanifolds. This improvement follows from observations in Groman and Solomon [18] on the dependence of the constant appearing in the isoperimetric inequality on these bounds.

Firstly, consider a loop $\gamma : \mathbb{R}/(2\pi\mathbb{Z}) \to B_{r_{\mathrm{inj}}(M)/2}(x)$, where $B_{r_{\mathrm{inj}}(M)/2}(x)$ is the metric ball in M of radius $\frac{1}{2}r_{\mathrm{inj}}(M)$ centered at some $x \in M$. Denote by $a(\gamma)$ the area, with respect to ω, of a disk extension of γ contained in a metric ball $B_{r_{\mathrm{inj}}(M)/2}(y)$. Here, y is not necessarily equal to x. This is of course independent on the choice of disk: if $u : \mathbb{D} \to B_{r_{\mathrm{inj}}(M)/2}(y)$ and $u' : \mathbb{D} \to B_{r_{\mathrm{inj}}(M)/2}(y')$ are two such extensions, then gluing them along their common boundary gives a sphere $u\#\overline{u}$ in the ball $B_{r_{\mathrm{inj}}(M)}(\gamma(0))$. Such a sphere must be nullhomotopic. Therefore, $\omega(u) - \omega(u') = \omega(u\#\overline{u}) = 0$, and $a(\gamma)$ is well-defined.

Let $L \in \mathscr{L}^{\star}_{\Lambda,\epsilon}(M)$. Consider now an arc $\gamma : ([0,\pi], \{0,\pi\}) \to (M, L)$ with image in a metric ball $B_\delta(x)$ for some $x \in L$, where

$$\delta := \min\left\{\epsilon, \frac{\epsilon}{2}r_{\mathrm{inj}}(L), \frac{\epsilon}{2}r_0, \frac{\pi}{4\sqrt{K_0}}\right\}. \tag{2.1}$$

Take a path $\alpha : [0,\pi] \to B^L_{\delta/\epsilon}(y)$ such that $\gamma(\pi i) = \alpha(\pi i)$ for each $i \in \{0,1\}$. Here, $B^L_{\delta/\epsilon}(y)$ denotes the metric ball in L, i.e. with respect to d_L, of radius $\frac{\delta}{\epsilon}$ centered at $y \in L$. We then define $a(\gamma)$ to be the area, again with respect to ω, of a disk extension of $\gamma\#\overline{\alpha}$ in the ball $B_\delta(y)$, i.e. $a(\gamma) = a(\gamma\#\overline{\alpha})$ as a loop. Here, $\overline{\alpha}(\theta) := \alpha(\pi - \theta)$ for all $\theta \in [0,\pi]$.

Note that such a path α always exists, since $d_M(x, \gamma(\pi i)) < \delta \leq \epsilon$. Thus, it must be so that $d_L(x, \gamma(\pi i)) < \frac{\delta}{\epsilon}$ if the tameness condition is to be fulfilled. We now show that $a(\gamma)$ is well-defined. Take $\alpha : [0,\pi] \to B^L_{\delta/\epsilon}(y)$ and $\alpha' : [0,\pi] \to B^L_{\delta/\epsilon}(y')$ to be two paths such that $\alpha(\pi i) = \alpha'(\pi i) = \gamma(\pi i)$. Take $u : \mathbb{D} \to B_\delta(y)$ and $u' : \mathbb{D} \to B_\delta(y')$ to be extensions of $\gamma\#\overline{\alpha}$ and $\gamma\#\overline{\alpha'}$, respectively. Then, gluing u and u' along γ gives a disk $u\#\overline{u'}$ with boundary $\overline{\alpha}\#\alpha'$. But $\overline{\alpha}\#\alpha'$ is contained in $B^L_{2\delta/\epsilon}(\gamma(0))$. Since $\frac{2\delta}{\epsilon} \leq r_{\mathrm{inj}}(L)$, it must be a contractible loop. The homotopy from $\overline{\alpha}\#\alpha'$ to a point extends to a homotopy of $u\#\overline{u'}$ to a topological sphere in $B_{2\delta/\epsilon}(\gamma(0))$. Since $\frac{2\delta}{\epsilon} \leq r_0 \leq r_{\mathrm{inj}}(M)$, this sphere must be nullhomotopic. Therefore, $\omega(u) - \omega(u') = \omega(u\#\overline{u'}) = 0$, and the definition of $a(\gamma)$ is again independent of choices.

Lemma 2.1 (Isoperimetric inequality). *Let M, L, and δ be as above. There exist constants $c = c(K_0, r_0, \epsilon) > 0$ and $c' = c'(K_0, r_0) > 0$ such that*

(i) *for all loops γ with image in a metric ball $B_{r_0/2}(x)$ for some $x \in M$, we have that*

$$a(\gamma) \le c'\ell(\gamma)^2;$$

(ii) *for all arcs γ with ends in L and image in a metric ball $B_\delta(x)$ for some $x \in L$, we have that*

$$a(\gamma) \le c\ell(\gamma)^2.$$

Proof. As noted by Groman and Solomon [18], the proof appearing in Remark 4.4.2 of [28] depends only on the constants above. This gives a proof of *(i)*. We give here the details of the proof of *(ii)*.

Take α to be the unique minimizing geodesic in L from $\gamma(0)$ to $\gamma(\pi)$; it exists since $d_L(\gamma(0), \gamma(\pi)) < \frac{2\delta}{\epsilon} \le r_{\text{inj}}(L)$. We define an extension $u : \mathbb{D} \to M$ of $\gamma \# \overline{\alpha}$ by

$$u(re^{i\theta}) := \exp_{\gamma(0)}(r\xi(\theta)),$$

where $\exp_{\gamma(0)}(\xi(\theta)) := (\gamma \# \overline{\alpha})(\theta)$ for all $\theta \in [0, 2\pi]$. Although this disk might not be entirely contained in $B_\delta(x)$ — if δ is larger than the convexity radius of M that is — it is contained in $B_{2\delta}(\gamma(0))$. Therefore, the same argument as above implies that $\omega(u) = a(\gamma)$.

By the Gauss lemma, we have that

$$|\partial_r u|(re^{i\theta}) = |(d\exp_{\gamma(0)})_{r\xi(\theta)}(\xi(\theta))| = |\xi(\theta)| = d_M(\gamma(0), (\gamma \# \overline{\alpha})(\theta)).$$

However, for all $\theta \in [0, \pi]$, we have that

$$d_M(\gamma(0), (\gamma \# \overline{\alpha})(\theta)) = d_M(\gamma(0), \gamma(\theta)) \le \ell(\gamma)$$

and

$$\begin{aligned}
d_M(\gamma(0), (\gamma \# \overline{\alpha})(2\pi - \theta)) &= d_M(\alpha(0), \alpha(\theta)) \le \ell(\alpha) \\
&= d_L(\gamma(0), \gamma(\pi)) \le \frac{1}{\epsilon} d_M(\gamma(0), \gamma(\pi)) \\
&\le \frac{1}{\epsilon} \ell(\gamma).
\end{aligned}$$

Therefore, we get that $|\partial_r u| \le \frac{1}{\epsilon} \ell(\gamma)$.

Likewise, we can use the Rauch comparison theorem to compare Jacobi fields on M with those on spaces of constant sectional curvature $\pm K_0$ to get

$$\begin{aligned}
|\partial_\theta u|(re^{i\theta}) &= |(d\exp_{\gamma(0)})_{r\xi(\theta)}(r\dot{\xi}(\theta))| \\
&\le \frac{\sinh(r|\xi(\theta)|\sqrt{K_0})}{|\xi(\theta)|\sqrt{K_0}} |\dot{\xi}(\theta)| \\
&= \frac{\sinh(r|\xi(\theta)|\sqrt{K_0})}{|\xi(\theta)|\sqrt{K_0}} \left| (d\exp_{\gamma(0)})_{\xi(\theta)}^{-1} \left(\frac{d}{d\theta}(\gamma \# \overline{\alpha})(\theta) \right) \right| \\
&\le \frac{\sinh(r|\xi(\theta)|\sqrt{K_0})}{\sin(|\xi(\theta)|\sqrt{K_0})} \left| \frac{d}{d\theta}(\gamma \# \overline{\alpha})(\theta) \right|.
\end{aligned}$$

Note that $\frac{\sinh(r|\xi(\theta)|\sqrt{K_0})}{\sin(|\xi(\theta)|\sqrt{K_0})} \leq \frac{\sinh(|\xi(\theta)|\sqrt{K_0})}{\sin(|\xi(\theta)|\sqrt{K_0})}$ since $r \leq 1$. Furthermore, we have the inequality $|\xi| \leq 2\delta \leq \min\{r_0, \frac{\pi}{2\sqrt{K_0}}\}$ by construction. But the map $s \mapsto \frac{\sinh(s)}{\sin(s)}$ is increasing on the interval $[0, \pi)$, so that we finally get

$$|\partial_\theta u|(re^{i\theta}) \leq \frac{\sinh\left(\min\{r_0\sqrt{K_0}, \frac{\pi}{2}\}\right)}{\sin\left(\min\{r_0\sqrt{K_0}, \frac{\pi}{2}\}\right)} \left| \frac{d}{d\theta}(\gamma \# \overline{\alpha})(\theta) \right|.$$

Therefore, we get that

$$\begin{aligned}
a(\gamma) &= \left| \int_0^{2\pi} \int_0^1 \omega(\partial_r u, \partial_\theta u) r \, dr \, d\theta \right| \\
&\leq \int_0^{2\pi} \int_0^1 |\partial_r u| |\partial_\theta u| r \, dr \, d\theta \\
&\leq \frac{\sinh\left(\min\{r_0\sqrt{K_0}, \frac{\pi}{2}\}\right)}{\epsilon \sin\left(\min\{r_0\sqrt{K_0}, \frac{\pi}{2}\}\right)} \ell(\gamma) \int_0^{2\pi} \int_0^1 \left| \frac{d}{d\theta}(\gamma \# \overline{\alpha})(\theta) \right| r \, dr \, d\theta \\
&= \frac{\sinh\left(\min\{r_0\sqrt{K_0}, \frac{\pi}{2}\}\right)}{2\epsilon \sin\left(\min\{r_0\sqrt{K_0}, \frac{\pi}{2}\}\right)} \ell(\gamma) \left(\ell(\gamma) + \ell(\alpha)\right) \\
&\leq \left(1 + \frac{1}{\epsilon}\right) \frac{\sinh\left(\min\{r_0\sqrt{K_0}, \frac{\pi}{2}\}\right)}{2\epsilon \sin\left(\min\{r_0\sqrt{K_0}, \frac{\pi}{2}\}\right)} \ell(\gamma)^2.
\end{aligned}$$

This concludes to proof of *(ii)*. Note that replacing $\ell(\alpha)$ by 0 in the above argument gives a proof of *(i)* with $c' = \frac{\sinh(\min\{r_0\sqrt{K_0}, \frac{\pi}{2}\})}{2\sin(\min\{r_0\sqrt{K_0}, \frac{\pi}{2}\})} \leq c$. $\qquad\square$

Note that we could have taken the $\frac{\pi}{4\sqrt{K_0}}$ part of δ (as defined in (2.1)) to actually be $\frac{\pi\eta}{2\sqrt{K_0}}$ for any $\eta \in (0, 1)$; we have just made a convenient choice. Moreover, δ can always be bounded away from zero by a constant depending only on K_0, r_0, and Λ, as shows the following lemma.

Lemma 2.2 ([18, 34]). *Let (M, g) be a complete Riemannian manifold with sectional curvature K such that $|K| \leq K_0$, and with injectivity radius such that $r_{\text{inj}}(M) \geq r_0 > 0$. Let L be a submanifold with second fundamental form B_L such that $||B_L|| \leq \Lambda$ for some $\Lambda \geq 0$. Then, there exists a constant $i_0 = i_0(K_0, r_0, \Lambda) > 0$ such that*

$$r_{\text{inj}}(L, g|_L) \geq i_0.$$

Remark 2.2. The constant c — and c' — we get in the proof depends continuously on K_0, r_0, and ϵ. Likewise, the constant i_0 appearing in Lemma 2.2 may also be chosen so that it depends continuously on K_0, r_0, and Λ. This follows directly from the proof in Groman and Solomon's paper [18].

Proposition 2.1 (Monotonicity lemma). *Let M, L, δ be as above. Let Σ be a compact Riemann surface with boundary $\partial\Sigma$ with corners. Consider a nonconstant J-holomorphic curve $u : (\Sigma, \partial\Sigma) \to (B(x, r), \partial B(x, r) \cup L)$ for some $x \in L$ and*

$r \leq \delta$ *such that* $x \in u(\Sigma)$. *Suppose that* u *sends the corners of* Σ *to* $\partial B(x,r) \cap L$. *Then,*

$$\omega(u) \geq Cr^2,$$

where $C = \frac{1}{4c}$, *and* c *is the constant of Lemma 2.1.*

Proof. The proof is that of Proposition 4.7.2 in [35], but using the version of the isoperimetric inequality above. We still give the details here for the sake of completeness.

Set $\Sigma_t := u^{-1}(B(x,t))$ and $a(t) := \omega(u|_{\Sigma_t})$. By Sard's theorem, there is a subset of full measure Ω of $(0, r)$ such that for all $t \in \Omega$, Σ_t is a subsurface of Σ with piecewise smooth boundary $\partial \Sigma_t = u^{-1}(\partial B(x,t) \cup L)$. The discontinuities of the boundary are then contained in $u^{-1}(\partial B(x,t) \cap L)$.
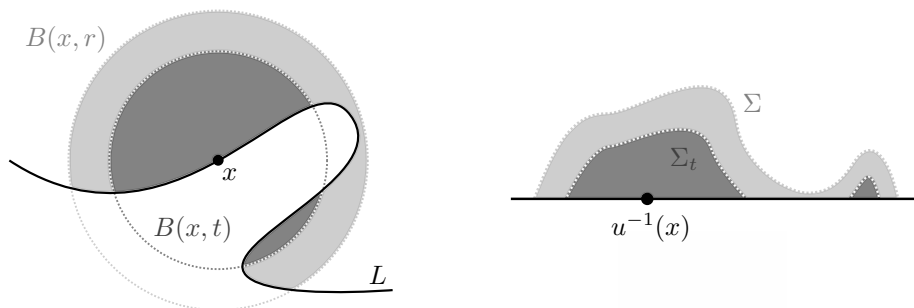

Fig. 1.   Visualization in two dimensions, both in the codomain (left) and domain (right).

We begin by noting that, for $t \in \Omega$, we have the inequality

$$a(t) \leq c\ell(t)^2,$$

where $\ell(t)$ is the length of $u|_{\partial \Sigma_t - u^{-1}(L)}$. Indeed, write the boundary of Σ_t as

$$\partial \Sigma_t = \left(\bigsqcup_i \gamma_i \# \overline{\alpha_i} \right) \sqcup \left(\bigsqcup_j \beta_j \right),$$

where the γ_i's are arcs in the interior of Σ with extremities in $u^{-1}(L)$, the α_i's are the segment of $u^{-1}(L)$ between the extremities of γ_i, and the β_j's are loops. Finally, choose disk extensions $v_i : \mathbb{D} \to B_t(x)$ and $w_j : \mathbb{D} \to B_t(x)$ of $u|_{\gamma_i \# \overline{\alpha_i}}$ and $u|_{\beta_j}$, respectively.

Since $B_t(x)$ is contractible, we may take a primitive λ of ω on it. Then, by

Stokes' theorem,

$$
\begin{aligned}
a(t) &= \int_{\Sigma_t} u^*\omega = \int_{\partial\Sigma_t} u^*\lambda \\
&= \sum_i \int_{\gamma_i \# \overline{\alpha_i}} u^*\lambda + \sum_j \int_{\beta_j} u^*\lambda \\
&= \sum_i \int_{\partial\mathbb{D}} v_i^*\lambda + \sum_j \int_{\partial\mathbb{D}} w_j^*\lambda \\
&= \sum_i \int_{\mathbb{D}} v_i^*\omega + \sum_j \int_{\mathbb{D}} w_j^*\omega.
\end{aligned}
$$

Hence, by Lemma 2.1,

$$
\begin{aligned}
a(t) &= \sum_i a(\gamma_i) + \sum_j a(\beta_j) \\
&\leq \sum_i c\ell(\gamma_i)^2 + \sum_j c'\ell(\beta_j)^2 \\
&\leq c\ell(t)^2.
\end{aligned}
$$

Take $f := \rho \circ u$, where ρ is the distance from x in M. Fix $t \in \Omega$. Then, in a neighborhood of $\partial\Sigma_t$, u^*g_J is a metric, and f is its distance function from $u^{-1}(x)$. In particular, $|\operatorname{grad} f|_{u^*g_J} \equiv 1$. Therefore, by the coarea formula (see for example [9]), for s near enough t, we have that

$$
\begin{aligned}
a(t) - a(s) &= \int_{\{s \leq f \leq t\}} |\operatorname{grad} f|_{u^*g_J} da_{u^*g_J} \\
&= \int_s^t \int_{\{f=\tau\}} d\ell_{u^*g_J} d\tau \qquad\qquad (2.2) \\
&= \int_s^t \ell(\tau) d\tau.
\end{aligned}
$$

In particular, a is absolutely continuous on $[0, r]$, differentiable on Ω, and $a'(t) = \ell(t)$ for all $t \in \Omega$.

Therefore, for all such t, we have that

$$
\left(\sqrt{a(t)}\right)' = \frac{a'(t)}{2\sqrt{a(t)}} = \frac{\ell(t)}{2\sqrt{a(t)}} \geq \frac{1}{2\sqrt{c}}
$$

by Lemma 2.1. Since Ω has full measure and a is absolutely continuous, we can integrate the lower bound to get $\omega(u) = a(r) \geq \frac{1}{4c}r^2$. □

Remark 2.3. Both Lemma 2.1 and Proposition 2.1 work for a larger class of almost complex structures J and of metrics g — not necessarily equal to g_J — respecting the hypotheses of Lemma 2.2. Indeed, suppose that there exist constants $C_1, C_2 > 0$ such that $\omega(X, Y) \leq C_1|X||Y|$ and $|X|^2 \leq C_2\omega(X, JX)$ for all X and Y. In that

case, let a denote the area with respect to ω as before, and let A denote the area with respect to g. Then, Lemma 2.1 gives the inequality $a(\gamma) \leq C_1 c' \ell(\gamma)$ at *(i)* and the inequality $a(\gamma) \leq C_1 c \ell(\gamma)$ at *(ii)*. Furthermore, Equation (2.2) now applies to A, and thus $A'(t) = \ell(t)$. Therefore, Proposition 2.1 becomes

$$\omega(u) \geq \frac{1}{C_2} A(r) \geq \frac{1}{4 C_1 C_2 c} r^2.$$

2.3. *Proof of Theorem 2.1*

We are now ready to give the proof of the main result. Remember that we have fixed a symplectic manifold (M, ω) and an ω-compatible almost complex structure J such that (M, J) is either closed or convex at infinity. Likewise, we have fixed families of closed connected Lagrangian submanifolds $\mathscr{F}$ and $\mathscr{F}'$ such that $(\overline{\cup_{F \in \mathscr{F}} F}) \cap (\overline{\cup_{F' \in \mathscr{F}'} F'})$ is discrete. We also assume from now on that all Lagrangian submanifolds are in some fixed collection $\mathscr{L}^{\star}_{\Lambda, \epsilon}(M)$ (as defined in Subsec. 1.1). In what follows, $\hat{d}^{\mathscr{F}, \mathscr{F}'}$ denotes a Chekanov-type metric (as defined in Subsec. 2.1) on that collection.

For subsets $A, B \subseteq M$, we take

$$s(A, B) := \sup_{x \in A} d_M(x, B) := \sup_{x \in A} \inf_{y \in B} d_M(x, y).$$

Therefore, the Hausdorff metric is given by $\delta_H(A, B) = \max\{s(A, B), s(B, A)\}$.

Take $L, L' \in \mathscr{L}^{\star}_{\Lambda, \epsilon}(M)$ and $\delta > 0$. Let $F_1, \ldots, F_k$ be the Lagrangian submanifolds in $\mathscr{F}$ given by the definition of a Chekanov-type pseudometric. Take also C^0- and Hofer-small Hamiltonian perturbations $\widetilde{L}$, $\widetilde{L}'$, and $\widetilde{F}_1, \ldots, \widetilde{F}_k$ of these manifolds making them pairwise transverse. Then, for any $x \in \widetilde{L} \cup \widetilde{L}'$, there exists a nonconstant J-holomorphic polygon $u : S_r \to M$ with the following properties:

(i) it has boundary along $\widetilde{L}$, $\widetilde{L}'$, and $\widetilde{F}_1, \ldots, \widetilde{F}_k$;
(ii) it passes through x;
(iii) $\omega(u) \leq d^{\mathscr{F}}(L, L') + \delta$.

Assuming that the perturbations of L and L' are also C^2-small, we get that $\widetilde{L}, \widetilde{L}' \in \mathscr{L}^{\star}_{\widetilde{\Lambda}, \widetilde{\epsilon}}(M)$ for some $\widetilde{\Lambda} \geq \Lambda$ and $\widetilde{\epsilon} \leq \epsilon$.

Take $x \in \widetilde{L} - (\widetilde{L}' \cup \widetilde{F}_1 \cup \cdots \cup \widetilde{F}_k)$ and

$$\tilde{\delta} := \epsilon \min \left\{ 1, \frac{1}{2} r_0, \frac{1}{2} i_0(K_0, r_0, \tilde{\Lambda}) \right\},$$

where i_0 is the constant appearing in Lemma 2.2. By Sard's theorem, there is an open dense subset of $(0, \min\{\tilde{\delta}, d_M(x, \widetilde{L}' \cup \widetilde{F}_1 \cdots \cup \widetilde{F}_{|r|-2})\})$ such that, for all ρ in this subset, $\Sigma := u^{-1}(B_\rho(x))$ is a smooth submanifold of S_r with boundary with corners. In particular, $u|_\Sigma$ respects the hypotheses of Proposition 2.1. We thus get

$$d^{\mathscr{F}}(L, L') + \delta \geq \omega(u) \geq \omega(u|_\Sigma) \geq \tilde{C} \rho^2,$$

where $\tilde{C} = C(K_0, r_0, \tilde{\Lambda}, \tilde{\epsilon})$ is the constant appearing in Proposition 2.1. Since this holds for all $\delta > 0$ and for all ρ in a dense subset, we get

$$d^{\mathscr{F}}(L, L') \geq \tilde{C} \min \left\{ \tilde{\delta}, d_M(x, \widetilde{L}' \cup \widetilde{F}_1 \cdots \cup \widetilde{F}_{|r|-2}) \right\}^2.$$

In particular, whenever $d^{\mathscr{F}}(L, L') < \tilde{C}\tilde{\delta}^2$, we get

$$d^{\mathscr{F}}(L, L') \geq \tilde{C} s(\widetilde{L}, \widetilde{L}' \cup \widetilde{F}_1 \cdots \cup \widetilde{F}_{|r|-2})^2 \tag{2.3}$$

by taking the supremum over all x's.

We now must get rid of the Hamiltonian perturbations on the right-hand side of (2.3). In order to do so, choose sequences of generic C^2- and Hofer-small Hamiltonian diffeomorphisms $\{\phi_n\}_{n \geq 1}$ and $\{\phi_n^0\}_{n \geq 1}$, which converges to the identity in the C^2 sense. Likewise, for each $i \in \{1, \ldots, k\}$, choose a sequence of generic C^0- and Hofer-small Hamiltonian diffeomorphisms $\{\phi_n^i\}_{n \geq 1}$ which converges to the identity in the C^0 sense. For each n, there is a J-holomorphic polygon $u_n : S_{r_n} \to M$ as above.

By Remark 2.2, as n tends to infinity, the corresponding constants $\tilde{\delta}$ and $\tilde{C}$ converge to

$$\delta_0 := \min \left\{ \epsilon, \frac{\epsilon}{2} r_0, \frac{\epsilon}{2} i_0(K_0, r_0, \Lambda), \frac{\pi}{4\sqrt{K_0}} \right\}$$

and $C = C(K_0, r_0, \Lambda, \epsilon)$, respectively. Furthermore, for any $n \geq 1$, $x \in L$, and $y \in L' \cup F_1 \cdots \cup F_k$, we have that

$$d_M(x, y) \leq d_M(x, \phi_n(x)) + d_M(\phi_n(x), \phi_n^i(y)) + d_M(\phi_n^i(y), y)$$
$$\leq d_M(\phi_n(x), \phi_n^i(y)) + d_{C^0}(\mathbb{1}, \phi_n) + \max_i d_{C^0}(\mathbb{1}, \phi_n^i),$$

for all $i \in \{0, 1, \ldots, k\}$. Therefore, by taking the infimum over all y and then, the supremum over all x, we have that

$$s(\phi_n(L), \phi_n^0(L') \cup \phi_n^1(F_1) \cup \cdots \cup \phi_n^k(F_k)) \geq s(L, L' \cup F_1 \cup \cdots \cup F_k)$$
$$- d_{C^0}(\mathbb{1}, \phi_n)$$
$$- \max_i d_{C^0}(\mathbb{1}, \phi_n^i).$$

We can thus finally put this back into (2.3), and take the limit $n \to \infty$ to get

$$d^{\mathscr{F}}(L, L') \geq C s(L, L' \cup F_1 \cdots \cup F_k)^2 \geq C s(L, L' \cup (\cup_{F \in \mathscr{F}} F))^2, \tag{2.4}$$

whenever $d^{\mathscr{F}}(L, L') < C\delta_0^2$. In other words, Inequality (2.3) holds without any perturbation. One gets similarly the inequalities

$$d^{\mathscr{F}}(L, L') \geq C s(L', L \cup (\cup_{F \in \mathscr{F}} F))^2$$
$$d^{\mathscr{F}'}(L, L') \geq C s(L, L' \cup (\cup_{F' \in \mathscr{F}'} F'))^2 \tag{2.5}$$
$$d^{\mathscr{F}'}(L, L') \geq C s(L', L \cup (\cup_{F' \in \mathscr{F}'} F'))^2.$$

We must now turn the inequalities in (2.4) and (2.5) into an inequality in terms of $\delta_H(L, L')$.

Since $(\overline{\cup F}) \cap (\overline{\cup F'})$ is discrete by the definition of a Chekanov-type metric, there exists a constant $\eta' > 0$ such that $B_{\eta'}(x) \cap B_{\eta'}(y) = \emptyset$ for all $x \neq y \in (\overline{\cup F}) \cap (\overline{\cup F'})$. Furthermore, there exists a constant $\eta'' = \eta''(K_0, r_0, \Lambda) > 0$ such that any closed manifold L which is contained in a metric ball $B_{\eta''}(x)$, for some $x \in M$, must have $||B_L|| > \Lambda$ — we refer the reader to Corollary 3.1 in the next section for a precise estimate of η''. We set $\eta := \min\{\eta', \eta''\}$.
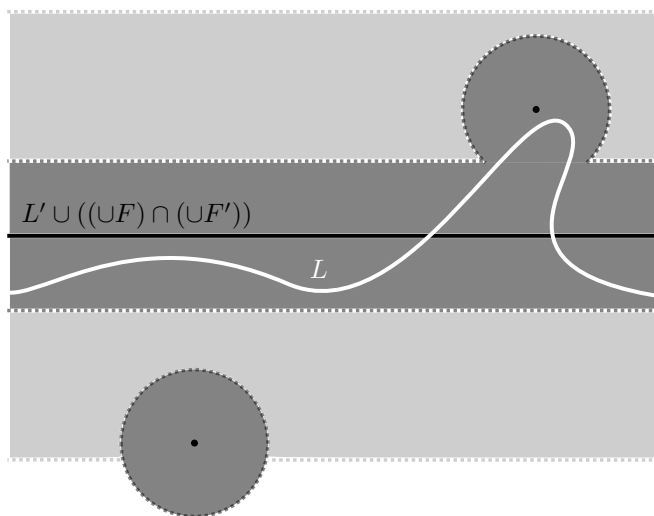

Fig. 2.   The 3σ-neighborhood of L' (in light grey) contains the connected component containing L' of the σ-neighborhood of $L' \cup ((\overline{\cup F}) \cap (\overline{\cup F'}))$ (in dark grey).

Suppose now that $\hat{d}^{\mathscr{F},\mathscr{F}'}(L,L') < \min\{C\delta_0^2, C\eta^2\} =: R$. By the inequalities in (2.4) and (2.5), we have that

$$\sigma := \max\{s(L, L' \cup (\cup F)), s(L, L' \cup (\cup F'))\} < \eta.$$

But, by definition of s, L must be in the σ-neighborhood of $L' \cup (\overline{\cup F})$ and $L' \cup (\overline{\cup F'})$, and thus of $L' \cup ((\overline{\cup F}) \cap (\overline{\cup F'}))$. However, this neighborhood is composed of a disjoint union of an open neighborhood of L' and metric balls of radius $< \eta''$. By the choice of η'', L must then be in the component containing L'. However, this component is itself contained in the 3σ-neighborhood of L'. Therefore, $3\sigma \geq s(L, L')$. By taking the maximum of the inequality in (2.3) and the second inequality of (2.5), we get

$$\hat{d}^{\mathscr{F},\mathscr{F}'}(L,L') \geq C\sigma^2 \geq \frac{C}{9}s(L,L')^2.$$

Doing the same thing for the first and third inequalities of (2.5) finishes the proof. The C appearing in the statement of Theorem 2.1 is thus equal to $\frac{1}{36c}$, where c is the constant of Lemma 2.1.

Remark 2.4.

- If one is only interested in the statement of Theorem A and is willing to restrict oneself to the case when M is compact, then the standing hypothesis that $(\overline{\cup F}) \cap (\overline{\cup F'})$ is discrete can be loosened up. More precisely, we can suppose that $(\overline{\cup F}) \cap (\overline{\cup F'})$ is only totally disconnected.

  In that case, a sequence $\{L_n\}_{n \geq 1}$ must have a converging subsequence; this is because the collection of closed subsets of M with the Hausdorff metric is compact. Denote by E its limit set. Since the inequalities in (2.4) and (2.5) still stands — with L replaced by L_n and L' by L_0 — the limit E must lie in $L_0 \cup ((\overline{\cup F}) \cap (\overline{\cup F'}))$. However, a sequence of connected subsets must converge to a connected one (see [36] for instance). But, by the argument above, it cannot converge to a point. Therefore, we must have $E \subseteq L_0$. Using an analogous argument for the first and last inequality in (2.5), we get $L_0 \subseteq E \cup ((\overline{\cup F}) \cap (\overline{\cup F'}))$. Thus, $L_0 \subseteq E$ by the same connectivity argument. Therefore, any converging subsequence of $\{L_n\}_{n \geq 1}$ converges to $E = L_0$, which means that $\{L_n\}_{n \geq 1}$ itself converges to L_0 in the Hausdorff metric.

- Likewise, to prove Theorem A, there is no need for the weak-exactness hypothesis on $\mathscr{L}^{L_0}(M)$ in the spectral case. Indeed, for n large, we will have $\gamma(L_0, L_n) < \hbar(M, L_0, J)$. Therefore, we will still have a J-holomorphic strip between Hamiltonian deformations of L_0 and L_n. The rest of the proof then follows as above. The same is true of γ_{ext}.

3. The two-dimensional case

We now explain how to prove the second part of Theorem A. More precisely, we will prove the following.

Theorem 3.1. *Suppose that* $\dim M = 2$. *For any Chekanov-type metric* $\hat{d}^{\mathscr{F},\mathscr{F}'}$ *on* $\mathscr{L}_\Lambda^\star(M)$, *there exist constants* $R' = R'(K_0, r_0, \Lambda, (\overline{\cup F}) \cap (\overline{\cup F'})) > 0$ *and* $C' = C'(K_0, r_0) > 0$ *such that whenever* $\hat{d}^{\mathscr{F},\mathscr{F}'}(L, L') < R'$, *then*

$$\hat{d}^{\mathscr{F},\mathscr{F}'}(L, L') \geq C' \delta_H(L, L')^2.$$

As noted in the introduction, the key to getting rid of the dependency of both R and C on ϵ is to use the two-dimensionality of M in order to make a better choice of a metric ball in M before applying the following version of the monotonicity lemma.

Proposition 3.1 (Monotonocity lemma, absolute version). *Let* (M, ω, J, g_J) *be a symplectic manifold equipped with a compatible almost complex structure and the corresponding metric. Let* Σ *be a compact Riemann surface with boundary* $\partial \Sigma$. *Consider a nonconstant* J-*holomorphic curve* $u : (\Sigma, \partial \Sigma) \to (B(x, r), \partial B(x, r))$, *where* $x \in M$ *and* $r \leq \frac{1}{2} r_0$. *Suppose that* $x \in u(\Sigma)$. *Then,*

$$\omega(u) \geq C' r^2,$$

where $C' = \frac{1}{4c'}$, *with* c' *the constant of Lemma 2.1.*

Proof. We make use of the same notation as in the proof of Proposition 2.1. Since u does not have boundary components along some Lagrangian submanifold L, the boundary $\partial\Sigma_t$ is only composed of disjoint loops β_j. Therefore, we now get

$$a(t) = \sum_j a(\beta_j) \le \sum_j c'\ell(\beta_j)^2 = c'\ell(t)^2.$$

The rest of the proof than follows as in Proposition 2.1. $\square$

Let L and L' be Lagrangian submanifolds of M, i.e. closed connected curves on M, and let $\widetilde{L}$ and $\widetilde{L}'$ be sufficiently small generic Hamiltonian deformations of them. Take $x \in \widetilde{L} \cup \widetilde{L}'$. Let $u : S_r \to M$ be the J-holomorphic polygon with boundary along L and L' — and potentially some other Lagrangian submanifolds — passing through x given by the definition of a Chekanov-type metric. If u is a sphere, i.e. $|r| = 0$, we can just apply Proposition 3.1 directly on it and the rest of the proof is as before. If u is not a sphere, i.e. $|r| \ge 1$, we can still use Proposition 3.1 if we find an open disk which is entirely contained in $u(\text{int } S_r)$. To prove Theorem 3.1, it thus suffices to prove the following purely geometric result.

Theorem 3.2. *Let M be a complete surface with Gaussian curvature $|K| \le K_0$ and injectivity radius $r_{\text{inj}}(M) \ge r_0 > 0$. Let $L, K_1, \ldots, K_k$ be connected curves on M such that L has geodesic curvature uniformly bounded by Λ. Let $x \in L - \cup_i K_i$. Suppose that there exists a smooth map $u : S_r \to M$, $1 \le |r| \le k+1$, with boundaries along $L, K_1, \ldots, K_k$ passing through x (in the notation of Subsec. 2.1). Then, there exists a constant*

$$\rho_0 = \rho_0(K_0, r_0, \Lambda, d_M(x, \cup_i K_i)) > 0$$

such that $u(\text{int } S_r)$ contains a metric ball of radius ρ_0.

We allow k to be zero, in which case it is understood that $S_r = \mathbb{D}$, $\cup_i K_i = \emptyset$, and $d_M(x, \cup_i K_i) = \infty$ for any $x \in L$. Note that the curves $L, K_1, \ldots, K_k$ in Theorem 3.2 need not be closed. This is because everything is happening in $u(S_r)$, and thus the global properties of the curves do not matter. In particular, we could very much take $|s| = k + 1$, $L = u(C_0)$, and $K_i = u(C_i)$ for $i > 0$.

Although it was developed independently, our proof of Theorem 3.2 uses a similar approach to a recent proof by Petrunin and Zamora Barrera of the so-called Moon in a puddle theorem [31]. We recover their result by taking M to be the Euclidean plane and k to be zero. Indeed, one can check that, in this case, $\rho_0 = \frac{1}{\Lambda}$.

Remark 3.1. Much like the constants C, C' and i_0, the constant ρ_0 depends continuously on K_0, r_0, Λ, and $d_M(x, \cup_i K_i)$.

3.1. *Finding a good disk*

The proof of Theorem 3.2 relies mostly on the following technical lemma, whose proof we will delay until the next subsection.

Lemma 3.1. *Let M be a complete Riemannian manifold of dimension $n \ge 2$ with*

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$|K| \leq K_0$ *and* $r_{\mathrm{inj}}(M) \geq r_0 > 0$. *There exist constants* $\rho_1 = \rho_1(K_0, r_0) > 0$ *and* $\alpha = \alpha(K_0, r_0) > 0$ *with the following property. Let* $\Gamma : [0, \ell] \to M$ *be a unit-speed curve with image in* $B_\rho(x)$ *for some* $0 < \rho \leq \rho_1$ *and some* $x \in M$. *Consider the map*

$$d \colon [0, \ell] \longrightarrow \mathbb{R}_{\geq 0}$$
$$s \longmapsto d_M(x, \Gamma(s)).$$

Suppose that d *has a local maximum at* $s_0 \in (0, \ell)$. *Then,*

$$\left| \frac{D}{ds} \dot{\Gamma}(s_0) \right| \geq \frac{\alpha}{\rho}.$$

By taking $M = \mathbb{R}^2$ and the limit $K_0 \to 0$, we recover the classical fact that, on a loop contained in a disk of radius $\rho > 0$, there is a point were its curvature is at least $\frac{1}{\rho}$.

Before going in the proof of Theorem 3.2, we note that Lemma 3.1 gives a precise bound on the smallest metric ball that a submanifold L with second fundamental form $||B_L|| \leq \Lambda$ can be contained in.

Corollary 3.1. *Let* M, K_0, ρ_0, *and* α *be as above. Let* L *be a closed submanifold of* M *contained in a metric ball* $B_\rho(x)$ *for some* $0 < \rho \leq \rho_0$ *and some* $x \in M$. *Then, its second fundamental form respects*

$$||B_L|| \geq \frac{\alpha}{\rho}.$$

Proof. Since L is closed, there is a point $y \in L$ such that the map $x' \mapsto d_M(x, x')$, seen as a map $L \to \mathbb{R}$, achieve its maximum at y. Let $\gamma : (-\epsilon, \epsilon) \to L$ be a unit-speed geodesic of L such that $\gamma(0) = y$. By Lemma 3.1, we have that

$$\left| B_L \left(\dot{\gamma}(0), \dot{\gamma}(0), \frac{\frac{D}{ds}\dot{\gamma}}{|\frac{D}{ds}\dot{\gamma}|}(0) \right) \right| = \left| \frac{D}{ds} \dot{\gamma}(0) \right| \geq \frac{\alpha}{\rho}. \qquad \square$$

We now begin the proof of Theorem 3.2 in the case $k = 0$. This implies in particular that $S_r = \mathbb{D}$ and that $L = u(\partial \mathbb{D})$ is a contractible loop. We will explain how to adapt the proof to the case $k > 0$ afterwards. Note that $u(\mathbb{D})$ must cover a contractible region bounded by L. Without loss of generality, we can suppose that $u(\text{int } \mathbb{D}))$ is entirely contained in the interior of that region.

Let us first reduce to the case where M is simply connected. Suppose that it is not. We then consider its universal cover $\pi : \widetilde{M} \to M$. Note that $\widetilde{M}$ comes naturally equipped with a Riemannian metric $\pi^* g$, which turns π into a local isometry. Therefore, we have that the Gaussian curvature of $\widetilde{M}$ respects $|\tilde{K}| \leq K_0$ and that $r_{\mathrm{inj}}(\widetilde{M}) \geq r_{\mathrm{inj}}(M) \geq r_0$. The inequality between the injectivity radii follows from the classical result of Klingenberg [24] that

$$r_{\mathrm{inj}}(M) = \min \left\{ r_{\mathrm{conj}}(M), \frac{1}{2} \ell(M) \right\}. \tag{3.1}$$

Here, $r_{\text{conj}}(M)$ is the length of the shortest geodesic segment $\gamma : [0, T] \to M$ such that there exists a normal Jacobi field J along γ with $J(0) = J(T) = 0$. Likewise, $\ell(M)$ is the length of the shortest *immersed* geodesic loop on M. Therefore, the metric bounds on M are also respected by $\widetilde{M}$. Likewise, since $\mathbb{D}$ is contractible, u admits a lift $\tilde{u} : \mathbb{D} \to \widetilde{M}$.

It thus indeed suffices to prove Theorem 3.2 on the universal cover: if $B_{\rho_0}(\tilde{x})$ is the metric ball in $\widetilde{M}$ given by the theorem, then $B_{\rho_0}(\pi(\tilde{x})) = \pi(B_{\rho_0}(\tilde{x}))$ will be the sought-after ball in M. Note that this is indeed a topological ball as we may take $\rho_0 \leq r_0$. Therefore, for the rest of the proof, we will assume that M is simply connected.

We now fix an injective unit-speed parametrization $\Gamma : [0, \ell] \to M$ of $L = u(\partial \mathbb{D})$. We may suppose that $\Gamma(0) = \Gamma(\ell) \neq x$.

The proof has three main steps.

(i) We define a notion of an "osculating disk" $D_\rho(s)$ of the curve Γ at s. This disk is a closed metric ball of M which has the property that $D_\rho(s) \cap \Gamma([s - \epsilon, s + \epsilon]) \subseteq \partial D_\rho(s)$ for some $\epsilon > 0$.

(ii) We find $t_0, s_0 \in [0, \ell]$ with the following property: if there is some $s \in [t_0, s_0]$ such that $\text{int}(D_\rho(s)) \not\subseteq u(\text{int }\mathbb{D})$ (i.e. such that $D_\rho(s) \cap u(\partial \mathbb{D}) \not\subseteq \partial D_\rho(s)$), then there also exists $t \in [t_0, s_0]$ such that $\Gamma(t) \in \text{int}(D_\rho(s))$.

(iii) We suppose that $D_\rho(s) \cap u(\partial \mathbb{D}) \not\subseteq \partial D_\rho(s)$ for all $s \in [s_0, t_0]$, and we get a contradiction.

Denote by $r_{\text{conv}}(M)$ the convexity radius of M, i.e. the largest $\rho > 0$ such that, for all $x \in M$ and all $y, y' \in B_\rho(x)$, there exists a unique minimizing geodesic from y to y' in $B_\rho(x)$. It is a classical result from Berger [4] that

$$r_{\text{conv}}(M) \geq \frac{1}{2} \min\left\{ r_{\text{inj}}(M), \frac{\pi}{\sqrt{K_0}} \right\}.$$

By taking a smaller ρ_0 if necessary, we may assume that $\rho_0 \leq \min\{\frac{r_0}{2}, \frac{\pi}{2\sqrt{K_0}}, \rho_1\}$. Therefore, $\rho \leq \rho_0$ implies that $\rho_0 \leq r_{\text{conv}}(M)$ and that ρ respects Lemma 3.1.

Remark 3.2. In fact, a more recent result of Dibble [13] gives

$$r_{\text{conv}}(M) = \min\left\{ r_{\text{foc}}(M), \frac{1}{4}\ell(M) \right\}, \tag{3.2}$$

where $r_{\text{foc}}(M)$ is the length of the shortest geodesic segment $\gamma : [0, T] \to M$ such that there exists a normal Jacobi field J along γ with $J(0) = \langle J', J \rangle(T) = 0$. This can be used to give a better estimate on the optimal ρ_0 for a given (M, g_J).

We thus begin with the definition of $D_\rho(s)$. Let $s \in (0, \ell)$ and $\rho \in (0, \rho_0)$. We define $D_\rho(s)$ to be the closed metric ball $\overline{B_\rho(\gamma_s(\rho))}$. Here, $\gamma_s(t) := \exp_{\Gamma(s)}(tN(s))$ for $t \in [0, r_{\text{inj}}(M)]$, and N is the unit-length vector field along Γ which is orthogonal to $\dot{\Gamma}$ and pointing toward the interior of the topological disk $\overline{u(\mathbb{D})}$.

Lemma 3.2. *Let $s \in (0, \ell)$ and $\rho \in (0, \min\{\frac{\alpha}{\Lambda}, \rho_1\})$, where ρ_1 and α are the con-*

stant appearing in Lemma 3.1. There exists $\epsilon > 0$ such that

$$D_\rho(s) \cap \Gamma([s - \epsilon, s + \epsilon]) \subseteq \partial D_\rho(s).$$

We also put off the proof of the lemma until the next subsection. In order to use Lemma 3.2 for the rest of the proof, we thus add the condition that $\rho_0 \leq \frac{\alpha}{\Lambda}$ and take $\rho \in (0, \rho_0)$.

We now make our choice of t_0 and s_0. Let $t_0 \in (0, \ell)$ be such that $\Gamma(t_0) = x$. If we have that

$$D_\rho(t_0) \cap \Gamma([0, \ell]) \subseteq \partial D_\rho(t_0) \tag{3.3}$$

then we have obviously proven the theorem.

Suppose therefore that $D_\rho(t_0) \cap \Gamma([0, \ell]) \not\subseteq \partial D_\rho(t_0)$. Fix $s_0 \in (0, \ell)$ such that

$$d_M(\Gamma(t_0), \Gamma(s_0)) = \min\{d_M(\Gamma(t_0), \Gamma(s)) \mid s \in [0, \ell], \Gamma(s) \in D_\rho(t_0)\}. \tag{3.4}$$

By shifting the parametrization Γ, we may assume that $s_0 \in (0, \ell)$. We must have that $\Gamma(s_0) \in \operatorname{int} D_\rho(t_0)$ and, by Lemma 3.2, $s_0 \neq t_0$. Without loss of generality, we may assume that $s_0 > t_0$.

Denote by $[\Gamma(t_0), \Gamma(s_0)]$ the unique minimizing geodesic segment in $D_\rho(t_0)$ from $\Gamma(t_0)$ to $\Gamma(s_0)$. Note that $[\Gamma(t_0), \Gamma(s_0)] \cap \Gamma([0, \ell]) = \{\Gamma(t_0), \Gamma(s_0)\}$ by minimality of the geodesic and the definition of s_0. Then, $\Gamma([t_0, s_0]) \cup [\Gamma(t_0), \Gamma(s_0)]$ is a continuously embedded loop in M. Therefore, since M is simply connected, it bounds a topological disk Δ_0 and divides M in two parts. Note that Δ_0 is a contractible subset of the topological disk $\overline{u(\mathbb{D})}$ by embeddedness of the geodesic segment $[\Gamma(t_0), \Gamma(s_0)]$.

We now get to the proof that $[t_0, s_0]$ has the desired property, i.e. the proof of the second step. Let $s \in [t_0, s_0]$ be such that $\operatorname{int}(D_\rho(s))$ is not contained in $u(\operatorname{int} \mathbb{D})$ and take

$$p \in \operatorname{int}(D_\rho(s)) \cap u(\partial \mathbb{D}) \subseteq \operatorname{int}(D_\rho(s)) - u(\operatorname{int} \mathbb{D}). \tag{3.5}$$

Since the curve $\partial \Delta_0$ divides M in two, the minimizing geodesic from $\Gamma(s)$ to p must intersect $\partial \Delta_0 = \Gamma([t_0, s_0]) \cup [\Gamma(t_0), \Gamma(s_0)]$ at some point y. Indeed, this geodesic exists, is unique, and is contained in $D_\rho(s)$, since $\rho < r_{\operatorname{conv}}(M) \leq r_{\operatorname{inj}}(M)$. Furthermore, since $p \in \operatorname{int}(D_\rho(s))$, it only intersects $\partial D_\rho(s)$ at $\Gamma(s)$. But remember that γ_s was chosen precisely so that $\dot{\gamma}_s(0)$ points towards the interior of $\overline{u(\mathbb{D})}$, and thus towards $\operatorname{int}(\Delta_0)$. In particular, for a small enough neighborhood U of $\Gamma(s)$, $U \cap \operatorname{int}(D_\rho(s))$ in contained in $\operatorname{int}(\Delta_0)$. This in turn implies that the segment $[\Gamma(s), p] \cap U$ must be inside $\operatorname{int}(\Delta_0)$. Since $p \notin u(\operatorname{int} \mathbb{D})$ and $\operatorname{int}(\Delta_0) \subseteq u(\operatorname{int} \mathbb{D})$, the geodesic must intersect $\partial \Delta_0$ at some point y as desired (cf. Figure 3). Note that we have that $y \in \operatorname{int}(D_\rho(s))$ and that $d_M(p, y) < d_M(p, \Gamma(s))$.

If $y \in \Gamma([t_0, s_0])$, then we have $y = \Gamma(t) \in D_\rho(s)$ for some $t \in [s_0, t_0]$ as desired. Suppose therefore that $y \in [\Gamma(t_0), \Gamma(s_0)]$. Clearly, it suffices to prove that either $\Gamma(t_0)$ or $\Gamma(s_0)$ is in $D_\rho(s)$ to prove the second step. We thus suppose that it is not the case and get a contradiction. In this case, $[\Gamma(t_0), \Gamma(s_0)]$ divides $D_\rho(s)$ in exactly two parts, both with nonempty interior. Indeed, $y \in [\Gamma(t_0), \Gamma(s_0)] \cap \operatorname{int}(D_\rho(s))$.

Moreover, if $D_\rho(s) - [\Gamma(t_0), \Gamma(s_0)]$ had more than two components, we would get a contradiction. To see this, note that it is equivalent to $[\Gamma(t_0), \Gamma(s_0)] \cap D_\rho(s)$ having more than one component. However, if $q, q' \in [\Gamma(t_0), \Gamma(s_0)] \cap D_\rho(s)$ were to belong to different components, there would still be a minimizing geodesic segment $[q, q']$ in $D_\rho(s)$. But then

$$\ell([\Gamma(t_0), q] \cup [q, q'] \cup [q', \Gamma(s_0)]) < \ell([\Gamma(t_0), \Gamma(s_0)]) = d_M(\Gamma(t_0), \Gamma(s_0)),$$

which is of course impossible. Furthermore, since

$$d_M(\Gamma(t_0), \Gamma(s_0)) \leq d_M(\Gamma(t_0), \gamma_{t_0}(\rho)) + d_M(\gamma_{t_0}(\rho), \Gamma(s_0)) < 2\rho,$$

the segment $[\Gamma(t_0), \Gamma(s_0)]$ cannot intersect $\gamma_s(\rho)$. Therefore, $\gamma_s(\rho)$ must lie in the interior of one of the component of $D_\rho(s) - [\Gamma(t_0), \Gamma(s_0)]$.

Suppose at first that $\gamma_s(\rho)$ lies in the same component as $\Gamma(s)$. We begin by showing that this implies that $p \in D_\rho(t_0)$. Suppose that it does not. Denote by A the intersection of $\partial D_\rho(s)$ and the component of $D_\rho(s) - [\Gamma(t_0), \Gamma(s_0)]$ not containing $\gamma_s(\rho)$. By minimality of $[\Gamma(t_0), \Gamma(s_0)]$, A is an embedded arc. Furthermore, since $\partial A \subseteq [\Gamma(t_0), \Gamma(s_0)]$, we have that $\partial A \subseteq D_\rho(t_0)$. However, since $p \notin D_\rho(t_0)$, A is not entirely contained in $D_\rho(t_0)$. In particular, the function $q \mapsto d_M(\gamma_{t_0}(\rho), q)$, $q \in A$, achieves its maximum at a point $q \in A - \partial A$. This implies that A must be normal to the geodesic segment $[\gamma_{t_0}(\rho), q]$ at q — to see this, one can consider the energy functional along the minimal geodesics from $\gamma_{t_0}(\rho)$ to A and use the fact that it has a critical point at q. But by construction, A is also normal to the geodesic segment $[\gamma_s(\rho), q]$. Since $\dim M = 2$ and $d(\gamma_{t_0}(\rho), q) > \rho = d(\gamma_s(\rho), q)$, this means that $[\gamma_s(\rho), q] \subsetneq [\gamma_{t_0}(\rho), q]$.

Denote by q' the intersection of the geodesic segments $[\gamma_{t_0}(\rho), q]$ and $[\Gamma(t_0), \Gamma(s_0)]$. Suppose at first that $d_M(\Gamma(t_0), q') \leq \rho$. Then, $[\gamma_{t_0}(\rho), q'] \subseteq \overline{B_\rho(\Gamma(t_0))}$, since $\rho < r_{\mathrm{conv}}(M)$. However, $\gamma_s(\rho) \in [\gamma_{t_0}(\rho), q']$, and thus $d_M(\Gamma(t_0), \gamma_s(\rho)) \leq \rho$. This leads to a contradiction since we have supposed that $\Gamma(t_0) \notin D_\rho(s)$. Therefore, it must be so that $d_M(\Gamma(t_0), q') > \rho$. In particular,

$$d_M(\Gamma(s_0), q') = d_M(\Gamma(s_0), \Gamma(t_0)) - d_M(q', \Gamma(t_0)) < 2\rho - \rho = \rho$$

However, by definition of s_0, $d_M(\gamma_{t_0}(\rho), \Gamma(s_0)) < \rho$. Therefore, we have that $[\gamma_{t_0}(\rho), q'] \subseteq B_\rho(\Gamma(s_0))$ similarly as before. Thus, $d_M(\gamma_s(\rho), \Gamma(s_0)) < \rho$ and $\Gamma(s_0) \in D_\rho(s)$, which is again a contradiction. Therefore, it must be so that $p \in D_\rho(t_0)$.

We now finally prove that $\gamma_s(\rho)$ cannot lie in the same component of $D_\rho(s) - [\Gamma(t_0), \Gamma(s_0)]$ as $\Gamma(s)$. Recall that y is the intersection of the geodesic segments $[\Gamma(t_0), \Gamma(s_0)]$ and $[\gamma_s(\rho), p]$. Denote by z the intersection of $[y, \Gamma(s_0)]$ and $\partial D_\rho(s)$ — once again, minimality of the geodesic ensures that this is indeed a point. By

definition of y, we have that

$$
\begin{aligned}
d_M(y,p) &= d_M(\gamma_s(\rho),p) - d_M(\gamma_s(\rho),y) \\
&\leq \rho - d_M(\gamma_s(\rho),y) \\
&\leq d_M(\gamma_s(\rho),y) + d_M(y,z) - d_M(\gamma_s(\rho),y) \\
&< d_M(y,\Gamma(s_0)).
\end{aligned}
$$

Indeed, $[\gamma_s(\rho),y] \cup [y,z]$ is a path from the center of a ball of radius ρ to its boundary, and thus must be of length at least ρ.
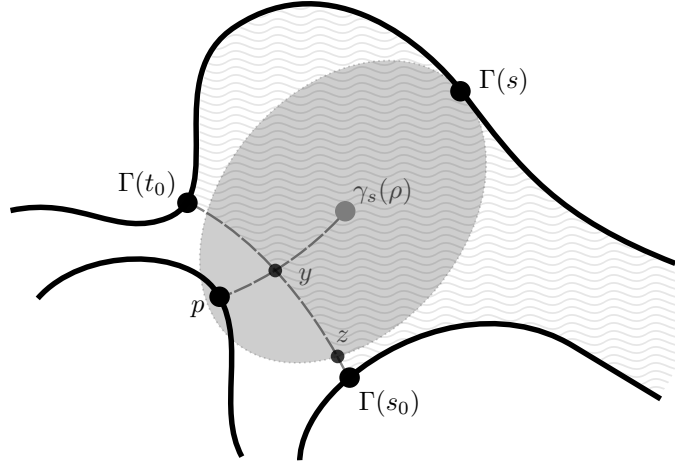

Fig. 3.   Some relevant geodesic segments (hatched lines) and points in the disk $D_\rho(s)$ (in light grey). The wavy region represents the interior of the topological disk Δ_0.

Therefore, we get

$$
\begin{aligned}
d_M(\Gamma(t_0),p) &\leq d_M(\Gamma(t_0),y) + d_M(y,p) \\
&< d_M(\Gamma(t_0),y) + d_M(y,\Gamma(s_0)) \\
&= d_M(\Gamma(t_0),\Gamma(s_0)).
\end{aligned}
$$

By minimality of s_0, this implies that

$$
p \notin \Gamma([0,\ell]) \tag{3.6}
$$

which is of course a contradiction since $\Gamma([0,\ell]) = u(\partial \mathbb{D})$.

Therefore, $\gamma_s(\rho)$ and $\Gamma(s)$ must lie in different components of $D_\rho(s) - [\Gamma(t_0),\Gamma(s_0)]$. But this once again leads to a contradiction by the same logic as what we have just done: it suffices to replace every occurrence of p by $\Gamma(s)$, and vice versa. Thus, it must be so that either $\Gamma(t_0)$ or $\Gamma(s_0)$ is in $D_\rho(s)$, and the second step of the proof is done.

For the third and final step of the proof, we suppose that $D_\rho(s) \cap \Gamma([t_0, s_0]) \nsubseteq \partial D_\rho(s)$ for all $s \in [t_0, s_0]$. By the previous step of the proof, if we get a contradiction, then we will have completed the proof. Fix $\epsilon \in (0, \rho_0 - \rho)$. Suppose that there exist $s \neq t \in [t_0, s_0]$ such that $\Gamma(t) \in \text{int}(D_\rho(s))$ and $\Gamma([t, s]) \subseteq B_{\rho+\epsilon}(\gamma_t(\rho))$. By Lemma 3.2, the function $d(\tau) = d_M(\gamma_s(\rho), \Gamma(\tau))$ has a minimum at s. Since $d_M(\gamma_s(\rho), \Gamma(s)) < \rho = d_M(\gamma_s(\rho), \Gamma(t))$, this thus implies that d has a maximum at some point $s' \in (t, s)$. Therefore, by Lemma 3.1, we have that

$$\left| \frac{D}{ds} \dot{\Gamma}(s') \right| \geq \frac{\alpha}{\rho + \epsilon} > \Lambda,$$

which is a contradiction.

Fig. 4.   Multiple disks of the form $D_\rho(t_i)$ (in light grey) with the inevitable point in $B_{\rho+\epsilon}(\gamma_{t_n}(\rho))$ breaking the curvature constraint (both in dark grey).

Summarizing what we have shown, for all $s \in [t_0, s_0]$, there exists $t \in [t_0, s_0]$ such that $\Gamma(t) \in \text{int}(D_\rho(s))$. However, for all such t and s, we have that $\Gamma([t, s]) \nsubseteq B_{\rho+\epsilon}(\gamma_t(\rho))$. In particular, we have that $s_0 - t_0 \geq 2\epsilon$. Take $t_1 := t_0 + \epsilon$. Note that, for any $s \in [t_0, t_1)$, we cannot have $\Gamma(s) \in \text{int}(D_\rho(t_1))$, since $\Gamma([s, t_1]) \subseteq B_{\rho+\epsilon}(\gamma_{t_1}(\rho))$. Therefore, we may take $s_1 \in (t_1, s_0]$ such that

$$d_M(\Gamma(t_1), \Gamma(s_1)) = \min\{d_M(\Gamma(t_1), \Gamma(s)) | s \in [t_0, s_0], \Gamma(s) \in D_\rho(t_1)\}.$$

Note that the second step of the proof stays true when we replace $[t_0, s_0]$ for $[t_1, s_1]$. In particular, we can similarly define $t_2 := t_1 + \epsilon$ and $s_2 \in [t_1, s_1]$. Continuing like

this, we inductively define $t_{n+1} := t_n + \epsilon$ and $s_{n+1} \in (t_{n+1}, s_n]$ such that

$$d_M(\Gamma(t_{n+1}), \Gamma(s_{n+1})) = \min\{d_M(\Gamma(t_{n+1}), \Gamma(s)) | s \in [t_n, s_n], \Gamma(s) \in D_\rho(t_{n+1})\}$$

for all $n \geq 0$. But of course, by construction,

$$2\epsilon \leq s_n - t_n \leq s_0 - t_0 - n\epsilon.$$

Therefore, we run into a contradiction for n large enough. This finally concludes the proof of Theorem 3.2 for the case $k = 0$.

When $k > 0$, the preparation is very similar to the basic case; we mostly just need to replace every $\mathbb{D}$ by S_r. Indeed, we can still lift everything to the universal cover, and the inequality $d_{\widetilde{M}}(\tilde{x}, \tilde{y}) \geq d_M(\pi(\tilde{x}), \pi(\tilde{y}))$ ensures that estimates on $d_M(x, \cup_i K_i)$ will lift to appropriate estimates upstairs. Likewise, we can still assume that $u(\operatorname{int} S_r)$ is entirely contained in the interior of the region bounded by $u(\partial \mathbb{D})$ — which may now no longer be contractible however (cf. Figure 5).

Note however that Γ now parametrizes the closure of the segment of $u(\partial S_r)$ in L containing x, i.e. $\overline{u(C_0)}$ in the notation of Subsec. 2.1. In particular, $\Gamma(0), \Gamma(\ell) \in L \cap (\cup_{i=1}^k K_i)$. By assuming that $2\rho_0 \leq d_M(x, \cup_i K_i)$, we make sure that

$$\operatorname{int}(D_\rho(t_0)) \cap \big(u(\partial S_r) - \Gamma([0, \ell])\big) = \emptyset.$$

Therefore, the fact that (3.3) directly implies the result is still true. Likewise, we have that $\Gamma(0), \Gamma(\ell) \notin D_\rho(t_0)$ for all $\rho < \rho_0$. Thus, it must be that $s_0 \in (0, \ell)$ by (3.4).

As for the proof proper, the first and third steps stay unchanged. However, in the second step, we need to change (3.5) by

$$p \in \operatorname{int}(D_\rho(s)) \cap \big(u(\partial S_r) - u(\operatorname{int} S_r)\big).$$

This is because the various K_i's may intersect L, and thus some part of $u(\partial S_r)$ may lie inside Δ_0 without that causing a contradiction (see Fig. 5 for an example of this phenomenon).

Since we have that

$$[\Gamma(t_0), \Gamma(s_0)] \cap (u(\partial S_r) - \Gamma([0, \ell])) \subseteq D_\rho(t_0) \cap (u(S_r) - \Gamma([0, \ell])) = \emptyset.$$

much of the proof stays nonetheless the same as before. Note however that we no longer have the equality $\Gamma([0, \ell]) = u(\partial S_r)$ in (3.6). We still get a contradiction by noting that

$$d_M(\Gamma(t_0), p) \leq d_M(\Gamma(t_0), \Gamma(s_0)) < 2\rho < d_M(\Gamma(t_0), \cup_i K_i),$$

thus also forcing $p \notin u(\partial S_r) - \Gamma([0, \ell])$.

Remark 3.3. Summarizing all the choices that have been made for ρ_0, we can take

$$\rho_0 = \min\left\{\frac{r_0}{2}, \frac{\pi}{2\sqrt{K_0}}, \rho_1, \frac{\alpha}{\Lambda}, \frac{1}{2}d_M(x, \cup_i K_i)\right\},$$

where ρ_1 and α are the constants appearing in Lemma 3.1. As we will see below, $\rho_1 \to \infty$ and $\alpha \to 1$ as $K_0 \to 0$. Therefore, when $k = 0$, we get $\rho_0 \to \frac{1}{\Lambda}$ as $K_0 \to 0$ and $r_0 \to \infty$, thus indeed recovering the Moon in a puddle theorem.
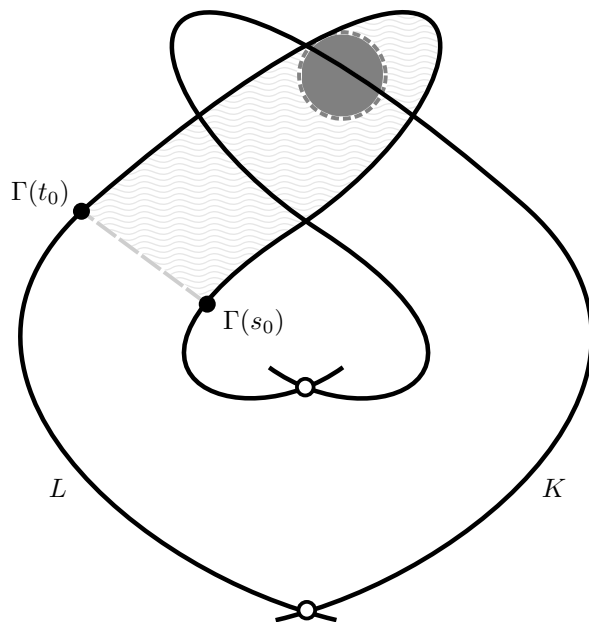
Fig. 5.   An osculating disk (in dark grey) in the topological disk Δ_0 (wavy region) satisfying the conclusion of Theorem 3.2, but intersecting $u(\partial S_r)$ nontrivially.

3.2. *Proof of technical results*

We now give the two proofs we had omitted in the previous subsection.

We now give the two proofs we had omitted in the previous subsection.

Proof of Lemma 3.1. The proof is done in two steps.

(i) For a well-chosen variation of the minimal geodesic from x to $\Gamma(s)$, we use the second variation formula for the energy functional to get that

$$\left| \frac{D}{ds} \dot{\Gamma}(s_0) \right| \geq \frac{I}{\rho}$$

for some $I > 0$ depending on the variation.

(ii) We use results on Jacobi fields to get a lower bound on I.

We begin with the proof of the first step. For $\epsilon > 0$ small enough, we consider the variation

$$h \colon [0,1] \times (-\epsilon, \epsilon) \longrightarrow M$$

$$(t,s) \longmapsto \gamma_s(t) := \exp_x(t\tilde{\Gamma}(s_0 + s)),$$

where $\exp_x(\tilde{\Gamma}(s_0 + s)) = \Gamma(s_0 + s)$. In particular, $d(s) = \ell(\gamma_s)$. Therefore, since the length and energy functionals have the same critical points, and that d achieve its

maximum at s_0, we have that

$$0 = \frac{1}{2}E'(0) = \left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t} \right\rangle (1,0) - \left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t} \right\rangle (0,0) \tag{3.7}$$

and

$$0 \geq \frac{1}{2}E''(0) = I\left(\frac{\partial h}{\partial s}, \frac{\partial h}{\partial s} \right) + \left\langle \frac{D}{\partial s}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t} \right\rangle (1,0) - \left\langle \frac{D}{\partial s}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t} \right\rangle (0,0). \tag{3.8}$$

Here, I denotes the index form of $\gamma := \gamma_0$. For vector fields V and W along γ, it is defined as

$$I(V,W) = \int_0^1 \left(\left\langle \frac{DV}{dt}, \frac{DW}{dt} \right\rangle - \langle R(\dot{\gamma}, V)\dot{\gamma}, W \rangle \right) dt.$$

Note that the last term on the right-hand side of (3.8) is zero since $h(0,s) = x$ for all s. Furthermore, the middle term can be bounded from below:

$$\left\langle \frac{D}{\partial s}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t} \right\rangle (1,0) = \left\langle \frac{D}{ds}c'(s_0), \dot{\gamma}(1) \right\rangle$$

$$\geq - \left| \frac{D}{ds}c'(s_0) \right| |\dot{\gamma}(1)|$$

$$= - \left| \frac{D}{ds}c'(s_0) \right| d(s_0)$$

$$\geq - \left| \frac{D}{ds}c'(s_0) \right| \rho.$$

Therefore, (3.8) turns into the desired bound of $\left| \frac{D}{ds}c'(s_0) \right|$ in terms of $I = I(\frac{\partial h}{\partial s}, \frac{\partial h}{\partial s})$ and ρ.

We now turn to the second step of the proof. First of all, note that $J(t) := \frac{\partial h}{\partial s}(t,0)$ is a Jacobi field along γ. Therefore, since $\rho < r_{\text{inj}}(M)$, we have that $J(t) \neq 0$ for all $t > 0$. Furthermore, (3.7) implies that $J(1)$ and $\dot{\gamma}(1)$ are orthogonal, since the last term on the right-hand side is zero, as noted previously. The same is then true of $J(t)$ and $\dot{\gamma}(t)$ for all t by standard results on Jacobi fields. Therefore, the index form simplifies slightly:

$$I = \int_0^1 \left(|\dot{J}|^2 - K\left(\dot{\gamma}, J\right)|J|^2|\dot{\gamma}|^2 \right) dt \geq \int_0^1 \left(|\dot{J}|^2 - K_0 d(s_0)^2|J|^2 \right) dt,$$

where $\dot{J} := \frac{D}{dt}J$.

However, we have that

$$|\dot{J}||J| \geq |\langle \dot{J}, J \rangle| = |\dot{\gamma}|\left| (\text{Hess}\,\rho)(J,J) \right| = d(s_0)\left| (\text{Hess}\,\rho)(J,J) \right|,$$

where ρ is the distance function from x. Therefore, by the Hessian comparison

theorem (see for example [16]) and the Rauch comparison theorem, we get that

$$I \geq \int_0^1 \left(\cot^2(\sqrt{K_0}d(s_0)t) - 1 \right) K_0 d(s_0)^2 |J(t)|^2 dt$$

$$\geq |\dot{\tilde{\Gamma}}(s_0)|^2 \int_0^1 \left(\cot^2(\sqrt{K_0}d(s_0)t) - 1 \right) \sin^2(\sqrt{K_0}d(s_0)t) dt$$

$$= \frac{\sin(2\sqrt{K_0}d(s_0))}{2\sqrt{K_0}d(s_0)} |\dot{\tilde{\Gamma}}(s_0)|^2.$$

But, using the Rauch comparison theorem again, we get

$$1 = |\dot{\Gamma}(s_0)| = |(d\exp_x)_{\tilde{\Gamma}(s_0)}(\dot{\tilde{\Gamma}}(s_0))| \leq \frac{\sinh(\sqrt{K_0}d(s_0))}{\sqrt{K_0}d(s_0)} |\dot{\tilde{\Gamma}}(s_0)|.$$

Therefore, this finally implies that

$$I \geq \frac{\sqrt{K_0}d(s_0)\sin(2\sqrt{K_0}d(s_0))}{2\sinh^2(\sqrt{K_0}d(s_0))}.$$

However, the function $\tau \mapsto (\tau \sin(2\tau))/(2\sinh^2(\tau))$ is positive and decreasing on $(0, \frac{\pi}{2})$. Therefore, if we take

$$\rho_1 := \min\left\{ r_0, \frac{\pi}{2\sqrt{K_0}} \right\} - \epsilon$$

for any $\epsilon > 0$ small enough, than we will have

$$I \geq \frac{\sqrt{K_0}\rho_1\sin(2\sqrt{K_0}\rho_1)}{2\sinh^2(\sqrt{K_0}\rho_1)} =: \alpha > 0,$$

which concludes the proof of the lemma. □

Proof of Lemma 3.2. We suppose the contrary and get a contradiction as follows.

(i) Since $D_\rho(s) \cap \Gamma([s-\epsilon, s+\epsilon]) \nsubseteq \partial D_\rho(s)$ for all $\epsilon > 0$, there must exist a decreasing sequence $\{\epsilon_n\}_{n \geq 1} \subseteq \mathbb{R}_{>0}$ converging to 0 such that $\Gamma(s \pm \epsilon_n) \in D_\rho(s)$ for all $n \geq 1$. By passing to a subsequence and changing the orientation of the parametrization Γ if needed, we may assume that $\Gamma(s + \epsilon_n) \in D_\rho(s)$ for all $n \geq 1$.

(ii) Since Γ is parametrized by arclength, we have the inclusion $\Gamma((s, s + \epsilon_n)) \subseteq B_{\rho+\epsilon_n}(\gamma_s(\rho))$. We recall that $\gamma_s(\rho) = \exp_{\Gamma(s)}(\rho N(s))$, and N is the unit-length vector field along Γ which is orthogonal to $\dot{\Gamma}$ and pointing toward the interior of the topological disk $\overline{u(S_r)}$.

(iii) Suppose that for all $n \geq 1$, there exists $s_n \in (s, s + \epsilon_n)$ such that

$$d_M(\gamma_s(\rho), \Gamma(s_n)) \geq \rho.$$

Note that $s_n \to s$, because $\epsilon_n \to 0$. Furthermore, s_n is a local maximum of $d(\sigma) = d_M(\gamma_s(\rho), \Gamma(\sigma))$ since $d_M(\gamma_s(\rho), \Gamma(s)) = d_M(\gamma_s(\rho), \Gamma(s + \epsilon_n)) = \rho$ by

construction. By applying Lemma 3.1 to $\Gamma|_{[s,s+\epsilon_n]}$, we get

$$\begin{aligned}
\left| \frac{D}{ds}\dot\Gamma(s) \right| &= \lim_{n\to\infty} \left| \frac{D}{ds}\dot\Gamma(s_n) \right| \\
&\geq \lim_{n\to\infty} \frac{\alpha}{\rho + \epsilon_n} \\
&= \frac{\alpha}{\rho} \\
&> \Lambda,
\end{aligned}$$

which is of course a contradiction. Therefore, it must be that $\Gamma([s, s + \epsilon_n]) \subseteq D_\rho(s)$ for all $n \geq 1$.

(iv) Note that s must be a critical point of the function d of Lemma 3.1 for $x = \gamma_s(\rho)$. Indeed, those correspond to the critical points of the energy functional along the variation $h(\tau, t) = \exp_{\gamma_s(\rho)}(\tau\tilde\Gamma(s + t))$, where $\exp_{\gamma_s(\rho)}(\tilde\Gamma(s + t)) = \Gamma(s + t)$. But, for such a variation,

$$\begin{aligned}
\frac{1}{2}E'(0) &= \left\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial \tau} \right\rangle (1, 0) \\
&= \left\langle \dot\Gamma(s), \frac{d}{d\tau}\exp_{\gamma_s(\rho)}(\tau\tilde\Gamma(s)) \Big|_{\tau=1} \right\rangle \\
&= \frac{1}{\rho}\left\langle \dot\Gamma(s), -N(s) \right\rangle \\
&= 0.
\end{aligned}$$

Indeed, the path $\tau \mapsto \exp_{\gamma_s(\rho)}(\tau\tilde\Gamma(s))$ is the unique minimizing geodesic of speed $|\tilde\Gamma(s)| = \rho$ from $\gamma_s(\rho)$ to $\Gamma(s)$. This is just γ_s parametrized in the opposite orientation and with a different speed.

Furthermore, the fact that $\Gamma([s, s + \epsilon_n]) \subseteq D_\rho(s)$ for all $n \geq 1$ implies that d must have nonpositive second derivative at s. This in turn implies that $E''(0) \leq 0$. Therefore, all the proof of Lemma 3.1 still works, and we get a contradiction:

$$\left| \frac{D}{ds}\dot\Gamma(s) \right| \geq \frac{\alpha}{\rho} > \Lambda. \qquad\qquad \square$$

4. Badly-behaved sequences

We conclude this paper with examples of sequences of Lagrangian submanifolds. These sequences show that bounds of curvature type are needed to ensure convergence in the Hausdorff metric.

Consider the sequence of Hamiltonians $\{H_n(x, y) := \frac{1}{n}\sin(nx)\}_{n\geq 1}$ on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. We equip the torus with the standard symplectic form ω_0, the standard complex structure J_0, and the flat metric $g_0 = g_{J_0}$. A quick calculation shows that the induced Hamiltonian flow of H_n is given by

$$\phi_n^t(x, y) = (x, y + t\cos(nx)),$$

for all $t \geq 0$, $n \geq 1$ and $(x,y) \in \mathbb{T}^2$. We then set

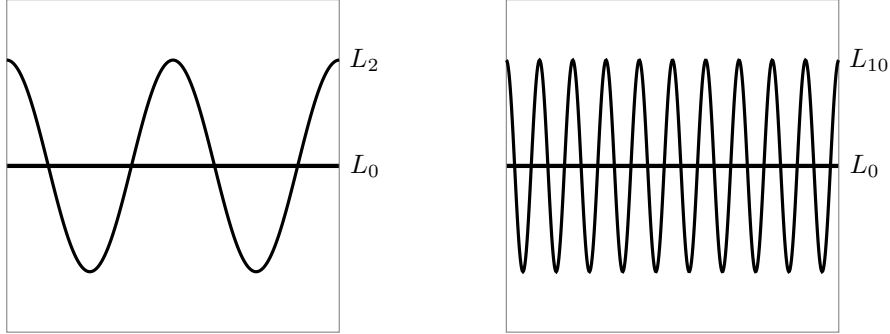$$L_0 := \{y = 0\} \quad \text{and} \quad L_n := \phi_n^1(L_0) = \{y = \cos(nx)\}.$$

Fig. 6. L_n as n gets larger

Note that $d_H(L_0, L_n) = \frac{2}{n}$, and thus L_n tends to L_0 in the Lagrangian Hofer metric. Indeed, $\frac{2}{n}$ is an upper bound of $d_H(L_0, L_n)$, since it is the oscillation of H_n. On the other hand, by Corollary 3.13 of [3], $d_H(L_0, L_n)$ is bounded from below by the area of a strip. However, it is clear that such a strip must have area at least $\frac{2}{n}$.

On the other hand, as Fig. 6 suggests, L_n tends to the full band $\{-1 \leq y \leq 1\}$ in the Hausdorff metric δ_H induced by the flat metric. Therefore, even though the sequence converges in both metric, the limits are quite different. Actually, we have that $\delta_H(L_0, L_n) \equiv 1$.

The conjecture does not apply to this sequence of Lagrangian submanifolds, as the norm of the second fundamental form is clearly unbounded. Actually, a quick calculation gives that

$$||B_{L_n}|| = \max_{p \in K_n} |\kappa(p)| = n,$$

where κ denotes the geodesic curvature.

Note that this example can easily be generalized to higher dimensional tori. Likewise, by multiplying H_n by a cutoff function, this example can be transposed to any symplectic manifold using a Darboux chart. Finally, since d_H bounds from above every other metric mentioned in the introduction, this problem applies to every known metric for which Theorem A holds. In other words, this is a universal example.

However, it is possible to get a sequence of Lagrangian submanifolds with exploding curvature, but where limits in a Chekanov-type metric and the Hausdorff metric coincide. For example, one can do the analogous construction as above, but with Hamiltonians $G_n := \frac{1}{\sqrt{n}} H_n$. Indeed, the associated Hamiltonian flows are

$$\psi_n^t(x,y) = \left(x, y + \frac{t}{\sqrt{n}} \cos(nx) \right),$$

and we get the Lagrangian submanifolds

$$K_n := \psi_n^1(L_0) = \left\{ y = \frac{1}{\sqrt{n}} \cos(nx) \right\}.$$

By arguments similar to the above, one gets $d_H(L_0, K_n) = 2n^{-3/2}$ and $\delta_H(L_0, K_n) = n^{-1/2}$. Therefore, K_n tends to L_0 in both the Lagrangian Hofer and the Hausdorff metric. However, it is easy to calculate that the maximum of the curvature of K_n is given by

$$\max_{p \in K_n} |\kappa(p)| = \sqrt{n}.$$

This, of course, tends to infinity as n tends to infinity.

Remark 4.1. Note that, in the sequence $\{L_n\}$ above, not only does $||B_{L_n}||$ tend to infinity, but it is also impossible to uniformly tame the Lagrangian submanifolds in the sequence. Indeed, the distance between two successive zeroes of $y = \cos(nx)$ is $\frac{\pi}{n}$ in M, but is at least 2 in L_n. Therefore,

$$\lim_{n \to \infty} \inf_{x \neq y \in L_n} \frac{d_M(x, y)}{\min\{1, d_L(x, y)\}} = 0.$$

We expect this phenomenon to be general: when M is simply connected, control over the second fundamental form should give enough control over tameness for the proof of Theorem 2.1 to still work. It would then be possible to pass to the universal cover to get the desired result, just as we have done in the proof of Theorem 3.1.

In order to clarify our intuition, let us note that L being ϵ-tame is equivalent to the following condition: for all $x \in L$, and for all $y \in B_\epsilon(x) \cap L$, we have that

$$d_M(x, y) \leq \frac{1}{\epsilon} d_L(x, y). \tag{4.1}$$

It is quite clear that any type of bound on curvature cannot stop ϵ from being arbitrarily small at some point $x \in L$. On the hand, as Fig. 4 from the previous section suggests, having this condition at every $x \in L$ should force a certain lower bound on $||B_L||$. Therefore, for any $L \in \mathscr{L}_\Lambda^\star(M)$, there should be some $x \in L$ where the optimal epsilon appearing in (4.1) is bounded from below by some constant $e = e(K_0, r_0, \Lambda) > 0$. We could then apply Proposition 2.1 on some appropriate metric ball centered at this x; the size of the ball would only depend on K_0, r_0, and Λ.

Acknowledgments

References

[1] M. Abouzaid, Nearby Lagrangians with vanishing Maslov class are homotopy equivalent, *Invent. Math.* **189**(2) (2012) 251–313.

[2] M. Abouzaid and T. Kragh, Simple homotopy equivalence of nearby Lagrangians, *Acta Math.* **220**(2) (2018) 207–237.

[3] J.-F. Barraud and O. Cornea, Homotopical dynamics in symplectic topology, in *Morse theoretical methods in non-linear analysis and symplectic topology*, eds. P. Biran, O. Cornea and F. Lalonde (Springer, Dordrecht, 2006), pp. 109–148.

[4] M. Berger, Some relations between volume, injectivity radius, and convexity radius in Riemannian manifolds, in *Differential geometry and relativity*, eds. M. Cahen and M. Flato (Springer, Dordrecht, 1976), pp. 33–42.

[5] P. Biran, O. Cornea and E. Shelukhin, Lagrangian shadows and triangulated categories, *Astérisque.* **426** (2021) 1–128.

[6] P. Biran, O. Cornea and J. Zhang, Triangulation and persistence: Algebra 101, preprint (2021), arXiv:2104.12258.

[7] D. Burago, Y. Burago and S. Ivanov, *A course in metric geometry* (American Mathematical Society, Providence, 2001).

[8] F. Charette, A geometric refinement of a theorem of Chekanov, *J. Symplectic Geom.* **10**(3) (2012) 475–491.

[9] I. Chavel, *Riemannian geometry* (Cambridge University Press, Cambridge, 2006).

[10] Y.V. Chekanov, Invariant Finsler metrics on the space of Lagrangian embeddings, *Math. Z.* **234**(3) (2000) 605–619.

[11] K. Cieliebak, V. Ginzburg and E. Kerman, Symplectic homology and periodic orbits near symplectic submanifolds, *Comment. Math. Helv.* **79**(3) (2004) 554–581.

[12] O. Cornea and E. Shelukhin, Lagrangian cobordism and metric invariants, *J. Diff. Geom.* **112**(1) (2019) 1–45.

[13] J. Dibble, The convexity radius of a Riemannian manifold, *Asian J. Math.* **21**(1) (2017) 169–174.

[14] K. Fukaya, P. Seidel and I. Smith, Exact Lagrangian submanifolds in simply-connected cotangent bundles, *Invent Math.* **172**(1) (2008) 1–27.

[15] K. Fukaya and P. Seidel and I. Smith, The symplectic geometry of cotangent bundles from a categorical viewpoint, in *Homological mirror symmetry*, eds. K.-G. Schlesinger, M. Kreuzer and A. Kapustin (Springer, Heidelberg, 2008), pp. 1–26.

[16] R.E. Greene and H. Wu, *Function theory on manifolds which possess a pole* (Springer, Heidelberg, 1979).

[17] Y. Groman and J.P. Solomon, A reverse isoperimetric inequality for *J*-holomorphic curves, *Geom. Funct. Anal.* **24**(5) (2014) 1448–1515.

[18] Y. Groman and J.P. Solomon, *J*-holomorphic curves with boundary in bounded geometry, *J. Symp. Geom.* **14**(3) (2016) 767–809.

[19] M. Gromov, *Structures métriques pour les variétés riemanniennes* (CEDIC/Fernand Nathan, Paris, 1981).

[20] V. Humilière, Hofer's distance on diameters and the Maslov index, *Int. Math. Res. Not.* **2012**(15) (2012) 3415–3433.

[21] A. Katsuda, Gromov's convergence theorem and its applications, *Nagoya Math. J.* **100** (1985) 11–48.

[22] M. Khanevsky, Hofer's metric on the space of diameters, *J. Topol. Anal.* **1** (2009) 407–416.

[23] A. Kislev and E. Shelukhin, Bounds on spectral norms and barcodes, *Geom. Topol.* **25**(7) (2022) 3257–3350.

[24] W. Klingenberg, Contributions to Riemannian geometry in the large, *Ann. Math.*

69(3) (1959) 654–666.

[25] T. Kragh, Parametrized ring-spectra and the nearby Lagrangian conjecture, *Geom. Topol.* **17**(2) (2013) 639–731.

[26] R. Leclercq, Spectral invariants in Lagrangian Floer theory, *J. Mod. Dyn.* **2**(2) (2008) 249–286.

[27] R. Leclercq and F. Zapolsky, Spectral invariants for monotone Lagrangians, *J. Topol. Anal.* **10**(3) (2018) 627–700.

[28] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology* (American Mathematical Society, Providence, 2012).

[29] D. Nadler, Microlocal branes are constructible sheaves, *Selecta Math.* **15**(4) (2009) 563–619.

[30] G. Perelman, Alexandrov spaces with curvatures bounded from below II, preprint (1991).

[31] A. Petrunin and S.Z. Barrera, Moon in a puddle and the four-vertex theorem, *Amer. Math. Monthly.* **129**(5) (2022) 475–479.

[32] P. Seidel, *Fukaya categories and Picard-Lefschetz theory* (European Mathematical Society, Zurich, 2008).

[33] S. Seyfaddini, Unboundness of the Lagrangian Hofer distance in the Euclidean ball, *Electron. Res. Announc. Math. Sci.* **21** (2009) 1–7.

[34] Z. Shen, A convergence theorem for Riemannian submanifolds, *Trans. Amer. Math. Soc.* **347**(4) (1995) 1343–1350.

[35] J.-C. Sikorav, Some properties of holomorphic curves in almost complex manifolds, in *Holomorphic curves in symplectic topology*, eds. M. Audin and J. Lafontaine (Birkhäuser, Basel, 1994), pp. 165–189.

[36] K. L. Stacy, On the Hausdorff hyperspace of a compact metric space, Master Thesis, Iowa State University (1967).

[37] C. Viterbo, Symplectic topology as the geometry of generating functions, *Math. Ann.* **292**(4) (1992) 685–710.

[38] F. Zapolsky, On the Hofer geometry for weakly exact Lagrangian submanifolds, *J. Symp. Geom.* **11**(3) (2013) 475–488.