Weinstein exactness of nearby Lagrangians: towards the Lagrangian C^0 flux conjecture

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Abstract

We address the following problem: if a Hamiltonian diffeomorphism maps a Lagrangian submanifold L to a small Weinstein neighborhood of L, is the image necessarily Hamiltonian isotopic to L inside that neighborhood? On one hand, we show that the question can have a negative answer in any symplectic manifold of dimension at least six. On the other hand, we answer an *a priori* weaker form of the question in the positive in various cases when L satisfies a rationality condition: we prove that the image of L is often exact inside the Weinstein neighborhood. We provide applications to the Lagrangian counterpart of the C^0 flux conjecture, to C^0-rigidity phenomena of Hamiltonian diffeomorphisms, and to topological properties of spaces of Lagrangians with the same rationality constraint. Moreover, we state and prove cases of an analogue of Viterbo's spectral norm conjecture for non-exact Lagrangians; in the process, we make progress on an old question of Viterbo regarding integer difference vectors between points of Lagrangians.

1 Introduction

This paper aims to study the local topological properties of natural sets of Lagrangians, most notably the Hamiltonian and symplectic orbits of a given Lagrangian L, respectively

$$\mathcal{L}\mathrm{Ham}(L) := \mathrm{Ham}(M)\cdot L = \{\varphi(L)\,|\,\varphi\in\mathrm{Ham}(M)\}\,,$$
$$\mathcal{L}\mathrm{Symp}_0(L) := \mathrm{Symp}_0(M)\cdot L = \{\psi(L)\,|\,\psi\in\mathrm{Symp}_0(M)\}\,.$$

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Note that for the sake of conciseness, we will refer throughout the paper to a *closed connected Lagrangian submanifold of a connected symplectic manifold without boundary* as a "Lagrangian in a symplectic manifold".

To enable our study, we first fix a metric g on the underlying manifold L. Recall that the Weinstein neighbourhood theorem ensures that there exist $r > 0$ and a symplectomorphism $\Psi : D^*_r L \to \mathcal{W}_r(L)$ from the codisk bundle of L of radius r to a neighbourhood $\mathcal{W}_r(L)$ of L in M which maps the 0-section to L.

Therefore, understanding $\mathcal{L}\mathrm{Ham}(L)$ ***locally*** is intimately related to the nearby Lagrangian conjecture (or NLC for short), which completely characterizes Lagrangians which are in the Hamiltonian orbit of the 0-section in T^*L. Indeed, it states that those are precisely the exact Lagrangians. It is known to hold for S^1, S^2 [Hin04], $\mathbb{R}\mathrm{P}^2$ [HPW16, Ada22], and $\mathbb{T}^2$ [RGI16]. Without restriction on the diffeomorphism type, the most advanced result in the direction of the NLC states that the natural projection $\pi : T^*L \to L$ induces a simple homotopy equivalence between any exact closed Lagrangian and the 0-section [AK18]. This latter result will play a crucial role in our study of the local structure of $\mathcal{L}\mathrm{Ham}(L)$.

Inspired by this conjecture, we propose that if $L' \in \mathcal{L}\mathrm{Ham}(L)$ is close to L, then there is an accordingly small Hamiltonian isotopy from L to L'. More precisely, we consider the following speculation.

Speculation A *Let L be a Lagrangian in a symplectic manifold M. There exists a neighbourhood U of L with the following property. If L' is Hamiltonian isotopic to L in M and $L' \subseteq U$, then there exists a Hamiltonian isotopy $\{\varphi_t\}_{t\in[0,1]}$ supported in U such that $\varphi_1(L) = L'$.*

If this holds for L, then its Hamiltonian orbit $\mathcal{L}\mathrm{Ham}(L)$ is locally path connected via Hamiltonian isotopies. Although this statement is new in the Lagrangian context, there are some results towards its Hamiltonian counterpart. More precisely, the group $\mathrm{Ham}_c(M)$ of compactly supported Hamiltonian diffeomorphisms of a symplectic manifold M is locally path connected in the C^0 topology (via Hamiltonian isotopies) if M is a closed surface or the open ball B^{2n}. The 2-dimensional case case follows from Fathi's work on homeomorphisms preserving a volume form [Fat80] and the folkloric fact that a path of such homeomorphisms on a closed surface can be C^0 approximated by a path of symplectomorphisms, see [Sey13] for a proof. The case of the open ball was proved by Seyfaddini, also in [Sey13].

Note that the local path connectedness of $\mathrm{Ham}(M)$ does not yield Speculation A for all Lagrangians of M. Theorem 1 below shows that, for instance in the standard symplectic ball M of dimension 6 or greater, there exist certain *irrational* tori for which Speculation A fails. It does, however, imply Speculation A for graphs of Hamiltonian symplectomorphisms of M. Conversely, note that even if Speculation A holds for all graphs in $M \times M$, it does not imply local path connectedness of $\mathrm{Ham}(M)$, since the Hamiltonian isotopy given by the speculation need not be through graphs.

In the positive direction, we prove the existence of neighbourhoods of ***local exactness*** for several classes of Lagrangians, by which we mean a Weinstein neighbourhood $\mathcal{W}(L)$ of a Lagrangian L so that any Lagrangian Hamiltonian isotopic to L contained in this neighbourhood is exact in $\mathcal{W}(L)$. When the NLC is known to hold for L, we can then deduce that L satisfies Speculation A. This indicates, however, that Speculation A is likely extremely hard to prove in great generality.

However, the following weaker form of the speculation holds in greater generality and is easier to prove.

Speculation B *Let L be a displaceable Lagrangian in a symplectic manifold M. There exists a neighbourhood U of L with the following property. If L' is Hamiltonian isotopic to L in M and $L' \subseteq U$, then $L \cap L' \neq \emptyset$.*

The Lagrangians which admit a neighbourhood of ***local exactness*** as described above obviously satisfy that weaker speculation. In fact, such Lagrangians must intersect L in many points: for instance, if the intersection is transverse, then there are at least $\sum_{i=0}^{n} \beta_i(L) \geq 2$ intersection points, where $\beta_i(L)$ is the i-th Betti number of L.

Actually, our methods allow us to prove that a large class of Lagrangians satisfy a slightly strengthened version of Speculation B, namely that if L' is the image of L under any ***symplectomorphism*** and $L' \subseteq U$, then $L \cap L' \neq \emptyset$. In what follows, we will refer to satisfying Speculation B (respectively its symplectic version) as ***having a neighbourhood of Hamiltonian nondisplacement*** (respectively of ***symplectic nondisplacement***).

To give an idea of the methods we use to prove our results, we introduce a Viterbo-style conjecture. To do so, we need the following rationality notions.

Definition *Let L be a Lagrangian of a symplectic manifold (M, ω). We say that L is H-**rational** in (M, ω) if the group of relative periods $\omega(H_2(M, L; \mathbb{Z})) = \tau_L \mathbb{Z} \subset \mathbb{R}$ is a discrete subgroup. We call $\tau_L \geq 0$ the H-**rationality constant** of L. We say that L is H-**exact** if $\tau_L = 0$.*

*Restricting to disk classes, that is, the image $H_2^D(M, L; \mathbb{Z})$ of $\pi_2(M, L)$ under the Hurewicz morphism, L is called **rational** if $\omega(H_2^D(M, L; \mathbb{Z})) \subset \mathbb{R}$ is a discrete subgroup — **weakly exact** if it vanishes — and we define the **rationality constant** of L analogously.*

We shall see in Section 3.2 below that the homological and homotopical conditions are equivalent in many important cases, e.g. when $\pi_1(M) = 0$. With these definitions in mind, we make the following conjecture, which we prove in several cases below.

Conjecture C *Let L be a closed connected manifold. Suppose that K is an H-rational Lagrangian inside D_r^*L such that the map $(\pi_K)_* : H_1(K; \mathbb{R}) \to H_1(L; \mathbb{R})$ induced by $\pi_K = \pi|_K$ is not surjective. Here, $\pi : T^*L \to L$ denotes the canonical projection. Then, the H-rationality constant τ_K of K satisfies*

$$\tau_K \leq Cr$$

for a constant C depending only on L and the choice of an auxiliary metric on it.

Remark 1. *We note that the non-surjectivity of $(\pi_K)_*$ can be replaced by the inequality of first Betti numbers $b_1(K; \mathbb{R}) \leq b_1(L; \mathbb{R})$.*

One can also pose a homotopy version of Conjecture C, where one replace either H-rational with rationality or $H_1(\,\cdot\,;\mathbb{R})$ with π_1 — or both. We will also explore such variant below.

We remark that it would be interesting to see if there is a version of Conjecture C that also applies to irrational Lagrangians, with the role of τ_K being replaced by $\hbar(K, J) = \min\{\omega(u) > 0\}$, where the minimum runs over all the non-constant J-holomorphic disks $u : \mathbb{D} \to M$ with boundary on K for a fixed compatible almost complex structure J. We leave this question for future work.

1.1 Main results

The following result provides a counterexample to Speculations A and B (and to Speculation D below, as well as its C^1 variant). It is not, however, a counterexample to the Viterbo-type conjecture. This shows that there is no hope to prove the above speculations in full generality, and that a rationality condition like above is required.

Theorem 1 *In any symplectic manifold of dimension $2n \geq 6$, there exists a Lagrangian torus whose Hamiltonian orbit*

- (i) *admits arbitrarily Hausdorff-close disjoint elements,*
- (ii) *is not closed in Hausdorff topology inside the set of Lagrangian tori.*

Both claims actually hold for any reasonable notion of C^1 topology — see Section 2 below.

The existence of such tori follows directly from the characterization of product tori in the Hamiltonian orbit of a given product Lagrangian torus in $\mathbb{C}^n$ by Chekanov [Che96] and in large enough balls by Chekanov and Schlenk [CS16]. We give the details in Section 2 below.

Again, we make the crucial observation that these Lagrangian tori are not ***rational***, and therefore do not contradict our positive results below.

1.1.1 Rationality bound on nearby Lagrangians

We first describe positive results related to Conjecture C.

Theorem 2 *Conjecture C holds for $L = L_0 \times L_1 \times \cdots \times L_k$, where $H_1(L_0;\mathbb{R}) = 0$, and L_i, $i \geq 1$, satisfies $H_1(L_i;\mathbb{R}) = \mathbb{R}$ and admits a Lagrangian embedding in a Liouville domain W_i with $SH(W_i) = 0$. Furthermore, if L is any manifold covered by a closed connected manifold L' for which Conjecture C holds, then it also holds for L.*

Note that, in the above statement, L_0 may be a point.

Remark 2. *The homotopy version of Theorem 2 also holds, that is, when H-rationality is replaced with rationality on disks and H_1 with π_1. When L has the diffeomorphism type of $L_0 \times \mathbb{T}^m$ with $H_1(L_0;\mathbb{R}) = 0$, it is possible, via an alternative argument, to prove a stronger version of the conjecture where K is only assumed to be rational on disks and $(\pi_k)_*$ is not surjective on homology; this is proven in Section 4.5, as Theorem 39.*

Theorem 39 and an examination of its proof implies the following result, which is a partial answer to an old question of Viterbo. Consider the subset $X \subset \mathbb{R}^n$ given by $X = \{(p_1, \ldots, p_n) \mid \min_j |p_j| \leq 1\}$.

Proposition 3 *Let K be a rational Lagrangian inside $\mathbb{C}^n$ with rationality constant $\tau_K > 2$. If K is contained in $\mathbb{R}^n + iX$ then there exist two distinct points on K whose difference vector is real with integer coordinates.*

To provide context, Viterbo's original question consists in proving the similar statement that there exists a constant $A \geq 1$ such that every rational Lagrangian K inside $\mathbb{C}^n$ with rationality constant $\tau_K > A$ must have two distinct points whose difference vector has Gaussian integer coordinates. This is equivalent to finding the maximal rationality constant of a nullhomotopic rational Lagrangian in the standard torus $\mathbb{C}^n/(\mathbb{Z}^n + i\mathbb{Z}^n)$. We prove a stronger statement with the difference vector being also real, under the additional assumption that K is contained in $\mathbb{R}^n + iX$.

1.1.2 Existence of nondisplacement neighbourhoods

We denote by $\mathcal{L}(\tau)$ the space of all H-rational Lagrangian submanifolds of M which have H-rationality constant $\tau \geq 0$, and by $\mathcal{L}(L, \tau)$ its subspace formed by those Lagrangians which have the same diffeomorphism type as L.

We now study Speculations A and B. First, we prove the existence of ***neighbourhoods of homological exactness*** for several classes of Lagrangians by reducing it to Conjecture C — this is Theorem 4 below. Second, we show that such neighbourhoods lead to ***Weinstein neighbourhoods of exactness*** for Hamiltonian isotopic Lagrangians. We also get such neighbourhoods for Lagrangians with a given H-rationality constant under an extra homological condition — this is the content of Theorem 5.

Theorem 4 *Suppose that L is a Lagrangian submanifold of (M, ω) for which Conjecture C holds. Then, for each $\tau \geq 0$, there exists a Weinstein neighbourhood $\mathcal{W}(L)$ of L such that all $L' \in \mathcal{L}(L, \tau)$ included in $\mathcal{W}(L)$ is H-exact in $\mathcal{W}(L)$.*

Theorem 5 *Let L be a H-rational Lagrangian in (M, ω) and let $L' \in \mathcal{L}\mathrm{Ham}(L)$ be a Lagrangian included in a Weinstein neighbourhood $\mathcal{W}_r(L)$ of size $r > 0$ such that L' is H-exact in $\mathcal{W}_r(L)$. Then, for a maybe smaller r, L' is exact in $\mathcal{W}_r(L)$.*

Moreover, if the inclusion of L into M induces the 0-map $H_1(L; \mathbb{R}) \to H_1(M; \mathbb{R})$, then the same result holds with $\mathcal{L}\mathrm{Ham}(L)$ replaced by $\mathcal{L}(\tau)$, where τ is the H-rationality constant of L.

The existence of nondisplacement neighbourhoods follows, as a direct combination of Theorems 4 and 5.

Corollary 6 *A H-rational Lagrangian L in a symplectic manifold M satisfying the assumptions of Theorem 2 admits a Hamiltonian nondisplacement neighbourhood. If furthermore the map $H_1(L; \mathbb{R}) \to H_1(M; \mathbb{R})$ is zero, then it also admits a neighbourhood of symplectic nondisplacement.*

In other words, such a Lagrangian L satisfies Speculation B (or its slightly strengthened symplectic version). Furthermore, if the nearby Lagrangian conjecture holds in T^*L, we also get Speculation A.

Remark 3. *We expect that one could use the McDuff–Siegel higher capacities to obtain Theorem 4 for certain $K(\pi, 1)$ spaces, including the n-torus. Moreover, we expect a slightly different approach to produce a class of examples invariant under all maps of non-zero degree, rather than coverings. Finally, we expect to extend our results to a large class of fibrations over tori. However, the arguments are quite involved and will be investigated in further work.*

Interestingly enough:

1. *Yet another approach one could take to the above questions for $L = \mathbb{T}^2$ is based on the resolution [RGI16] in this case of the nearby Lagrangian conjecture. However, since this does not fit in the general framework developed here, we omit them from this version of the paper. Nonetheless, in Appendix A, we present a standalone proof of H-exactness for Lagrangian Klein bottles in the cotangent bundle of a Klein bottle as the proof is short and could be of independent interest.*
2. *In their work in progress on the C^0 flux conjecture for Hamiltonian diffeomorphisms [AS24] the first and last named authors get a similar result of local exactness for graphs in $M \times M$ of C^0-small Hamiltonian diffeomorphisms for M closed. In this case, there is no requirement that M be rational, contrary to the setup of the present work.*

1.1.3 Lagrangian flux conjectures

We now move on to another speculation about Lagrangian submanifolds.

Speculation D (Lagrangian C^0 flux conjecture) *Let L be a Lagrangian in a symplectic manifold M. Its Hamiltonian orbit $\mathcal{L}\mathrm{Ham}(L)$ is Hausdorff-closed in the space $\mathcal{L}\mathrm{Lag}(L)$ of all Lagrangians which are Lagrangian isotopic to L.*

As far as the authors know, this version of the conjecture has not been studied previously — we will talk about its C^1 cousin, which has been studied, below. The name that we give it here is in analogy to the famous C^0 flux conjecture for Hamiltonian diffeomorphisms, which states that the group $\mathrm{Ham}(M)$ of Hamiltonian diffeomorphisms of a closed symplectic manifold M is C^0 closed in the identity component $\mathrm{Symp}_0(M)$ of the group of symplectomorphisms of M. This conjecture is only known to hold in some fairly specific cases [LMP98, Buh15] and further results in this direction will appear in [AS24], which still do not resolve this question completely. This is in stark contrast with its C^1 cousin, which is known to hold in full generality [Ono06].

We note that, similarly to Speculation A above, the Lagrangian C^0 flux conjecture does not imply the one for Hamiltonian diffeomorphisms. Indeed, suppose that $\{\varphi_i\} \subseteq \mathrm{Ham}(M)$ C^0-converges to $\psi \in \mathrm{Symp}_0(M)$. Then, all that the Lagrangian flux conjecture ensures is that there is some Hamiltonian diffeomorphism Φ of $M \times M$ such that $\mathrm{graph}(\psi) = \Phi(\Delta)$, where $\Delta \subseteq M \times M$ is the diagonal. However, we cannot be sure that Φ can be chosen of the form

$\mathbb{1} \times \varphi$ for some $\varphi \in \mathrm{Ham}(M)$, which is what the flux conjecture for Hamiltonian diffeomorphisms would require.

To study this speculation, we can use the techniques developed to prove Theorems 4 and 5. In fact, they imply the following continuity result.

Theorem 7 *Let $\{L_i\}$ be a sequence of H-rational Lagrangians of a tame symplectic manifold M such that*

(i) *$\{L_i\}$ Hausdorff-converges to a n-dimensional smooth submanifold L;*

(ii) *$\inf \tau_i > 0$, where τ_i denotes the H-rationality constant of L_i.*

Then, L is itself Lagrangian.

Moreover, if L_i is H-exact in a Weinstein neighbourhood $\mathcal{W}(L)$ for i large, then $\lim \tau_i$ *exists and is the H-rationality constant of L. This is in particular the case if the L_i's respect the hypotheses of Theorem 4.*

By ***tame***, we mean that M admits an almost complex structure J making $g_J := \omega(\cdot, J\cdot)$ into a complete Riemannian metric whose sectional curvature is bounded and whose injectivity radius is bounded away from zero.

The first part of the theorem is a fairly direct application of Laudenbach and Sikorav's result on the displaceability of non-Lagrangian submanifolds [LS94] — we mostly write it here for the reader's convenience. Furthermore, the second part of the theorem is very reminiscent of Theorem 1 of [MO21] — the proof is in fact very inspired by what appears in that paper. The strength of our result is that it applies to sequences $\{L_i = \varphi_i(L)\}$ where the sequence of Hamiltonian diffeomorphisms $\{\varphi_i\}$ need not C^0-converge. See Section 5.2 for more details.

Before moving on to corollaries of this result, note that, in the formulation above, one could also ask for C^0-closure of $\mathrm{Ham}(M)$ in larger groups than $\mathrm{Symp}_0(M)$, most notably in $\mathrm{Symp}(M)$.

Following this logic, we can replace $\mathcal{L}\mathrm{Lag}(L)$ in Speculation D with larger spaces. Most notably, we will also be interested in the spaces $\mathsf{SMan}(L)$, of all submanifolds of M with the same diffeomorphism type as L, and SMan_n, of all n-dimensional submanifolds of M^{2n}. By Theorem 7, closure of $\mathcal{L}\mathrm{Ham}(L)$ in the two latter spaces is equivalent to closure in the subspace formed by Lagrangian submanifolds.

To address these many spaces, we will make use of the following weaker form of Theorem 5.

Proposition 8 *Let L be a H-rational Lagrangian submanifold of M with H-rationality constant τ. There exists $r_0 > 0$ with the following property. Assume that $L' \in \mathcal{L}(\tau)$ is a Lagrangian included in a Weinstein neighbourhood $\mathcal{W}_r(L)$ of radius $r \in (0, r_0]$ such that L' is H-exact in $\mathcal{W}_r(L)$. Then, there is a symplectic isotopy $\{\psi_t\}_{t\in[0,1]}$ of M such that $\psi_1(L')$ is exact in $\mathcal{W}_r(L)$. Furthermore, the size of the isotopy is controlled by r.*

The last sentence corresponds in actuality to a precise estimate on the flux of the Lagrangian isotopy $\{\psi_t(L')\}$, but we do not want to get into all the details here. We refer the interested reader to Section 5.3.

Combining Theorem 4, Theorem 5, and Proposition 8, we thus get the following — again, the precise proof is in Section 5.3.

Corollary 9 *Suppose that L satisfies the hypotheses of Theorem 2 or that $H_1(L;\mathbb{R}) = 0$. If the nearby Lagrangian conjecture holds in T^*L, then we have the following. Suppose that L' is a H-rational Lagrangian diffeomorphic to L. Its Hamiltonian orbit $\mathcal{L}\mathrm{Ham}(L')$ and symplectic orbit $\mathcal{L}\mathrm{Symp}_0(L')$ are Hausdorff-closed in $\mathcal{L}(L)$.*

In fact, we can prove the equivalence of the NLC and the Lagrangian C^0 flux conjecture in some cases — we refer the interested reader to Section 5.3.

Remark 4. *Note that one can upgrade from $\mathcal{L}(L)$ to $\mathsf{SMan}(L)$ under the additional condition that the ambient symplectic manifold is tame.*

Furthermore, one can upgrade from $\mathsf{SMan}(L)$ to SMan_n in some contexts as, for example, if $n = 2$. Indeed, any H-exact Lagrangian in the cotangent bundle of a surface has the same diffeomorphism type as that surface (see Lemma 24 below). This is a nontrivial update: Polterovich [Pol93] constructed Lagrangian tori in the cotangent bundle of any flat manifold; these tori can be made to be arbitrarily close to the zero-section. We discuss these examples in more details at the very end of Section 3.2.

We will explore in Section 1.2 below examples where these conditions are all satisfied.

The Lagrangian C^1 flux conjecture A natural variant of Speculation D is obtained by replacing closedness in the Hausdorff topology with closedness in the C^1 topology. We call this the Lagrangian C^1 flux conjecture.

By C^1 topology, we mean the one constructed as follows. Fix a Riemannian metric g on M. We say that a closed connected half-dimensional submanifold N' is ε-C^1-close to another one N if N' is in a tubular neighbourhood of N and there is a normal vector field ν along N such that $\|\nu\| < \varepsilon$ and $\exp(\nu(N)) = N'$. We then set

$$B(N, \varepsilon) := \left\{ N' \,\middle|\, N' \text{ is } \varepsilon\text{-}C^1\text{-close to } N \text{ \& vice-versa} \right\}.$$

The C^1 topology on SMan_n is then the topology generated by the $B(N,\varepsilon)$'s. One can easily check that this is independent of the choice of Riemannian metric.

With our methods, we get the following.

Corollary 10 *Let L be a H-rational Lagrangian in a tame symplectic manifold M. Then, $\mathcal{L}\mathrm{Ham}(L)$ and $\mathcal{L}\mathrm{Symp}_0(L)$ are C^1-closed in SMan_n.*

The reason there is no restriction on the diffeomorphism type as in Corollary 9 is because Lagrangians which are C^1-close to L are graphs of 1-forms in $\mathcal{W}(L)$, and graphs are necessarily H-exact. Likewise, the NLC is not needed since exact graphs are Hamiltonian isotopic to the zero-section in $\mathcal{W}(L)$. Note that C^1-close Lagrangians are necessarily diffeomorphic so that closure in SMan_n is the same as closure in $\mathsf{SMan}(L)$.

The Lagrangian C^1 flux conjecture has been studied previously by Ono [Ono08] and Solomon [Sol13] in the case when M is closed or a cotangent bundle. They proved that it holds when L has Maslov class zero and is unobstructed in the sense of [FOOO09] and when the so-called Lagrangian flux group of L is discrete, respectively. When L is H-rational, the Lagrangian flux group is automatically discrete. Therefore, our improvement with regards to

Solomon's result is that we allow M to be open — otherwise, we only have proved a subcase. As for Ono's, our condition is somewhat orthogonal to his: he needs no bad disks, but we ask for a lot of them.

1.2 Examples

We give a few examples where Speculation B follows from the results above. We will only specify a Lagrangian submanifold in a symplectic manifold without worrying about it being rational or not, since given an irrational Lagrangian submanifold L' of a rational symplectic manifold, we can construct a nearby rational Lagrangian L by an arbitrarily C^1-small perturbation.

First, from Theorem 2 we know that Conjecture C holds for closed connected manifolds L which are covered by $L_0 \times \mathbb{T}^m$, where $H_1(L_0; \mathbb{R}) = 0$ and m is allowed to be equal to zero. For example, this includes every manifold admitting a flat metric by Bieberbach's theorem. Curiously, such manifolds also include every mapping torus L of a diffeomorphism $f \in \mathrm{Diff}(L_0)$ of a simply connected manifold L_0 that has finite order in the smooth mapping class group, i.e. f^k is smoothly isotopic to id for some integer k. Indeed, the multiplication by k map $S^1 \to S^1$ pulls the bundle $L \to S^1$ to a bundle diffeomorphic to $L_0 \times S^1$. Furthermore, note that the class of manifolds covered by $L_0 \times \mathbb{T}^m$ (where m and L_0 are not fixed) is also closed under products, so that Conjecture C also holds for all products of the above examples.

Second, we give a few examples of Lagrangian submanifolds L where Speculation B holds. By Corollary 6, this includes those L such that $H_1(L; \mathbb{R}) = \mathbb{R}$ which admit embeddings to a Liouville domains W with $SH(W) = 0$. For instance if $TL_0 \otimes \mathbb{C}$ is trivial and $H_1(L_0; \mathbb{R}) = 0$, then $L = L_0 \times S^1$ embeds as a Lagrangian in $\mathbb{C}^n$ by the Gromov–Lees h-principle [Gro70, Lee76] and a result of Audin, Lalonde, and Polterovich [ALP94]. In another direction, Ekholm, Eliashberg, Murphy, and Smith [EEMS13] showed that, given any 3-manifold L_0, $L = L_0\#(S^1 \times S^2)$ embeds as a Lagrangian in $\mathbb{C}^3$. But, by the van Kampen theorem, $\pi_1(L) = \pi_1(L_0) * \pi_1(S^1 \times S^2)$, so that $H_1(L; \mathbb{R}) = H_1(L_0; \mathbb{R}) \oplus \mathbb{R}$. Therefore, Corollary 6 covers $L = L_0\#(S^1 \times S^2)$ with $H_1(L_0; \mathbb{R}) = 0$, e.g. L_0 can be a (connected sum of) lens spaces. Finally, the Lagrangian Grassmannian $L = \Lambda_n$ admits a Lagrangian embedding in $\mathrm{Sym}(\mathbb{C}^n) = \mathbb{C}^{n(n+1)/2}$ (see, for example, [ALP94]), and $\pi_1(\Lambda_n) = \mathbb{Z}$.

Specializing to the case of the 2-torus, where the NLC is indeed known to hold [RGI16], we get the following.

Corollary 11 *Speculations A and D hold for all H-rational Lagrangian 2-tori.*

Again, recall that the class for which Corollary 6 holds is closed under covering. But note that the Klein bottle admits a displaceable Lagrangian embedding in $S^2 \times \mathbb{C}$. It is obtained from the usual Lagrangian Klein bottle in $S^2 \times S^2$ (see, for example, [Eva22] and [AE24]) by removing a point on the second copy of S^2 and identifying $S^2 \setminus \{pt\}$ with $\mathbb{D} \subseteq \mathbb{C}$. Again, it is displaceable, because $\mathbb{D}$ is. In fact, the Klein bottle can even be made to be monotone.

We conclude with one additional case when we can establish Speculation A: when L is a 2-sphere or a projective plane. Indeed, any other such Lagrangian in $\mathcal{W}(L)$ is then automatically exact in that neighbourhood, so there is no need for Theorems 4 or 5. Since the NLC is known to hold in T^*S^2 [Hin04] and $T^*\mathbb{R}P^2$ [HPW16], we thus directly get the following.

Corollary 12 *Speculations A and D hold for Lagrangian* 2*-spheres or projective planes.*

1.3 Applications

We end this introduction with applications of our results. These are divided in four parts: additional rigidity results on Lagrangians with regards to Hamiltonian diffeomorphisms, further study on the local topology of $\mathcal{L}\mathrm{Ham}(L)$, new results on the space of (H-)rational Lagrangians with a fixed rationality constant, and some computations of numerical invariants. The next to last part has further implications when it comes to the space of all Lagrangians of a given symplectic manifold.

Except for a couple of references to further results, this last part of the introduction is intended to be self-contained; we directly give the proofs when these are not direct.

C^0 rigidity of Hamiltonian diffeomorphisms There is a natural variant of Speculation B where we ask not that L' be close to L, but rather that the Hamiltonian diffeomorphism sending L to L' be small. More precisely, we can make the following conjecture.

Conjecture E *Let L be a displaceable Lagrangian in a symplectic manifold M. There exists $\delta > 0$ with the following property. If φ is a Hamiltonian diffeomorphism of M and $d_{C^0}(\mathbb{1}, \varphi) < \delta$, then $L \cap \varphi(L) \neq \emptyset$.*

In other words, any Hamiltonian diffeomorphism displacing L is uniformly C^0-bounded away from 0.

The existence of such a bound is not at all trivial: if L is a displaceable n-dimensional submanifold which is not Lagrangian, then it can be displaced by an arbitrarily C^0-small Hamiltonian diffeomorphism [LS94]. Moreover, this does not follow from the fact that Lagrangians have positive displacement energy, since there are Hamiltonian diffeomorphisms which are arbitrarily C^0-small, but arbitrarily Hofer-large.

However, this is not expected to be the case for the spectral metric, that is, C^0-small Hamiltonian diffeomorphisms should also have small spectral norm. More precisely, Conjecture E follows from the fact that Lagrangians have positive spectral displacement energy [AAC23] in the cases where it is known that the spectral metric is C^0-continuous, i.e. when M is $\mathbb{C}^n$ [Vit92], a closed surface [Sey13], closed and symplectically aspherical [BHS21], $\mathbb{C}P^n$ [She22], or closed and negative monotone [Kaw22] or symplectically Calabi-Yau [SW].

In the context of this paper, Conjecture E is implied by Speculation A or by the Hamiltonian version of Speculation B above when it holds. However, it turns out to be much easier to prove than either one of these speculations. This is because we have the following lemma.

Lemma 13 *For every Lagrangian L, there exists $\delta > 0$ with the following property. Suppose that $\psi : M \to M$ is a map such that $d_{C^0}(\mathbb{1}, \psi) < \delta$ and $\psi(L)$ is Lagrangian. Then, $\psi(L)$ is H-exact in some $\mathcal{W}(L)$.*

Proof. Take a Riemannian metric g on M which corresponds to a Sasaki metric on T^*L on a Weinstein neighbourhood $\mathcal{W}(L)$. With such a metric, the geodesics starting at L and going to $L' = \psi(L)$ stay in $\mathcal{W}(L)$ (see Lemma A.4 of [Cha24] for example). Therefore, if we assume that δ is smaller than the injectivity radius $r_{\mathrm{inj}}(TM|_L)$ of the Riemannian exponential of g restricted to $TM|_L$, we get for every $x \in L$ a unique geodesic $\gamma_x : [0, 1] \to M$ such that $\gamma_x(0) = x$, $\gamma_x(1) = \psi(x)$, and $\gamma_x([0, 1]) \subseteq \mathcal{W}(L)$. Moreover, γ_x smoothly depends on x. Therefore, $(x, t) \mapsto \gamma_x(t)$ defines a smooth homotopy in $\mathcal{W}(L)$ from the inclusion $\iota : L \hookrightarrow \mathcal{W}(L)$ to $\psi\iota$. Since ι is a homotopy equivalence, then so must be $\psi\iota$. In particular, $H_2(\mathcal{W}(L), L') = 0$, and L' is H-exact. □

Then, it suffices to use Theorem 5 to get the following.

Corollary 14 *Conjecture E holds for all H-rational Lagrangians.*

In fact, at the price of making the constant δ depend on the size of the compact support of φ, a similar, more careful approach proves much more. We delay the proof of the Section 5.4 below.

Proposition 15 *Let L be a displaceable Lagrangian of M, and take a compact K containing L in its interior. Suppose furthermore that one of the following holds:*

- (*a*) *M is rational, i.e. $\omega(\pi_2(M))$ is discrete, or*
- (*b*) *the image of $\pi_1(L) \to \pi_1(M)$ is torsion.*

Then, there exists $\delta > 0$ with the following property. If φ is a Hamiltonian diffeomorphism supported in K and $d_{C^0}(\mathbb{1}, \varphi) < \delta$, then $L \cap \varphi(L) \neq \emptyset$.

Furthermore, we get a rigidity result for sequences of Hamiltonian or symplectic diffeomorphisms from Theorem 7 and Corollary 9.

Corollary 16 *Let $\{\psi_i\}$ be a sequence of symplectomorphisms with (weak) C^0 limit $\psi \in C^0(M, M)$, and let $L \in \mathcal{L}(\tau)$. If $\psi(L)$ is a smooth n-submanifold, then $\psi(L) \in \mathcal{L}(\tau)$.*

If, furthermore, the NLC holds on T^*L and

- (*a*) *if $\{\psi_i\} \subseteq \mathrm{Ham}(M)$, then $\psi(L) \in \mathcal{L}\mathrm{Ham}(L)$;*
- (*b*) *if $\{\psi_i\} \subseteq \mathrm{Symp}_0(M)$, then $\psi(L) \in \mathcal{L}\mathrm{Symp}_0(L)$.*

Note that a similar result about the continuity of the area spectrum under C^0 limits was shown by Membrez and Opshtein [MO21].

Local contractibility of $\mathcal{L}\mathrm{Ham}(L)$ in the Hausdorff topology While we need the NLC in all cases where we can prove Speculation A, the fact that $\mathcal{L}\mathrm{Ham}(L)$ is locally path connected in the Hausdorff topology turns out to be easier to prove. More precisely, we get the following.

Corollary 17 *Suppose that L is H-rational and respects the hypotheses of Theorem 4. Then $\mathcal{L}\mathrm{Ham}(L)$ is locally contractible in the Hausdorff topology.*

Note that this is not quite the type of results we mentioned earlier in the introduction. Indeed, we do not claim that the Hausdorff-continuous path from L' to L (as subsets of the symplectic manifold) in a small neighbourhood of L is generated by an actual Hamiltonian isotopy, simply that it consists at all times of elements of $\mathcal{L}\mathrm{Ham}(L)$.

Proof. Note that it suffices to prove this statement at L. Fix a Weinstein neighbourhood $\Psi : D_r^*L \to \mathcal{W}(L)$ as given by the conclusions of Theorems 4 and 5. Then, every $L' \in \mathcal{L}\mathrm{Ham}(L)$ which is in $\mathcal{W}(L)$ is exact in that neighbourhood. We can thus take

$$(L', t) \mapsto \Psi(t\Psi^{-1}(L'))$$

to be the contraction. Indeed, exactness in $\mathcal{W}(L)$ ensures that this is a Hamiltonian isotopy for all $t > 0$. In particular $\Psi(t\Psi^{-1}(L'))$ is Hamiltonian isotopic to L' and hence it is in $\mathcal{L}\mathrm{Ham}(L)$. But exactness also implies that the projection $\Psi^{-1}(L') \to T^*L \to L$ is a homotopy equivalence [AK18]. In particular, that projection must be surjective, otherwise $H_n(L') \to H_n(L) \neq 0$ would be zero. Therefore, L' being close to L implies that L is close to L' and hence that L' and L are Hausdorff-close. This means that the Hausdorff limit of $\Psi(t\Psi^{-1}(L'))$ as $t \to 0$ is precisely L, i.e. the above contraction is indeed Hausdorff-continuous. □

Spaces of Lagrangians with fixed H-rationality constant We now turn our attention to the space $\mathcal{L}(L, \tau)$ of all Lagrangians of M with the diffeomorphism type of L and H-rationality constant τ.

From Theorems 4 and 5, we get the following.

Corollary 18 *Let L be a H-rational Lagrangian in a symplectic manifold, and denote by τ its H-rationality constant. Then, $\mathcal{L}\mathrm{Symp}_0(L)$ is open in $\mathcal{L}(L, \tau)$ in the C^1 topology. If moreover L respects the hypotheses of Theorem 4 or $H_1(L; \mathbb{R}) = 0$ and the NLC holds on T^*L, then the same holds in the Hausdorff topology.*

Proof. Note that it suffices to prove that there is an open neighbourhood of L in $\mathcal{L}(L, \tau)$ which is fully in $\mathcal{L}\mathrm{Symp}_0(L)$. Let thus $\Psi : D_r^*L \to \mathcal{W}_r(L)$ be the Weinstein neighbourhood given by Proposition 8. Then, every graph in $\mathcal{W}(L)$ must be, up to a global symplectic isotopy, exact. Since exact graphs are Hamiltonian isotopic to the zero-section, such a graph must thus be in $\mathcal{L}\mathrm{Symp}_0(L)$. This proves the C^1 case.

For the Hausdorff case, suppose that r is also small enough so that Theorem 4 and Proposition 8 hold in $\mathcal{W}_r(L)$. Then, any $L' \in \mathcal{L}(L, \tau)$ such that $L' \subseteq \mathcal{W}(L)$ must be, up to some global symplectic isotopy, exact in $\mathcal{W}(L)$ — we still denote by L' its image under the isotopy. As in the proof of Corollary 17, we note that the path $t \mapsto \Psi(t\Psi^{-1}(L'))$, $t \in [0, 1]$, is continuous in the Hausdorff topology. Furthermore, it is a Hamiltonian isotopy for all $t > 0$. In particular, L must be in the Hausdorff closure of $\mathcal{L}\mathrm{Ham}(L') \subseteq \mathcal{L}\mathrm{Symp}_0(L')$. But $\mathcal{L}\mathrm{Symp}_0(L')$ is Hausdorff closed by Corollary 9 and the hypotheses on L. Therefore, $L' \in \mathcal{L}\mathrm{Symp}_0(L)$. □

Together with the Lagrangian flux conjectures, this yields the following result.

Corollary 19 *Let L and τ be as above. The (path) connected components of $\mathcal{L}(L,\tau)$ in the C^1 topology are precisely the orbits of $\mathrm{Symp}_0(M)$. In particular, the quotient $\mathcal{L}(L,\tau)/\mathrm{Symp}_0(M)$ is discrete in the induced topology. If moreover L respects the hypotheses of Theorem 4 or $H_1(L;\mathbb{R}) = 0$ and the NLC holds on T^*L, then the same holds in the Hausdorff topology.*

For example, this means that a ρ-monotone Clifford torus can never be reached from a Chekanov torus (or any monotone special torus) by a Hausdorff-continuous path in $\mathcal{L}(\mathbb{T}^2, 2\rho)$. Contrast this with the fact that all these tori are Lagrangian isotopic [RGI16].

Proof. Combining Corollaries 10 and 18, we get that for all $L \in \mathcal{L}(L,\tau)$, the orbit $\mathcal{L}\mathrm{Symp}_0(L)$ is both closed and open in $\mathcal{L}(L,\tau)$ with the C^1 topology. Therefore, $\mathcal{L}\mathrm{Symp}_0(L)$ must be a union of connected components of $L \in \mathcal{L}(L,\tau)$ by point-set topology. Since $\mathcal{L}\mathrm{Symp}_0(L)$ is obviously path connected, it must be both a connected component and a path connected component of $L \in \mathcal{L}(L,\tau)$. The proof in the Hausdorff setting is completely analogous. □

Note that, when $H_1(L;\mathbb{R}) \to H_1(M;\mathbb{R})$ is zero, the role of $\mathcal{L}\mathrm{Symp}_0(L)$ in the above proof can be replaced by $\mathcal{L}\mathrm{Ham}(L)$. In particular, both $\mathcal{L}\mathrm{Symp}_0(L)$ and $\mathcal{L}\mathrm{Ham}(L)$ are the connected component of $\mathcal{L}(\tau)$ containing L in the C^1 topology, so they must be equal. This can be seen as a generalization that $\mathrm{Symp}_0(M) = \mathrm{Ham}(M)$ for closed manifolds with $H_1(M;\mathbb{R}) = 0$.

Corollary 20 *Let L be H-rational and such that $H_1(L;\mathbb{R}) \to H_1(M;\mathbb{R})$ is zero. There is a symplectic isotopy $\{\psi_t\}$ of M such that $\psi_1(L) = L'$ if and only if there is a Hamiltonian isotopy $\{\varphi_t\}$ such that $\varphi_1(L) = L'$. In other words, $\mathcal{L}\mathrm{Symp}_0(L) = \mathcal{L}\mathrm{Ham}(L)$.*

We also have the following.

Corollary 21 *The space $\cup_{\tau\geq 0}\mathcal{L}(L,\tau)/\mathrm{Symp}_0(M)$ is Hausdorff in the topology induced by the C^1 topology. In particular, the quotient $\mathcal{L}\mathrm{Lag}(L)/\mathrm{Symp}_0(M)$ can only be non-Hausdorff at orbits corresponding to H-irrational Lagrangians. The same holds for $\mathrm{Symp}_0(M)$ replaced by $\mathrm{Ham}(M)$.*

The part on $\mathrm{Symp}_0(M)$ follows directly from Corollary 19. The part with $\mathrm{Ham}(M)$ is a finer result that also makes use of the local description of L in $\mathcal{L}(L,\tau)$ given by Corollary 43 below.

It has been proven by Ono [Ono08] and Solomon [Sol13] that the quotient $\mathcal{L}\mathrm{Lag}(L)/\mathrm{Ham}(M)$ is Hausdorff in the C^1 topology in different settings. Most notably, they both ask that $H_1(L;\mathbb{R}) \to H_1(M;\mathbb{R})$ be injective, which makes L automatically H-exact. Corollary 21 shows the difficulty of relaxing the condition that $H_1(L;\mathbb{R}) \to H_1(M;\mathbb{R})$ be injective: H-irrational Lagrangians can create non-Hausdorff points in the quotient. In fact, Theorem 1 shows that in dimension $2n \geq 6$, this always happens. That this is a problem was already mentioned by Ono in his work on the subject.

Quantitative symplectic topology When Theorem 4 holds, it allows for a new measurement associated with a Lagrangian embedding $Q \hookrightarrow M$ with image L and a Riemannian metric g on Q:

$$c^e_{(M,L)}(Q,g) := \sup\left\{ r \geq 0 \,\middle|\, \text{all } L' \in \mathcal{L}\mathrm{Ham}(L) \text{ in } \; W^g_r(L) \text{ are exact} \right\}.$$

By writing $\mathcal{W}^g_r(L)$, we want to underline that it is the image of a Weinstein neighbourhood $\Psi : D^*_r Q \to \mathcal{W}_r(L)$, where the radius r of the codisk bundle is computed using g. We write $c^e_{(M,L)}(Q,g) = 0$ if L has no neighbourhood of local exactness, e.g. for the example given by Theorem 1.

Note that $c^e_{(M,L)}(Q,g)$ is invariant under symplectomorphisms, so it is truly a symplectic quantity. Furthermore, $c^e_{(M,L)}(Q,g)$ is bounded from above by the size of the largest Weinstein neighbourhood of L in M, i.e. by the relative capacity

$$c^{\mathcal{W}}_{(M,L)}(Q,g) := \sup\left\{ r > 0 \,\middle|\, L \text{ admits a neighbourhood } \mathcal{W}^g_r(L) \right\}.$$

This can in turn be bounded in terms of Poisson bracket invariants of L in M [MO21].

When $L = S^1$, a direct computation gives the following estimate.

Corollary 22 *Let L be a closed curve in a surface M. If L bounds an embedded disk, let A be the smallest area of such a disk. If there are no such disks, we set $A = +\infty$. We have that*

$$c^e_{(M,L)}(S^1, g_0) = \min\left\{ \frac{A}{2}, c^{\mathcal{W}}_{(M,L)}(S^1, g_0) \right\},$$

where g_0 is the flat metric so that S^1 has length 1.

Note that $\frac{A}{2}$ is precisely half the radius of the largest Weinstein neighbourhood of the circle $T(A)$ enclosing area A in $\mathbb{C}$, i.e. $c^e_{(\mathbb{C},S^1(A))}(S^1, g_0) = \frac{1}{2} c^{\mathcal{W}}_{(\mathbb{C},S^1(A))}(S^1, g_0)$.

In general, however, it is hard to get an estimate on $c^e_{(M,L)}(Q,g)$, as it is hard to get one on the neighbourhood for which Theorem 4 holds. One exception to this is when $Q = K$ is the Klein bottle: in this case, the theorem holds on every Weinstein neighbourhood (see Theorem 45 below). Therefore, the bound comes only from the proof of Theorem 5 — more precisely, from Lemma 40 and Proposition 41. In particular, we have the following bound.

Corollary 23 *Let L be a H-rational Lagrangian Klein bottle with H-rationality constant τ. We have that*

$$c^e_{(M,L)}(K,g) \geq \min\left\{ \frac{\tau}{\ell^{\min}_g(\beta)}, c^{\mathcal{W}}_{(M,L)}(K,g) \right\},$$

where $\ell^{\min}_g(\beta)$ denotes the minimal length in g of a curve representing the generator β of the free factor of $H_1(K;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$.

Remark 5. *There are of course many variations of $c^e_{(M,L)}(Q,g)$ that one could take. For example, one could be interested in $c^A_{(M,L)}(Q,g)$ or $c^B_{(M,L)}(Q,g)$, the largest neighbourhood on which Speculation A or Speculation B, respectively, holds. However, if one believes in the NLC, then we should always have $c^A = c^e$. Moreover, we have not found an example where $c^B \neq c^{\mathcal{W}}$. Therefore, c^e seems to be the more fruitful version of the relative capacity.*

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2 Irrational counterexamples

We now explain the construction of the Lagrangian tori from Theorem 1. These tori support the fact that we need in general to require some type of ***rationality*** condition on our Lagrangians for Speculations A, B or D to hold.

We start with the case when $M = \mathbb{C}^3$. Consider the product torus $L = T(1,2,1+\alpha) := T(1) \times T(2) \times T(1+\alpha)$, where $\alpha > 0$ is an irrational number and $T(A) \subseteq \mathbb{C}$ denotes the round circle enclosing area $A > 0$. By work of Chekanov [Che96], another product torus $T(a, a+b, a+c)$ with $a,b,c > 0$ is Hamiltonian isotopic to L in $\mathbb{C}^3$ if and only if $a = 1$ and $\mathrm{span}_{\mathbb{Z}}\{b,c\} = \mathrm{span}_{\mathbb{Z}}\{1,\alpha\} =: G$.

Let p_i/q_i be the ith convergent to α obtained from its infinite continued fraction. In particular p_i, q_i are coprime and $|p_i - q_i\alpha| < \frac{1}{q_{i+1}}$. Fix $\varepsilon > 0$. Then for all $i \geq i_0$ sufficiently large $|p_i - q_i\alpha| < \varepsilon$. Moreover, the matrix

$$\begin{pmatrix} p_i & q_i \\ p_{i+1} & q_{i+1} \end{pmatrix}$$

has determinant ± 1 and hence for $i \geq i_0$, $b = p_i - q_i\alpha$, $c = p_{i+1} - q_{i+1}\alpha$ satisfy $\mathrm{span}_{\mathbb{Z}}\{b,c\} = G$ while also $|b| < \varepsilon, |c| < \varepsilon$. By changing the signs of b, c if necessary, which preserves both conditions, we can ensure that $b > 0, c > 0$.

This means that we can take $T(1, 1+b, 1+c)$ which are all in the Hamiltonian orbit of L but are arbitrarily C^1-close to the monotone torus $T(1,1,1)$. This proves *(ii)* of Theorem 1 in $\mathbb{C}^3$ and shows that, without the H-rational hypothesis on L, not even the Lagrangian C^1 flux conjecture is true in $\mathbb{C}^3$.

Note that a similar argument as above actually implies that the set of $b, c > 0$ such that $T(1, 1+b, 1+c)$ is Hamiltonian isotopic to L is dense in $\mathbb{R}^2_{>0}$. This means that any neighbourhood U of such a torus $T(1, 1+b, 1+c)$ contains infinitely many $T(1, 1+b', 1+c')$ in the same Hamiltonian orbit. But $T(1, 1+b, 1+c) \cap T(1, 1+b', 1+c') = \varnothing$ if $b \neq b'$ or $c \neq c'$. This proves *(i)* of Theorem 1 in $\mathbb{C}^3$ and shows that, without the H-rational hypothesis on L, not even Speculation B is true in $\mathbb{C}^3$. Note that this also shows that the Lagrangian flux group of Lagrangian isotopies need not be discrete since that of $T(1,2,1+\alpha)$ is not.

We now explain how to generalize the result to any manifold of dimension $2n \geq 6$. First note that by taking a product with $T(1)^{n-3}$, we get a counterexample to our speculations in $\mathbb{C}^n$ whenever $n \geq 3$. Furthermore, by Theorem 1.1(ii) of [CS16], the Hamiltonian isotopy from $T(1,\ldots,1,1+b,1+c)$ to $T(1,\ldots,1,1+b',1+c')$ can be taken to be fully supported in the ball $B^{2n}(A)$ of capacity $A = n+1+\max\{b+c, b'+c'\}$, i.e. of radius $\sqrt{\frac{A}{\pi}}$. In particular, for b, b', c, and c' small enough, it can be supported in the ball of capacity $n+2$. Therefore, we get a counterexample in $M = B^{2n}(n+2)$ But then, by simply rescaling the ball, we get a counterexample in the ball $B^{2n}(A)$ for any $A > 0$. By the Darboux theorem, any symplectic manifold M^{2n} admits a symplectic embedding of the ball $B^{2n}(A)$ for A small enough, which gives the counterexample for every M with $\dim M \geq 6$.

Remark 6. *Interestingly enough, the above counterexample does not work in dimension* 4*. Indeed, Chekanov's classification of product tori implies that every product torus L in $\mathbb{C}^2$ has a C^1 neighbourhood U such that $\mathcal{L}\mathrm{Ham}(L) \cap U = \{L\}$. In particular, the C^1 version of Speculation A holds for L, and if its Hamiltonian orbit is not closed, then the limit cannot be a product or a Chekanov torus. By Theorem* 1.3 *of* [CS16], *the same holds for product tori in small enough Darboux balls in subtame symplectically aspherical symplectic* 4*-manifolds.*

However, we expect that one can use Theorem 1.5 *of* [CS16] *to construct — in a similar fashion as above — a counterexample to Speculation B in any (spherically) irrational symplectic* 4*-fold.*

3 Relations between homological rationality and exactness and their standard counterparts

In this section, we discuss relations between standard rationality/exactness and H-rationality/H-exactness. In Section 3.1, we prove the general following fact: for a closed Lagrangian of the cotangent bundle, being H-exact is equivalent to being isotopic to an exact Lagrangian. In Section 3.2, we explain some specific situations in which H-rationality reduces to rationality. Finaly in Section 3.3, we give an example which illustrates why we generally work with the H-rationality condition rather than the standard rationality one.

3.1 The central lemma

The following lemma will prove to be quite useful in order to prove the main results of this work.

Lemma 24 *Let L be a closed Lagrangian in T^*Q, the following are equivalent:*

(*i*) *L is H-exact;*

(*ii*) *the Liouville form $[\lambda_0|_L]$ is in the image of the map in $H^1(\cdot;\mathbb{R})$ induced by the composition $L \to T^*Q \to Q$;*

(*iii*) *the composition $L \to T^*Q \to Q$ is a homotopy equivalence;*

(*iv*) *L is isotopic to an exact Lagrangian through Lagrangian submanifolds;*

(*v*) *L is symplectically isotopic to an exact Lagrangian;*

(*vi*) *L is a shift of an exact Lagrangian by a closed one-form on Q.*

Proof. We first show that *(i)* and *(ii)* are equivalent. To see this, note that H-exactness is equivalent to $\lambda_0|_L$ vanishes on the image of $H_2(T^*Q, L; \mathbb{R}) \to H_1(L; \mathbb{R})$. But this is equivalent to $\delta^*[\lambda_0|_L] = 0$, where $\delta^* : H^1(L; \mathbb{R}) \to H^2(T^*Q, L; \mathbb{R})$ is the coboundary operator. By the long exact sequence in cohomology, this is equivalent to $[\lambda_0|_L]$ being in the image of the map induced by the inclusion $i : L \hookrightarrow T^*Q$. We then conclude using the fact that the projection $\pi : T^*Q \to Q$ is a homotopy equivalence.

Suppose we have *(ii)*. This means that there exists a closed 1-form $\sigma \in \Omega^1(Q)$ such that $[i^*\lambda] = [i^*(\pi^*\sigma)]$. Now, σ induces a fibrewise symplectomorphism ψ_σ of T^*Q which satisfies $[\psi_\sigma^*(i^*\lambda)] = 0$ so that ψ_σ maps L to an exact Lagrangian. Furthermore, ψ_σ generates a shift by σ. This shows that *(ii)* yields *(vi)*, which obviously yields *(v)* and thus *(iv)*.

Note also that *(iv)* implies by definition that the inclusion $i : L \hookrightarrow T^*Q$ is homotopic to the inclusion of an exact Lagrangian. Indeed, L being Lagrangian isotopic to L' means precisely that there is a smooth map $F : [0,1] \times L \to T^*Q$ such that $F_0 = i$, F_t for all t is a Lagrangian embedding, and $F_1(L) = L'$, where $F_t : L \to T^*Q$ is given by $F_t(x) = F(t, x)$. But, when L is exact, the composition $L \to T^*Q \to Q$ is a (simple) homotopy equivalence [AK18], i.e. *(iii)* holds. Finally, note that *(iii)* directly implies *(ii)*. □

Remark 7. *The notions (vi) and (ii) appear in [AS24] under the names of* almost exact *and H_1*-standard *Lagrangians in T^*Q, respectively.*

3.2 When does H-rationality reduce to rationality?

Obviously, H-rationality implies usual rationality, i.e. $\omega(H_2(M, L))$ being discrete implies that $\omega(\pi_2(M, L))$ also is. Furthermore, in many cases, these conditions are equivalent. This is the case, for example, when $\pi_1(M) = 0$. Indeed, in this case, the relative Hurewicz morphism $\pi_2(M, L) \to H_2(M, L; \mathbb{Z})$ can be shown to be surjective. Expanding on this idea, we get the following.

Lemma 25 *Suppose that $[\pi_1(M), \pi_1(M)]$ is finite. Then, we have that*

$$N\omega(H_2(M, L; \mathbb{Z})) \subseteq \omega(\pi_2(M, L)) + \omega(H_2(M; \mathbb{Z})),$$

where N is the order of $[\pi_1(M), \pi_1(M)]$. In particular, if $\pi_1(M)$ is abelian, then we have equality.

Proof. We consider the following commutative diagram.

$$\begin{array}{ccccccc}
\pi_2(M) & \xrightarrow{j} & \pi_2(M, L) & \xrightarrow{\partial} & \pi_1(L) & \xrightarrow{i} & \pi_1(M) \\
\downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_2''} & & \downarrow{\scriptstyle h_1'} & & \downarrow{\scriptstyle h_1} \\
H_2(M) & \xrightarrow{j} & H_2(M, L) & \xrightarrow{\partial} & H_1(L) & \xrightarrow{i} & H_1(M)
\end{array}$$

Here, the rows are the long exact sequences of the pair (M, L) in homotopy and homology with integer coefficients, respectively, and the columns are the various Hurewicz morphisms; it commutes by naturality of the Hurewicz map.

The proof follows from a straightforward diagram chasing argument, but we still give the details. Let $A \in H_2(M, L)$. Since h_1' is surjective — the Hurewicz morphism in first degree is simply the abelianization morphism — there is some $a \in \pi_1(L)$ such that $\partial(A) = h_1'(a)$. But note that

$$h_1 i(a) = i h_1'(a) = i\partial(A) = 0.$$

Therefore, $i(a) \in \operatorname{Ker} h_1 = [\pi_1(M), \pi_1(M)]$. By hypothesis, $i(Na) = 0$. Therefore, there is some $u \in \pi_2(M, L)$ such that $\partial(u) = Na$. But note that

$$\partial(NA - h_2''(u)) = h_1'(Na) - h_1'(\partial u) = 0.$$

There is thus some $B \in H_2(M)$ such that $NA = j(B) + h_2''(u)$. To conclude, we only note that $\omega(j(B)) = \omega(B)$ and $\omega(h_2''(u)) = \omega(u)$. □

Note that this corollary recovers the statement at the start of the subsection that H-rationality and rationality are the same when $\pi_1(M) = 0$. In the case $M = D_r^* L$, the condition that $\omega(H_2(M; \mathbb{Z})) = 0$ is automatically satisfied since ω is exact, and the condition on the commutator subgroup of $\pi_1(M)$ becomes that $[\pi_1(L), \pi_1(L)]$ be finite. Therefore, we get directly the following result.

Corollary 26 *Suppose that $[\pi_1(L), \pi_1(L)]$ is finite. Then, Conjecture C is equivalent to its homotopical version, i.e. the one where K is only rational, and we get a bound on its rationality constant.*

Remark 8. *The homotopical version of Theorem 2, that is, Theorem 39 below, proves the homotopical version of Conjecture C when $L = L_0 \times \mathbb{T}^m$ with $H_1(L_0; \mathbb{R}) = 0$. This intersects with the above corollary precisely in the case when $\pi_1(L_0)$ is finite.*

Using Corollary 26, we can obtain a homotopical version of Theorem 4. However, the proof of Theorem 5 truly requires H-exactness. One exception to this is when $\pi_2(M, L) \to \pi_1(L)$ is surjective, but an argument similar to the above shows that rationality is then equivalent to H-rationality as long as $q\omega(H_2(M; \mathbb{Z})) = \omega(\pi_2(M, L))$ for some $q \in \mathbb{Q}$, which is the case generically (and is always the case if $[\pi_1(L), \pi_1(L)]$ is finite).

3.3 Why H-rationality in general?

We end this section with an example which showcases the need to work with H-rational Lagrangians and not just rational Lagrangians for many applications. Note that this example is such that $[\pi_1(L), \pi_1(L)]$ is not finite.

In [Pol93], Polterovich constructs for any vector $v \in \mathbb{R}^n$ and any flat manifold Q a Lagrangian torus L_v in T^*Q. This torus has the property that

(i) for a contractible open $U \subseteq Q$, $L_v \cap T^*Q|_U = U \times \{v\} \subseteq U \times \mathbb{R}^n$;

(ii) the map $L_v \to Q$ given by restriction of $\pi : T^*Q \to Q$ is a covering.

We concentrate our efforts on the simplest case: $n = 2$ and $Q = K$ is the Klein bottle. In that case, $L_v \to K$ is the 2:1 cover.

First note that L_v is weakly exact in T^*K. To see this, denote by $p : \mathbb{T}^2 \to K$ the 2:1 cover and take $\widetilde{p} : T^*\mathbb{T}^2 \to T^*K$ to be its lift using the flat metrics on $\mathbb{T}^2$ and K. Point (i) gives that $\widetilde{p}^{-1}(L_v) = \mathbb{T}^2 \times \{v\} \subseteq T^*\mathbb{T}^2 = \mathbb{T}^2 \times \mathbb{R}^2$. But any disk u with boundary along L_v admits a lift $\widetilde{u}$ in $T^*\mathbb{T}^2$ with boundary along $\mathbb{T}^2 \times \{v\}$. Since $\mathbb{T}^2 \times \{v\} \hookrightarrow T^*\mathbb{T}^2$ is a homotopy equivalence, $\pi_2(T^*\mathbb{T}^2, \mathbb{T}^2 \times \{v\}) = 0$, and the lift $\widetilde{u}$ is contractible. But then, so must be u, and we have that $\pi_2(T^*K, L_v) = 0$.

On the other hand, L_v is ***not*** H-exact. Indeed, let $\gamma : S^1 \to K$ be a loop admitting a lift to L_v, that is, $[\gamma] \in p_*(\pi_1(\mathbb{T}^2))$. Since $L_v \to K$ is a 2:1 cover, there are two lifts $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ of γ. Furthermore, each lift $\widetilde{\gamma}_i$ defines a cylinder C_i in T^*K by taking $C_i(s,t) = t\widetilde{\gamma}_i(s)$, $(s,t) \in S^1 \times [0,1]$. Note that $\partial C_i = \widetilde{\gamma}_i \sqcup -\gamma$, where the minus sign denotes the reversal of orientation. Therefore, $C := C_1 \cup_\gamma -C_2$ is a cylinder in T^*K with boundary along L_v. Furthermore, it has area

$$\omega_0(C) = \lambda_0(\widetilde{\gamma}_1) - \lambda_0(-\widetilde{\gamma}_2) = 2\int_{S^1} \langle v, \dot{\gamma}(s) \rangle ds,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product. In particular, if we take γ to be a simple loop corresponding to a straight line in the fundamental domain of $\mathbb{R}^2$ defining $K = \mathbb{R}^2/\pi_1(K)$ and v to be positively proportional to $\dot{\gamma}$, then $\omega_0(C) = 2|v| > 0$. Therefore, such an L_v is indeed not H-exact.

Finally, note that, as $v \to 0$, $L_v \to K$ in the Hausdorff topology. Therefore, however small we take a neighbourhood of the zero-section of T^*K, there is a weakly exact Lagrangian in that neighbourhood which is not exact. Therefore, there is no homotopical equivalent of Theorem 5 with $\mathcal{L}\mathrm{Ham}(L)$ replaced by the space of all τ-rational Lagrangians if $[\pi_1(L), \pi_1(L)]$ is not finite. In particular, many applications in the introduction do not have equivalents in spaces of τ-rational Lagrangians.

4 Proofs of Theorems 2 and 4

We now turn our attention to Theorem 2. To obtain a proof, we first introduce some capacities inspired by work of Cieliebak and Mohnke [CM18] (Section 4.1), and we explain how their finiteness implies the first half of the theorem (Section 4.2). We show that this straightforwardly yields Theorem 4 (Section 4.3). We then prove that Conjecture C is closed under covering, which concludes the proof of Theorem 2 (Section 4.4). Finally, we state and prove its homotopical variant (Section 4.5). Note that the methods developed here will also be central to the proof of Theorem 7.

4.1 Some capacities *à la* Cieliebak–Mohnke

In [CM18], Cieliebak and Mohnke introduce — and compute in some cases — a capacity which measures, in a given domain, the largest possible area of a minimal disk with boundary along a Lagrangian torus. We start by introducing a small tweak in their definition, which will turn out to be quite useful in our setting.

Let Q be a closed connected n-manifold. For any $2n$-dimensional symplectic manifold (X,ω), we define three classes of Lagrangians:

$$\mathscr{L}_Q(X) := \{L = \mathrm{Im}(f : Q \hookrightarrow X) \mid f^*\omega = 0,\ \omega(H_2(X,L;\mathbb{Z})) \neq 0\}$$
$$\mathscr{L}_Q'(X) := \{L = f(Q) \in \mathscr{L}_Q(X) \mid \mathrm{im}(H_1(f)\otimes\mathbb{R}) \neq H_1(X;\mathbb{R}) \text{ or } H_1(X;\mathbb{R}) = 0\}$$
$$\mathscr{L}_Q^0(X) := \{L = f(Q) \in \mathscr{L}_Q(X) \mid H_1(f)\otimes\mathbb{R} = 0\}$$

where $H_1(f)\otimes\mathbb{R}$ is the map induced by f on first homology with real coefficients.

Lemma 27 *We always have the inclusions $\mathscr{L}_Q^0(X) \subset \mathscr{L}_Q'(X) \subset \mathscr{L}_Q(X)$.*
Moreover, we have the following particular cases.

- *If $H_1(X;\mathbb{R}) = 0$, we have $\mathscr{L}_Q^0(X) = \mathscr{L}_Q'(X) = \mathscr{L}_Q(X)$.*
- *If $\dim H_1(Q;\mathbb{R}) = 1$ and X is exact, we have $\mathscr{L}_Q^0(X) = \mathscr{L}_Q'(X) = \mathscr{L}_Q(X)$.*
- *If $\dim H_1(Q;\mathbb{R}) \leq \dim H_1(X;\mathbb{R})$ and X is exact, we have $\mathscr{L}_Q'(X) = \mathscr{L}_Q(X)$.*

Proof. The general statement and the first point are obvious.

Now, we fix Q and X as in the second point. We can assume that there is a Lagrangian embedding $f : Q \hookrightarrow X$; otherwise all sets are empty. Since $\dim H_1(Q;\mathbb{R}) = 1$, $H_1(f)\otimes\mathbb{R}$ is either 0 or injective. Suppose that it is injective. By the long exact sequence in homology, we then get that the boundary map $\partial : H_2(X,L;\mathbb{R}) \to H_1(L;\mathbb{R})$ is zero, where $L = f(Q)$. Since $\omega(H_2(X,L;\mathbb{R})) = \lambda(\partial(H_2(M,L;\mathbb{R})))$ whenever $\omega = d\lambda$, we then conclude that L is H-exact. In particular, $L \notin \mathscr{L}_Q(X)$. Therefore, we have that $\mathscr{L}_Q(X) = \mathscr{L}_Q^0(X)$.

Finally, for the third point, $\omega(H_2(X,L;\mathbb{Z})) \neq 0$ implies by the long exact sequence of a pair that $H_1(f)\otimes\mathbb{R}$ has non-trivial kernel and therefore $\dim \mathrm{im}(H_1(f)\otimes\mathbb{R}) \leq \dim H_1(X;\mathbb{R}) - 1$. □

In turn, this defines three capacities:

$$c_Q(X) := \sup\{A^H_{\min}(L,X) \mid L \in \mathscr{L}_Q(X)\} \in [0,+\infty];$$
$$c_Q'(X) := \sup\{A^H_{\min}(L,X) \mid L \in \mathscr{L}_Q'(X)\} \in [0,+\infty];$$
$$c_Q^0(X) := \sup\{A^H_{\min}(L,X) \mid L \in \mathscr{L}_Q^0(X)\} \in [0,+\infty],$$

where

$$A^H_{\min}(L,X) := \inf\{\omega(u) \mid u \in H_2(X,L;\mathbb{Z}), \omega(u) > 0\}.$$

We take the convention that $c_Q(X) = 0$ (respectively $c_Q^0(X) = 0$, $c_Q'(X) = 0$) if $\mathscr{L}_Q(X) = \varnothing$ (respectively $\mathscr{L}_Q^0(X) = \varnothing$, $\mathscr{L}_Q'(X) = \varnothing$). Obviously, we have that $c_Q^0 \leq c_Q' \leq c_Q$. Finally, we set

$$c_{\mathrm{all}}(X) := \sup c_Q(X),$$
$$c_{\mathrm{all}}'(X) := \sup c_Q'(X), \quad \text{and}$$
$$c_{\mathrm{all}}^0(X) := \sup c_Q^0(X),$$

where the supremum runs over all closed connected n-dimensional manifolds.

Remark 9. The main differences between our definition and Cieliebak–Mohnke's are that we work with homology instead of homotopy, we allow any Q and not only tori, and we only look at Lagrangians which do bound some homology class with nonvanishing area. The latter is central to our argument, as we will mainly be interested in the case $X = D^*Q$, but such a manifold obviously admits an exact Lagrangian Q. Therefore, without this restriction, $c_Q(D^*Q)$ would be infinite for trivial reasons, which runs counter to our purpose here.

However, we could develop an entirely analogous theory using homotopy and get some homotopical version of Theorem 4 — see Section 3.2 for a discussion.

The following properties follow directly from the definition of the capacities.

Lemma 28 *Let c denote c_Q, c'_Q, c^0_Q, c_{all}, c'_{all} or c^0_{all}. We have the two following properties.*

(*i*) *For all $\alpha \neq 0$, we have that $c(X, \alpha\omega) = |\alpha| c(X, \omega)$.*

(*ii*) *If there is a 0-codimensional symplectic embedding $\iota : X \hookrightarrow X'$ such that $H_2(X', \iota(X); \mathbb{R}) = 0$, then $c(X) \leq c(X')$.*

The problem with the monotonicity property (ii) when $H_2(X', \iota(X); \mathbb{R}) \neq 0$ is that there could then be homology classes in X' with smaller area than those in X — thus inverting the expected direction of the inequality. However, the capacity c^0_Q partially goes around that issue.

Lemma 29 *If there exists a 0-codimensional symplectic embedding $\iota : X \hookrightarrow X'$ and X' is exact, then $c^0_Q(X) \leq B c^0_Q(X')$, where $B \geq 1$ only depends on the torsion part of $H_1(X; \mathbb{Z})$.*

Proof. Let λ' be a primitive of the symplectic form of ω' on X'. Then, $\lambda = \iota^*\lambda'$ is a primitive of ω on X. Fix $L = f(Q) \in \mathscr{L}^0_Q(X)$. Since $H_1(f) \otimes \mathbb{R} = 0$, we must have that $f_*(H_1(Q; \mathbb{Z}))$ is a torsion subgroup of $H_1(X; \mathbb{Z})$. Take B to be the order of the torsion of $H_1(X; \mathbb{Z})$ if it is nonzero, i.e. if

$$H_1(X; \mathbb{Z}) = \mathbb{Z}^b \oplus \mathbb{Z}_{p_1^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_\ell^{k_\ell}},$$

then $B = p_1^{k_1} \dots p_\ell^{k_\ell}$. If $H_1(X; \mathbb{Z})$ has no torsion, then we simply set $B = 1$. We thus have $B \cdot f_*(H_1(Q; \mathbb{Z})) = 0$. By the homology long exact sequence of the pair (X, L), this is equivalent to saying that $\partial H_2(X, L; \mathbb{Z}) \supseteq B \cdot H_1(L; \mathbb{Z})$. Therefore, we have that

$$\begin{aligned} A^H_{\min}(L, X) &= \inf_{\substack{a \in \partial H_2(X,L;\mathbb{Z}) \\ \lambda(a) > 0}} \lambda(a) \\ &\leq \inf_{\substack{a \in B \cdot H_1(L;\mathbb{Z}) \\ \lambda(a) > 0}} \lambda(a) \\ &= B \cdot \inf_{\substack{a' \in H_1(\iota(L);\mathbb{Z}) \\ \lambda'(a') > 0}} \lambda'(a') \\ &\leq B \cdot A^H_{\min}(\iota(L), X'). \end{aligned}$$

Since $\iota(\mathscr{L}^0_Q(X)) \subseteq \mathscr{L}^0_Q(X')$, this gives the desired inequality. $\square$

From Lemma 27 above, we directly get.

Lemma 30
- *If $H_1(X;\mathbb{R})=0$, we have $c_Q(X)=c'_Q(X)=c^0_Q(X)$.*
- *If $\dim H_1(Q;\mathbb{R})=1$ and X is exact, we have $c_Q(X)=c'_Q(X)=c^0_Q(X)$.*
- *If $\dim H_1(Q;\mathbb{R})\leq \dim H_1(X;\mathbb{R})$ and X is exact, we have $c'_Q(X)=c_Q(X)$.*

We end this short list of properties of our capacities by proving that they behave relatively well under products.

Lemma 31 *Suppose that Q' admits a H-exact Lagrangian embedding in X'. Then, $c_Q(X)\leq c_{Q\times Q'}(X\times X')$. In particular, $c_{\mathrm{all}}(X)\leq c_{\mathrm{all}}(X\times X')$ as soon as X' admits a H-exact Lagrangian. The same holds for the corresponding c' capacities*

If Q' admits any Lagrangian embedding in an exact X' and $H_1(Q';\mathbb{R})=0$, then we have that $c^0_Q(X)\leq c^0_{Q\times Q'}(X\times X')$. In particular, $c^0_{\mathrm{all}}(X)\leq c^0_{\mathrm{all}}(X\times X')$ as soon as X' admits a Lagrangian with vanishing first Betti number.

Proof. Let L be the image of a Lagrangian embedding of Q in X, and let L' be the image of a H-exact Lagrangian embedding in X'. Note that we can suppose that L bounds some homology class, otherwise the inequality is trivial. Let thus $v:(\Sigma,\partial\Sigma)\longrightarrow(X\times X',L\times L')$ for some compact surface Σ with boundary. Projecting on each component gives maps $u:(\Sigma,\partial\Sigma)\longrightarrow(X,L)$ and $u':(\Sigma,\partial\Sigma)\longrightarrow(X',L')$. Furthermore, if ω is the symplectic form of X and ω' of X', we then have that

$$(\omega\oplus\omega')(v)=\omega(u)+\omega(u')=\omega(u),$$

since L' is H-exact. Taking infima over all v, we thus get

$$c_{Q\times Q'}(X\times X')\geq A^H_{\min}(L\times L',X\times X')=\inf_{\substack{u=pr_1\circ v\\ \omega(u)>0}}\omega(u)\geq A^H_{\min}(L,X).$$

We then get the inequality by taking the supremum over all possible L's.

The case $H_1(Q';\mathbb{R})=0$ is proven in much the same way. Indeed, exactness of X' along with $H_1(Q';\mathbb{R})=0$ ensures that we also have $(\omega\oplus\omega')(v)=\omega(u)$. Furthermore, the vanishing of the first Betti number ensures that $H_1(Q\times Q';\mathbb{R})\to H_1(X\times X';\mathbb{R})$ vanishes if and only if $H_1(Q;\mathbb{R})\to H_1(X;\mathbb{R})$ does. □

4.2 Finiteness of the capacities

Having enunciated the main properties of our capacities, we now explain how one can get the first part of Theorem 2 from their finiteness.

Proposition 32 *Let L be a closed connected manifold. There exists $R>0$ such that $c'_{\mathrm{all}}(D^*_RL)$ is finite if and only if Conjecture C holds.*

Proof. Suppose that $c'_{\mathrm{all}}(D^*_RL)$ is finite, and set $C:=c'_{\mathrm{all}}(D^*_RL)/R$. It follows from Property (i) of Lemma 28 that

$$c'_{\mathrm{all}}(D^*_rL)=\frac{r}{R}c'_{\mathrm{all}}(D^*_RL)=Cr.$$

Here, we have made use of the fact that $(D^*_r L, \omega_0)$ is symplectomorphic to $(D^*_{r/a} L, a\omega_0)$ via the map $(q, p) \mapsto (q, ap)$. Note that Property (ii) of Lemma 28 implies that our capacity is invariant under symplectomorphisms. Let K be a H-rational Lagrangian in $D^*_r L$, with H-rationality constant τ_K, such that $(\pi_K)_*$ is not surjective on $H_1(\,\cdot\,;\mathbb{R})$, i.e. K is in $\mathscr{L}'_K(D^*_r L)$. Then

$$\tau_K = A^H_{\min}(K, D^*_r L) \leq c'_{\text{all}}(D^*_L) = Cr,$$

which is what we wanted to show.

On the other hand, if Conjecture C holds, then $A^H_{\min}(K, D^*_R L) \leq CR$ for all H-rational Lagrangians in $D^*_R L$ such that $(\pi_K)_*$ is not surjective on $H_1(\,\cdot\,;\mathbb{R})$. Since $A^H_{\min}(K, D^*_R L) = 0$ when K is H-irrational, we directly get a bound on the capacity. □

Therefore, proving Theorem 2 reduces to proving finiteness of some capacity in cotangent bundles. In general, this turns out to be nontrivial, since even $c_{\mathbb{T}^n}(X)$ — the best-behaved version of our capacities — is only well understood when X is a convex or concave toric domain, which is far from the case we need. We will explore this further down, but we already note some interesting cases where finiteness is achievable.

Proposition 33 *If $c_{Q\times Q'}(D^*_R(Q \times Q')) < \infty$, then we have that $c_Q(D^*_R Q) < \infty$ and $c_{Q'}(D^*_R Q') < \infty$.*

Proof. It follows from Lemma 31 that

$$c_Q(D^*_R Q) \leq c_{Q\times Q'}(D^*_R Q \times D^*_R Q').$$

But $D^*_R Q \times D^*_R Q'$ embeds symplectically in $D^*_{2R}(Q\times Q')$ and that embedding is a homotopy equivalence. The proposition then follows directly from Property (ii) of Lemma 28, since finiteness for some $R > 0$ implies finiteness for every $R > 0$ by Property (i) of that lemma. □

Note that if L is a displaceable Lagrangian in a tame symplectic manifold, $A^H_{\min}(L)$ is a lower bound for its displacement energy — this follows from Chekanov's famous estimate [Che98]. In particular, $c_{\text{all}}(B^{2n})$ is bounded by the displacement energy of B^{2n}, and thus it is finite. Zhou [Zho20] proved a broad generalization of this result using a truncated version of Viterbo's transfer map.

Theorem 34 ([Zho20]) *Let X be a Liouville domain with $SH(X) = 0$. We have that $c_{\text{all}}(X) < \infty$.*

Note that $SH(D^*_R L) \neq 0$ because of Viterbo's isomorphism [Vit99]. Therefore, Zhou's theorem never directly implies Theorem 4. However, in some cases, we still manage to compare $c_Q(D^*L)$ to $c_{\text{all}}(X)$ as we shall see below.

Remark 10. Zhou actually works with homotopy — not homology like us — and allows for the possibility of weakly exact Lagrangians. He also allows some nonexact Liouville domains, but it will not be needed here. Therefore, his result is actually more general than what is cited here.

Theorem 35 *Let L_1, ..., L_k be closed manifolds with* $\dim H_1(L_i;\mathbb{R}) = 1$ *such that L_i embeds as a Lagrangian in a Liouville domain W_i such that $SH(W_i) = 0$. Let L_0 be a closed manifold with $H_1(L_0;\mathbb{R}) = 0$. We have that*

$$c'_{\mathrm{all}}\left(D^*_{r_0}L_0 \times \cdots \times D^*_{r_k}L_k\right) < \infty$$

for small enough r_i's. Moreover, the same is true for c_Q for a closed connected manifold Q with $\dim H_1(Q;\mathbb{R}) \leq k$.

Proof of Theorem 35. We start with the case L_0 is a point and remove it from the notation for now. Let L be a Lagrangian in the product of codisk bundles X diffeomorphic to Q and such that $\omega(H_2(X, L)) \neq 0$. Further suppose that $f_* : H_1(Q;\mathbb{R}) \to H_1(X;\mathbb{R})$ is not surjective. Denote by

$$p_i : L_1 \times \cdots \times L_k \to L_1 \times \cdots \times \widehat{L_i} \times \cdots \times L_k$$

the projection onto the product of all L_j's except L_i and by Π_i its lift between products of cotangent bundles.

Since f_* is not surjective, its image I is a vector subspace of $H_1(X;\mathbb{R})$ of dimension less than or equal to $k-1$. Therefore, there exists at least one $i \in \{1,\ldots,k\}$ such that, under the identification $H_1(X;\mathbb{R}) \cong \mathbb{R}^k$, the projection $\pi_i : \mathbb{R}^k \to R_i = \{x \in \mathbb{R}^k \mid x_i = 0\}$ is injective on I. Note that $(\Pi_i)_*$ is naturally identified with π_i. Therefore, for that i, we have that $\ker f_* = \ker(\Pi_i \circ f)_*$.

Take r_i small enough so that $D^*_{r_i}L_i$ symplectically embeds in W_i as in the statement of the theorem. Denote by

$$\Psi_i : X \hookrightarrow X'_i := D^*_{r_1}L_1 \times \cdots \times W_i \times \cdots \times D^*_{r_k}L_k$$

the symplectic embedding it induces and by ω'_i the symplectic form on X'_i. Note that $\ker(\Pi_i \circ f)_* = \ker(\Psi_i \circ f)_*$.

Suppose $u' \in H_2(X'_i, L;\mathbb{Z})$ such that $\omega'_i(u') \neq 0$. This implies that $\partial u'$ is in $\ker(\Psi_i \circ f)_* = \ker f_*$ in $H_1(Q;\mathbb{R})$. In particular, there is some $N \in \mathbb{Z}$ and some $u \in H_2(X, L;\mathbb{Z})$ such that $\partial u = N\partial u'$. Furthermore, some diagram chasing gives that N must divide the order B_i of the torsion part of $H_i(D^*_{r_i}L_i)$. Moreover, since Ψ_i is an exact symplectic embedding, we have $\omega(u) = k\omega'(u')$. Hence

$$A^H_{\min}(L, X) \leq B_i A^H_{\min}(L, X'_i) \leq B_i c_{\mathrm{all}}(X'_i)$$

Taking the supremum over all possible L, we thus get

$$c'_Q(X) \leq \max_i B_i c_{\mathrm{all}}(X'_i),$$

and finiteness follows from Zhou's theorem.

When L_0 is not a point, one can check that the entire proof above follows in the same way. □

4.3 Proof of Theorem 4

For completeness, we now give a quick proof of how Theorem 4 follows from Conjecture C when it holds.

Proof of Theorem 4. For all sufficiently small $r > 0$, there is a Weinstein neighbourhood $W_r(L) \cong D_r^*L$ of L in M. Let $L' \in \mathcal{L}(L, \tau)$, and suppose that the map $(\pi|_{L'})_* : H_1(L'; \mathbb{R}) \to H_1(L; \mathbb{R})$, induced by the restriction of the projection $\pi : D^*L \to L$ to L' is not surjective. Then, there is some positive integer k such that L' has rationality constant $k\tau$ in $\mathcal{W}_r(L)$. If Conjecture C holds, then we have that $k\tau \leq Cr$ for some $C = C(L) > 0$. We thus get a contradiction if $r < \tau/C$.

For such r, $(\pi|_{L'})_* : H_1(L'; \mathbb{R}) \to H_1(L; \mathbb{R})$ must thus be surjective whenever $L' \subseteq \mathcal{W}_r(L)$. Since $H_1(L'; \mathbb{R}) \cong H_1(L; \mathbb{R})$, this means that it is an isomorphism, and the result follows from the central lemma (Lemma 24). □

We remark that using the methods above and the c^0 variant of the capacities, we can show the following interesting partial result.

Theorem 36 *Let L be a Lagrangian submanifold in M. Suppose that, as an abstract manifold, L admits a Lagrangian embedding in a Liouville domain W with $SH(W) = 0$. For every $\tau \geq 0$, there exists a Weinstein neighbourhood $\mathcal{W}(L)$ of L in M, such that if $L' \in \mathcal{L}(\tau)$ is included in $\mathcal{W}(L)$, then the map $\pi_* : H_1(L'; \mathbb{R}) \to H_1(L; \mathbb{R})$ induced by the projection is nonzero.*

Sketch of proof. By adapting the proof of Proposition 32, one can see that finiteness of $c^0_{\text{all}}(D_R^*L)$ for some $R > 0$ is equivalent to the following version of Conjecture C: if K is an H-rational Lagrangian inside D_r^*L such that the map $(\pi|_K)_* : H_1(K; \mathbb{R}) \to H_1(L; \mathbb{R})$ is zero, then the H-rationality constant τ_K of K satisfies $\tau_K \leq Cr$ for a constant $C = C(L)$.

However, the finiteness of $c^0_{\text{all}}(D_R^*L)$ for some $R > 0$ follows directly from the Weinstein neighbourhood theorem, the monotonicity of the c^0 capacities (Lemma 29), and Zhou's Theorem.

Replicating the proof of Theorem 4, one gets the existence of a neighbourhood $\mathcal{W}(L)$ such that whenever $L' \in \mathcal{L}(\tau)$ is in $\mathcal{W}(L)$, then $H_1(L'; \mathbb{R}) \to H_1(L; \mathbb{R})$ is nonzero. □

Remark 11. *Since L embeds as a Lagrangian in a Liouville domain with $SH(W) = 0$, so does any nearby L'. The Viterbo transfer morphism then implies that $H_1(L; \mathbb{R})$ and $H_1(L'; \mathbb{R})$ are nonzero (see [Rit13]). Therefore, $H_1(L'; \mathbb{R}) \to H_1(L; \mathbb{R})$ being nonzero is consistent with the hypotheses on L and L'.*

4.4 Conjecture C and covering spaces

The condition on the Lagrangian $K \subset D_r^*L$ in Conjecture C, i.e. non-surjectivity of $(\pi_K)_*$, while technical, has the advantage of behaving well with covering spaces. This is highlighted by the following proposition, which proves the second part of Theorem 2.

Proposition 37 *Suppose that Conjecture C holds for a closed connected manifold L', then it holds for all L such that there exists a covering map $\pi : L' \to L$.*

Before we prove Proposition 37, we require the following lemma from group theory. For a group G, denote by $\mathrm{Ab}(G) = G/[G, G]$ its abelianization.

Lemma 38 *Suppose H is a subgroup of finite index in a group G. Then the inclusion $i : H \to G$ induces a surjection*

$$\mathrm{Ab}(H) \otimes \mathbb{Q} \to \mathrm{Ab}(G) \otimes \mathbb{Q}.$$

Proof. Observe that $\operatorname{rank} \mathrm{Ab}(G) = \dim_{\mathbb{Q}} \mathrm{Hom}(G, \mathbb{Q})$. It thus suffices to prove that the restriction map

$$r : \mathrm{Hom}(G, \mathbb{Q}) \to \mathrm{Hom}(H, \mathbb{Q})$$

is injective. Suppose $\nu \in \ker(r)$, i.e. $\nu|_H \equiv e_H$. Let $H = a_1H, a_2H, \ldots, a_kH$ be the right cosets of H in G. Then $\mathrm{im}(\nu) = \{e_H, \nu(a_2), ..., \nu(a_k)\} \subset \mathbb{Q}$ is a finite subset. Since $\mathbb{Q}$ has no non-zero elements of finite order, this yields that $\nu \equiv e_H$. □

Proof of Proposition 37. First, extend the covering $p : L' \to L$ to a covering

$$p : W'_r = D^*_r L' \to W_r = D^*_r L$$

for the metric $g' = p^*g$ on L'. Look at a connected component K' of $p^{-1}(K)$. Then $p_{K'} = p|_{K'} : K' \to K$ is a covering of K. Denote by $\omega' = p^*\omega$ and $\lambda' = p^*\lambda$ the symplectic and Liouville forms on W'_r, and set $\lambda'_{K'} = \lambda'|_{K'}$, $\lambda_K = \lambda|_K$. It is easy to see that K' is H-rational in $D^*_r L'$. Indeed, given $A \in H_2(K', W'_r)$, we have $\langle \omega', A\rangle = \langle \omega, p_* A\rangle$, which is an integer multiple of τ_K. Hence for a suitable integer $m \geq 1$, $\tau_{K'} = m\tau_K \geq \tau_K$.

It is now sufficient to prove that if $(\pi_{K'})_* : H_1(K';\mathbb{Q}) \to H_1(L';\mathbb{Q})$ is surjective, then $(\pi_K)_* : H_1(K;\mathbb{Q}) \to H_1(L;\mathbb{Q})$ also is. Note that as covering maps, p and $p_{K'}$, induce injective homomorphisms at the level of fundamental groups. As $H_1(X) = \mathrm{Ab}\,\pi_1(X)$ for any path connected space X, Corollary 38 implies that $p_* : H_1(L';\mathbb{Q}) \to H_1(L;\mathbb{Q})$ and $(p_K)_* : H_1(K';\mathbb{Q}) \to H_1(K;\mathbb{Q})$ are surjective. Hence if $(\pi_{K'})_* : H_1(K';\mathbb{Q}) \to H_1(L';\mathbb{Q})$ is surjective, then $(p \circ \pi_{K'})_* = p_* \circ (\pi_{K'})_* : H_1(K';\mathbb{Q}) \to H_1(L';\mathbb{Q})$ is surjective. However, as $p \circ \pi_{K'} = \pi_K \circ p_{K'}$ the image of $(p \circ \pi_{K'})_* = (\pi_K \circ p_{K'})_* = (\pi_K)_* \circ (p_{K'})_*$ is contained in that of $(\pi_K)_*$. Therefore $(\pi_K)_*$ is surjective. □

4.5 Homotopy version of Theorem 2

The argument in Proposition 37 implies that the homotopy version of Conjecture C is also invariant under coverings: if the rationality constant of all $K' \subseteq D^*_r L'$ with $(\pi_{K'})_* : H_1(K';\mathbb{Q}) \to H_1(L';\mathbb{Q})$ nonsurjective· is uniformly bounded and $L' \to L$ is a covering, then the rationality constant of all $K \subseteq D^*_r L$ with $(\pi_K)_* : H_1(K;\mathbb{Q}) \to H_1(L;\mathbb{Q})$ nonsurjective is uniformly bounded by the same constant. As noted in Remark 8, this thus gives the homotopy version of Conjecture C for anything covered by $L_0 \times \mathbb{T}^m$ with $\pi_1(L_0)$ finite.

Let us however finally give a direct proof of the homotopy version of Theorem 2, which we now state precisely.

Theorem 39 *Suppose that L is diffeomorphic to the product $L_0 \times \mathbb{T}^m$ of a closed manifold L_0 with $H_1(L_0;\mathbb{R}) = 0$ and a m-torus. Suppose that K is a rational Lagrangian in $D^*_r L$ with rationality constant τ_K and such that $(\pi_K)_*$ is not surjective, then*

$$\tau_K \leq Cr$$

for a constant $C > 0$ depending only on a choice of an auxiliary metric on L. Moreover, the same is true for any closed L' which is covered by L.

Proof of Theorem 39. Suppose f is a rational Lagrangian embedding of K into

$$D_a^*(T^m \times Q) \cong (D_a^* S^1)^m \times D_a^* Q,$$

with rationality constant $\tau_K > 0$. Observe that, as an open symplectic manifold, $D_a^* S^1 = (-a, a) \times S^1$ is symplectomorphic to the punctured disk $D^2(2a)' = D^2(2a) \setminus \{0\}$ of area $2a$. The symplectomorphism is given by

$$\psi : D_a^* S^1 \to D^2(2a)', \quad (p, q) \mapsto (r, \theta)$$

where $\pi r^2 = a + p$ and $\theta = 2\pi q$ — this identifies the zero section $0_{S^1} = \{0\} \times S^1$ with the circle $S(a) = \partial D(a)$. In particular $(D_a^* S^1)^m \times D_a^* Q$ is symplectomorphic via $\psi^m \times \mathrm{id}$ to $V = (D^2(2a)')^m \times D_a^* Q$.

Consider the induced map $f_* : H_1(K) \to H_1(V)$. If $R = \ker(f_*) = 0$, then K is H-exact by the central lemma (Lemma 24), and $\tau_K = 0$; see also Remark 1. Suppose therefore that $R = \ker(f_*) \neq 0$. Let

$$V_i = (D^2(2a)')^{i-1} \times D^2(2a) \times (D^2(2a)')^{m-i} \times D_\varepsilon^* Q$$

for $1 \leq i \leq m$. Let $g_i : V \to V_i$ denote the inclusion. We claim that $R = \ker(f_*) = \ker((g_i \circ f)_*)$ for at least one index $i \in \{1, \ldots, m\}$. Indeed, as the image $I \subset \mathbb{Z}^m$ of f_* is a free abelian subgroup of rank at most $m-1$, there exists at least one such i with the projection $\pi_i : \mathbb{Z}^m \to Z_i = \{k \in \mathbb{Z}^m \mid k_i = 0\}$ being injective on I.

However $(g_i)_* : H_1(V) \to H_1(V_i)$ is naturally identified with π_i. This means that the period groups

$$\mathcal{P}_{V,K} = \left\langle [\omega], H_2^D(V, K; \mathbb{Z}) \right\rangle, \; \mathcal{P}_{V_i,K} = \left\langle [\omega], H_2^D(V_i, K; \mathbb{Z}) \right\rangle,$$

coincide. In particular K is rational in V_i with the same rationality constant τ_K. For topological considerations, the same is true for K inside

$$\widehat{V}_i = (D^2(2a)')^{i-1} \times \mathbb{C} \times (D^2(2a)')^{m-i} \times D_\varepsilon^* Q.$$

Now, as $K \subset V_i$, the displacement energy of K inside $\widehat{V}_i$ is at most $2a$. By [Che98], this yields $\tau_K \leq 2a$. ◻

Note that we directly get Proposition 3 for rational Lagrangians in $\mathbb{R}^n \times [-1, 1]^n$ by setting $a = 1$ and considering by contradiction the embedded copy of the rational Lagrangian K inside $\mathbb{R}^n/\mathbb{Z}^n \times [-1, 1]^n$. Note, however, that in this case the map f_* from the proof above vanishes identically, whence we may choose any index i to run the argument. This implies that one may replace $[-1, 1]^n$ by X as in Proposition 3.

5 Theorem 5 and the Lagrangian C^0 flux conjecture

We start this section by proving Theorem 5 (Section 5.1). We then turn to the Lagrangian C^0 flux conjecture and first give a short proof of Theorem 7 (Section 5.2), which follows almost directly from the proof of Theorem 5. Finally, we prove a refined version of Proposition 8 and use it to properly show Corollary 9, that is, the Lagrangian C^0 flux conjecture (Section 5.3). We conclude the section with a proof of Proposition 15 (Section 5.4).

5.1 Proof of Theorem 5

We consider a H-rational Lagrangian submanifold L of (M, ω) of rationality constant $\tau \geq 0$. We fix a Riemannian metric g on L and a Weinstein neighbourhood $\mathcal{W}_r(L)$ in M of size $r > 0$. Let $L' \in \mathcal{L}(\tau)$ be a Lagrangian entirely contained and H-exact in $\mathcal{W}_r(L)$.

We want to prove that there exists $r' > 0$, such that L' is exact in $\mathcal{W}_{r'}(L)$ whenever one of the following conditions hold:

(a) $L' \in \mathcal{L}\mathrm{Ham}(L)$, or

(b) the map $H_1(i) \otimes \mathbb{R}$ induced by the inclusion $i : L' \hookrightarrow M$ vanishes.

To do so, we first claim that under any of these assumptions, the rationality constant of L' in $\mathcal{W}_r(L)$, seen as a subset of T^*L, is a fraction of that of L' in M.

Lemma 40 *Let L and L' be as above, and denote by $\Psi : D_r^*L \to \mathcal{W}_r(L)$ a Weinstein neighbourhood of L. There exists an integer $k = k(M, L)$ such that $\lambda_0(H_1(\Psi^{-1}(L'))) \subseteq \frac{\tau}{k}\mathbb{Z}$.*

This lemma, whose proof we postpone to § 5.1.2 below, directly shows that, when $\tau = 0$, H-exactness yields exactness.

When $\tau > 0$, we conclude by using the following additional estimate.

Proposition 41 *Let $L \hookrightarrow (D_r^*L, d\lambda_0)$ be a Lagrangian embedding whose image L' is H-exact. We have that*

$$\forall \beta' \in H_1(L'), \qquad |\lambda_0(\beta')| \leq r\ell_g^{\min}(\pi_*\beta')$$

where $\ell_g^{\min}(\beta)$ denotes the length of the shortest geodesic loop for g in L representing the class β.

Indeed, we choose a basis $\{\beta_1', \dots \beta_m'\}$ of $H_1(L')$ and we fix $r' < \frac{\tau}{k\ell}$ where

$$\ell = \max\{\ell_g^{\min}(\pi_*\beta_i') \,|\, 1 \leq i \leq m\}\,.$$

The proposition above gives that, for all i, $|\lambda_0(\beta_i')| \leq r'\ell < \frac{\tau}{k}$. Because of Lemma 40, we then get that λ_0 vanishes on $H_1(L')$, which proves the exactness of L'.

It only remains to prove the lemma and proposition above to conclude the proof of Theorem 5.

5.1.1 Proof of Proposition 41

We start with the proposition. First, let us remark that when $L = \mathbb{T}^n$, the estimate follows directly from Eliashberg's result on the shape of subsets of $T^*\mathbb{T}^n$ [Eli91]. With the additional hypothesis that L is also contained in a Weinstein neighbourhood of L', this is a result of Membrez and Opshtein [MO21]. However, as they themselves point out, there should be a proof of this result without their additional constraint using the theory of graph selectors — they even sketch out a proof, which we mostly follow here.

Proof of Proposition 41. In Theorem 6.1 of [PPS03], Paternain, Polterovich, and Siburg show that, for every Lagrangian submanifold $L' \subseteq T^*L$ Lagrangian isotopic to the zero-section and every fiberwise-convex neighbourhood W of L', there is a closed 1-form σ of L such that $\operatorname{graph}(\sigma) \subseteq W$ and $[\sigma] = [\lambda_0|_{L'}]$. However, inspecting the proof of that statement, we see that all that is truly required is the existence of a symplectic isotopy preserving fibres sending L' to an exact Lagrangian submanifold admitting a graph selector — we refer to that paper for the definition of a graph selector. On the one hand, we have shown in Lemma 24 that H-exact Lagrangians in T^*L indeed have associated symplectic isotopies preserving fibres which send them to exact ones. On the other hand, it is now known that every exact Lagrangian submanifold of T^*L admits a graph selector. This was proven using Floer theory by Amorim, Oh, and Dos Santos [AOS18] and using microlocal sheaves by Guillermou [Gui23]. Therefore, the result applies as is in our case.

But it follows from this that

$$\left\{[\iota^*\lambda_0] \,\middle|\, \iota : L \hookrightarrow D_r^*L \text{ is } H\text{-exact}\right\} = \left\{[\sigma] \in H^1(L;\mathbb{R}) \,\middle|\, |\sigma| < r\right\}.$$

In particular, for every H-exact Lagrangian embedding $\iota : L \hookrightarrow D_r^*L$ and every loop $\gamma : S^1 \to L$, we have that

$$|\lambda_0(\iota \circ \gamma)| < r\ell_g(\gamma),$$

where ℓ_g denotes the length in the metric g. By taking the infimum over all loops representing a class $\beta = \pi_*\beta'$, we get the desired inequality. ◻

5.1.2 Proof of Lemma 40

Recall that L is a H-rational Lagrangian with rationality constant $\tau \geq 0$, that $\Psi : D_r^*L \to \mathcal{W}_r(L)$ is a Weinstein neighbourhood of L in M of size $r > 0$, and that $L' \in \mathcal{L}(\tau)$ is a Lagrangian entirely contained and H-exact in $\mathcal{W}_r(L)$. The lemma states that, under one of the following conditions,

(a) $L' \in \mathcal{L}\mathrm{Ham}(L)$, or

(b) the map $H_1(i) \otimes \mathbb{R}$ induced by the inclusion $i : L' \hookrightarrow M$ vanishes

there exists an integer $k = k(M, L)$ such that $\lambda_0(H_1(\Psi^{-1}(L'))) \subseteq \frac{\tau}{k}\mathbb{Z}$.

For convenience, we denote by $\overline{X}$ the object in T^*L corresponding to X via Ψ^{-1}, e.g. $\Psi^{-1}(L) = \overline{L}$.

Proof of Lemma 40. Fix a representative $\overline{\beta} : S^1 \to \overline{L}$ of a class in $H_1(\overline{L})$. Since $\overline{L'}$ is H-exact in D_r^*L, the projection $\overline{L'} \hookrightarrow T^*L \to \overline{L}$ is a homotopy equivalence by Lemma 24. Therefore, there exist a loop $\overline{\beta'}$ in $\overline{L'}$ and a cylinder $\overline{C}$ in D_r^*L such that $\pi_*(\overline{\beta'}) = \overline{\beta}$ and $\partial\overline{C} = \overline{\beta'} \sqcup (-\overline{\beta})$. By Stokes Theorem and exactness of the 0-section $\overline{L}$ in T^*L, we thus have that

$$\omega(C) = d\lambda_0(\overline{C}) = \lambda_0(\overline{\beta'}) - \lambda_0(\overline{\beta}) = \lambda_0(\overline{\beta'})\,.$$

In case (a), take a Hamiltonian isotopy $\{\varphi_t\}_{t\in[0,1]}$ starting at identity and such that $\varphi_1(L) = L'$. Then $C'(s,t) := \varphi_t^{-1}(\beta'(s))$ defines a cylinder in M and $C'' := C \cup_{\beta'} C'$ represents a class in $H_2(M,L)$. In particular, $\omega(C) + \omega(C') = \omega(C'') \in \tau\mathbb{Z}$. But note that, since $\{\varphi_t^{-1}\}$ is Hamiltonian,

$$\omega(C') = \mathrm{Flux}(\{\varphi_t^{-1}\})(\beta') = 0\,.$$

Therefore, $\omega(C) = \lambda_0(\overline{\beta'}) \in \tau\mathbb{Z}$, and we can take $k = 1$.

In case (b), note that $H_1(L;\mathbb{R}) \to H_1(M;\mathbb{R})$ being zero is equivalent to the image of $H_1(L) \to H_1(M)$ being finite, since $H_1(\,\cdot\,;\mathbb{R}) = H_1(\,\cdot\,) \otimes \mathbb{R}$. By the long exact sequence of the pair (M,L), this is in turn equivalent to $H_2(M,L) \to H_1(L)$ having finite cokernel, whose size we denote by k. Then, $k\beta$ bounds some $u \in H_2(M,L)$, and we have that

$$k\lambda_0(\overline{\beta'}) = k\omega(C) = \omega(u\#kC) - \omega(u) \in \tau\mathbb{Z},$$

because $u\#kC \in H_2(M,L')$, and L and L' belong to $\mathcal{L}(\tau)$. Therefore, $\lambda_0|_{L'}$ must take values in $\frac{\tau}{k}\mathbb{Z}$. □

5.2 Proof of Theorem 7

We now turn our attention to Theorem 7 on limits of H-rational Lagrangians. As we shall see, the theorem follows pretty directly from the techniques that we developed to prove Theorem 4 and 5.

Proof of Theorem 7. We start with the first part of the statement: if L_i converges to L with L smooth and n-dimensional and $L_i \in \mathcal{L}(\tau_i)$ with $\inf \tau_i > 0$, then L is Lagrangian. This follows pretty directly from Laudenbach and Sikorav's result on displacement of non-Lagrangians [LS94].

Indeed, suppose L is not Lagrangian. Then, $L \times S^1 \subseteq M \times T^*S^1$ is also not Lagrangian and its normal bundle admits a nowhere vanishing section. Therefore, it follows from [LS94] that, for every $\varepsilon > 0$, there is a Hamiltonian diffeomorphism φ of $M \times T^*S^1$ such that $\varphi(L\times S^1) \cap L \times S^1 = \varnothing$ and with Hofer norm $||\varphi||_H < \varepsilon$. But then, there is a neighbourhood U of $L \times S^1$ such that $\varphi(U) \cap U = \varnothing$. In particular, for i large enough, $\varphi(L_i \times S^1) \cap (L_i \times S^1) = \varnothing$. Therefore, if $e(L_i \times S^1)$ is the displacement energy of $L_i \times S^1$, we have that

$$\varepsilon \geq \limsup e(L_i \times S^1) \geq \limsup \tau_i \geq \inf \tau_i > 0,$$

where the second inequality follows from Chekanov's estimate on displacement energy [Che98]. We get a contradiction by taking the limit $\varepsilon \to 0$.

The second part — that is, for when we know that the L_i's are H-exact in $\mathcal{W}(L)$ for i large — is proved in a very similar way as Theorem 5. Indeed, we know that the projection $L_i \to L$ is a homotopy equivalence by Lemma 24. In particular, any homology class $A \in H_2(M, L; \mathbb{Z})$ can be obtained by gluing a class $A_i \in H_2(M, L_i; \mathbb{Z})$ to a (union of) cylinder C_i with $\partial C = \partial A_i - \partial A$ and $\partial A_i = (\pi|_{L_i})^{-1}(\partial A)$. Here, we identify L_i with its preimage in T^*L under a Weinstein neighbourhood of L. If $L_i \subseteq D^*_{r_i}L$, then Proposition 41 gives

$$\lim |\lambda_0(\partial A_i)| \leq \lim r_i \ell_g^{\min}(\partial A) = 0,$$

where we have made use of the fact that we may take $\lim r_i = 0$ since $\{L_i\}$ Hausdorff-converges to L. Therefore,

$$|\omega(A)| = \liminf |\omega(A_i) + \lambda_0(\partial A_i)| = \liminf |\omega(A_i)|.$$

But by rationality, $|\omega(A_i)| = n_i \tau_i$ for some $n_i \in \mathbb{Z}_{\geq 0}$. Since $\inf \tau_i > 0$, $\{n_i\}$ must be bounded. Therefore, by passing to a subsequence if necessary, we may suppose that $n_i \equiv n$ for all i. Thus, if $\tau := \liminf \tau_i$, then $\omega(A) \in \tau\mathbb{Z}$. In particular, L is H-rational and its rationality constant τ_L is a multiple of τ.

We now prove that this multiple must be in fact 1. Pick a base $\{A^1, \ldots, A^k\}$ of the free part of $H_2(M, L; \mathbb{Z})$ such that $\omega(A^j) = \tau_L$ for all j, and construct A_i^j and C_i^j as above. By the same logic as above, we may suppose that $|\omega(A_i^j)| = n^j \tau_i$ for all i. Therefore, we get that

$$n^j \tau_i - r_i \ell \leq \tau_L \leq n^j \tau_i + r_i \ell,$$

where $\ell := \max_j \ell_g^{\min}(\partial A^j)$. Since $\inf \tau_i > 0$, for i large enough, $r_i \ell < \tau$, and thus we must have $\tau_L = n^j \tau$ for all j, i.e. $n \equiv n^j$. But since the projection $L_i \to L$ is a homotopy equivalence, $\{A_i^1, \ldots, A_i^k\}$ is a basis of the free part of $H_2(M, L_i; \mathbb{Z})$. This is only possible if $n = 1$ as $|\omega(A_i^j)| = n\tau_i$ and $\omega(H_2(M, L_i; \mathbb{Z})) = \tau_i \mathbb{Z}$. This thus implies that $\tau_L = \tau$.

Finally, to conclude that $\lim \tau_i$ must exists, note that the contrary would imply the existence of two subsequences of $\{L_i\}$ — thus still converging to L — such that their corresponding subsequences of $\{\tau_i\}$ converge to different values. But then, both these values would need to be equal to τ_L, which is not possible. ◻

5.3 Proof of Proposition 8 and Corollary 9

We now turn to the proof of Proposition 8, i.e. the partial result one gets instead of Theorem 5 when one does not know that $H_1(L; \mathbb{R}) \to H_1(M; \mathbb{R})$ is zero. In fact, we prove the following stronger statement.

Proposition 42 *Let L be a H-rational Lagrangian submanifold of M with H-rationality constant τ. There is some $r_0 > 0$ and some $C > 0$ with the following property. Assume that $L' \in \mathcal{L}(\tau)$ is a Lagrangian included in a Weinstein neighbourhood $\mathcal{W}_r(L)$ of size $r \in (0, r_0]$ such that L' is H-exact in $\mathcal{W}_r(L)$. Then, there is a symplectic isotopy $\{\psi_t\}_{t\in[0,1]}$ of M with $|\mathrm{Flux}(\{\psi_t(L')\})| \leq Cr$ such that $\psi_1(L')$ is exact in $\mathcal{W}_r(L)$.*

By $\mathrm{Flux}(\{L_t\}) \in H^1(L;\mathbb{R})$, we mean the Lagrangian flux of the Lagrangian isotopy $\{L_t\}$; it is defined as follow. Take $F : [0,1] \times L \to M$ such that $F(L,t) = L_t$. Then, $F^*\omega = dt \wedge \alpha_t$ for some time-dependent 1-form α_t on L, and we set $\mathrm{Flux}(\{L_t\})(\gamma) := \int_0^1 \alpha_t(\gamma)dt$ for any loop $\gamma : S^1 \to L$. This is precisely the area swept by γ through the isotopy — in particular, it is independent of the parametrization F of $\{L_t\}$.

Proof. Denote by V the image of the boundary map $H_2(M,L';\mathbb{R}) \to H_1(L';\mathbb{R})$. Pick a complement W of V in $H_1(L';\mathbb{R})$, and take loops $\{\gamma_1,\dots,\gamma_k\}$ which induce a basis of W. Similarly to Section 5.2 above, the proof of Theorem 5 still implies that $\lambda_0|_{L'}(V) = 0$ for r small enough. Therefore, we can take r_0 to ensure this is true for all $r \le r_0$.

We divide our isotopy in two parts. First, we consider the Lagrangian isotopy $F : t \mapsto [(\alpha-1)t+1]\cdot L'$ induced by the multiplication along the fibers of T^*L, where $\alpha \in [0,1]$. A direct computation gives that $F^*\omega = (\alpha-1)\lambda_0|_{L'} \wedge dt$, so that the flux associated to the isotopy is $(\alpha-1)[\lambda_0|_{L'}]$. Note that, by the above paragraph, this cohomology class is in the annihilator V^0 of V, which we can identify with the dual W^* of W in $H^1(L';\mathbb{R}) = \mathrm{Hom}(H_1(L';\mathbb{R}),\mathbb{R})$.

Second, take a closed 1-form σ on L such that $\sigma(V) = 0$ and $\sigma(\pi\circ\gamma_i) = \lambda_0(\gamma_i)$ for all i. It exists, since the projection $L' \to L$ is a homotopy equivalence by Lemma 24. Consider the symplectic isotopy $\{\psi'_t\}$ of T^*L generated by X such that $\iota_X\omega_0 = -\pi^*\sigma$, where $\pi : T^*L \to L$ is the canonical projection. It is easy to check that

(i) $\psi'_1(L')$ is exact in T^*L,

(ii) if $L' \subseteq D^*_rL$, then $\psi'_t(L') \subseteq D^*_{r+|\sigma|}L$ for all $t \in [0,1]$,

(iii) $\mathrm{Flux}(\{\psi'_t(L')\}) = (\iota')^*\mathrm{Flux}(\{\psi'_t\}) = -(\iota')^*\pi^*[\sigma] = -[\lambda_0|_{L'}]$.

We have made here the slight abuse of notation of identifying L' with its preimage in T^*L via the Weinstein neighbourhood. Again, *(iii)* implies that the flux of the isotopy is in W^*.

The Lagrangian isotopy $\{L'_t\}$ from L' to an exact Lagrangian L'' that we are interested in is the (smoothing of the) concatenation of Lagrangian isotopies as above. More precisely, start with $L' \subseteq D^*_rL$ and σ as above. Then, the first half of the isotopy is given by the scaling from L' to $\alpha L'$ for $\alpha = \frac{r}{r+|\sigma|}$. Note that then, $\alpha\sigma$ is a closed 1-form on L having the same properties as above for the Lagrangian $\alpha L'$. We thus get from it a symplectic isotopy $\{\psi'_t\}$ with properties *(i)*–*(iii)* for $\alpha L'$. In particular, $\psi'_t(\alpha L') \subseteq D^*_{\alpha r+|\alpha\sigma|}L = D^*_rL$ and $\mathrm{Flux}(\{\psi'_t(L')\}) = -\alpha[\lambda_0|_{L'}]$. Therefore,

$$\mathrm{Flux}(\{L'_t\}) = (\alpha-1)[\lambda_0|_{L'}] - \alpha[\lambda_0|_{L'}] = -[\lambda_0|_{L'}] \in W^*,$$

where we have made use of the additivity of the flux under concatenation. Furthermore, Proposition 41 then implies that $|\mathrm{Flux}(\{L'_t\})| \le r\max_i \ell_g^{\min}(\gamma_i)$, and it suffices to take $C := \max_i \ell_g^{\min}(\gamma_i)$.

We now show how $\{L'_t\}$ comes from a symplectic isotopy of M — this is essentially Lemma 6.6 of [Sol13]. Note that in the splitting $H^1(L';\mathbb{R}) = V^*\oplus W^*$, W^* corresponds to the image of the restriction homomorphism $\Psi^* : H^1(M;\mathbb{R}) \to H^1(\mathcal{W}_r(L);\mathbb{R})$ under the restriction isomorphism $H^1(\mathcal{W}_r(L);\mathbb{R}) \to H^1(L';\mathbb{R})$.

Here, we make use of the fact that L' is isotopic to an exact Lagrangian of T^*L, so that the inclusion $L' \to \mathcal{W}(L)$ induces an isomorphism on cohomology. In particular, since $[\lambda_0|_{L'}]$ belongs to W^*, there is a closed 1-form θ' of M such that $\theta'|_{L'} = \lambda_0|_{L'} + dF$ for some function $F : L' \to \mathbb{R}$. We then pick an extension $F' : M \to \mathbb{R}$ of F and set $\theta := \theta' - dF'$. Taking $\{\psi_t\}$ generated by θ gives the desired symplectic isotopy in M. □

Corollary 43 *By taking r_0 smaller if necessary, we have the following. If we have that* $\mathrm{Flux}(\{\psi_t(L')\}) \neq 0$*, then L' and $\psi_1(L')$ are in different Hamiltonian isotopy class in M.*

Moreover, if the NLC holds on T^*L, then $L', L'' \in \mathcal{L}(\tau)$ with $L', L'' \subseteq \mathcal{W}_r(L)$, $r \leq r_0$, are Hamitlonian isotopic in M if and only if their associated isotopy to an exact Lagrangian has the same flux.

Proof. Suppose that there is a Hamiltonian isotopy $\{\varphi_t\}$ of M sending L' to $\psi_1(L')$. Then, the concatenation $\{L''_t\}$ of $\{\psi_t(L')\}$ and $\{\varphi_t^{-1}(\psi_1(L'))\}$ is a loop, so that $\mathrm{Flux}(\{L''_t\}) \in H^1(L'; \tau\mathbb{Z})$. Indeed, for every loop γ of L', $\mathrm{Flux}(\{L''_t\})(\gamma) \in \tau\mathbb{Z}$, since it is the area of a cylinder with boundary in L'. If we take $r_0 < \frac{\tau}{C}$, then this is only possible if $\mathrm{Flux}(\{L''_t\}) = 0$. Since the flux of a Hamiltonian isotopy is zero, this implies the first result.

If the NLC holds on T^*L, we get an extension $\{\psi_t\}_{t\in[0,2]}$ of $\{\psi_t\}_{t\in[0,1]}$ to a symplectic isotopy with $\psi_2(L') = L$ and same flux. Let $\{\psi'_t\}_{t\in[0,2]}$ be the corresponding isotopy for L''. If L' and L'' are Hamiltonian isotopic, we can construct a loop similarly to above using that Hamiltonian isotopy, $\{\psi_t\}$ and $\{\psi'_t\}$. We then again get that the flux of this loop is zero, so that $\mathrm{Flux}(\{\psi_t(L')\}) = \mathrm{Flux}(\{\psi'_t(L'')\})$. If the fluxes are the same, then extension and concatenation as above give a symplectic isotopy in T^*L from L' to L'' with zero flux. By Proposition 2.3 of [Ono08] or Lemma 6.7 of [Sol13], that isotopy must be Hamiltonian. □

The Lagrangian C^0 flux conjecture We now give a proof of Corollary 9. To do so we first prove the following more technical, but stronger, version of the corollary.

Corollary 44 *Suppose that $L = L_0 \times L_1 \times \cdots \times L_k$, where $H_1(L_0; \mathbb{R}) = 0$ and, for $i \geq 1$, L_i satisfies $H_1(L_i; \mathbb{R}) = \mathbb{R}$ and admits a Lagrangian embedding in a Liouville domain W_i with $SH(W_i) = 0$. Here, we allow L_0 to be a point or $k = 0$. The following statements are equivalent.*

- (*a*) *The nearby Lagrangian conjecture holds in T^*L.*
- (*b*) *Suppose that L' is a Lagrangian diffeomorphic to L in a symplectic manifold M and that L' is in the Hausdorff closure of $\mathcal{L}\mathrm{Ham}(L'')$ of a H-rational Lagrangian L'' in M. Then, $L' \in \mathcal{L}\mathrm{Ham}(L'')$. The same holds if $\mathcal{L}\mathrm{Ham}(L'')$ is replaced by $\mathcal{L}\mathrm{Symp}_0(L'')$.*

Proof. Suppose that we are in Case *(a)*. The case of $\mathcal{L}\mathrm{Symp}_0(L'')$ follows directly from Proposition 42 together with Theorems 4 and 7. For the case of $\mathcal{L}\mathrm{Ham}(L'')$, take a sequence $\{L_i\}$ in that space with limit L' diffeomorphic to L. By Theorem 7, L'' is a H-rational Lagrangian with same rationality constant as the L_i's — the L_i's respect the hypotheses of Theorem 4, so that they are H-exact in $\mathcal{W}(L')$ for i large. Since all L_i are Hamiltonian isotopic to each other, their associated symplectic isotopy from Proposition 42 must all have the same flux by Corollary 43. But by that proposition, that flux must tend to 0 as $L_i \to L'$. Therefore, for i large, there is a symplectic isotopy in T^*L' sending L_i to L' with zero flux; again, we suppose that the NLC holds here. By Proposition 2.3 of [Ono08] or Lemma 6.7 of [Sol13], that isotopy must be Hamiltonian, and we have closure, i.e. *(a)* implies *(b)*.

Suppose that we are in Case *(b)*, and let L'' be an exact Lagrangian of T^*L. Then, $L_i := \frac{1}{i}L''$ defines a sequence of Lagrangians whose Hausdorff limit is the zero-section L. But note that L_i is Hamiltonian isotopic to L'' since it is the image of the exact Lagrangian L'' by the Liouville flow of T^*L. Since $\mathcal{L}\mathrm{Ham}(L'')$ is Hausdorff closed, $L \in \mathcal{L}\mathrm{Ham}(L'')$, i.e. the NLC holds in T^*L. □

The Lagrangian C^0 flux conjecture then follows directly.

Proof of Corollary 9. Case *(b)* of Corollary 44 implies the Hausdorff closure of $\mathcal{L}\mathrm{Ham}(L')$ and $\mathcal{L}\mathrm{Symp}(L')$ in $\mathcal{L}(L)$ whenever L' is H-rational and diffeomorphic to L. □

Remark 12. *Suppose that all exact Lagrangians of T^*L are known to be* diffeomorphic *to the zero-section. Then, the equivalence of (a) and (b) in Corollary 44 actually proves that the NLC for L and the Lagrangian C^0 flux conjecture for H-rational Lagrangians diffeomorphic to L are equivalent. This hypothesis is satisfied when simple homotopy type is enough to determine diffeomorphism type, e.g. when* $\dim L \leq 3$, $L = S^{2k+1}/\mathbb{Z}_m$ *for $m \geq 3$ (see [Mil66]), or $L = S^n$ for $n \in \{1, 2, 3, 5, 6, 12, 56, 61\}$ (see [WX17]).*

Remark 13. *If NLC holds for T^*L, Corollary 43 actually allows us to identify a Hausdorff neighbourhood of L in $\mathcal{L}(\tau)$ with a neighbourhood of $(L, 0)$ in* $\mathcal{L}\mathrm{Ham}(L) \times$ *W^*, where we recall that W is a complement of the image of the boundary map $H_2(M, L; \mathbb{R}) \to H_1(L; \mathbb{R})$. We do not know how much this extends to a global homeomorphism.*

5.4 Proof of Proposition 15

We now give a proof of Proposition 15, which appeared in the introduction. The proof is essentially an amalgam of the one of Lemma 13 and of Theorem 5.

Proof of Proposition 15. As in Lemma 13, take a Riemannian metric g on M which corresponds to a Sasaki metric on T^*L on a Weinstein neighbourhood $\mathcal{W}_r(L)$ contained in K. Let $r_{\mathrm{inj}}(TM|_K)$ be the injectivity radius of the Riemannian exponential of g restricted to $TM|_K$, and take $\delta = \min\{r, r_{\mathrm{inj}}(TM|_K)\}$. For $\varphi \in \mathrm{Ham}(M)$, we then get a homotopy $\{f_t\}$ from $f_0 = \mathbb{1}$ to $f_1 = \varphi$ using geodesics as in that lemma. Furthermore, by geodesic convexity of $\mathcal{W}_r(L)$, $f_t(L) \subseteq \mathcal{W}_r(L)$ for all t and $f_1(L) = \varphi(L)$ is H-exact in the neighborhood.

Pick a Hamiltonian isotopy $\{\varphi_t\}$ with $\varphi_1 = \varphi$ and denote by $c : S^1 \to C^\infty(M, M)$ the loop of smooth maps given by the concatenation of $\{f_t\}$ with $\{\varphi_{1-t}\}$. Let x_0 be a fixed point of φ such that $t \mapsto \varphi_t(x_0)$ is contractible — this always exists by Floer theory. Then, $f_t(x_0) = x_0$ for all t so that $t \mapsto c(t)(x_0)$ is a contractible loop. But if x is any point in M, there is a path α from x to x_0, so that $(t,s) \mapsto c(t)(\alpha(s))$ defines a free homotopy from the loop defined by x to the one defined by x_0. In particular, all loops $t \mapsto c(t)(x)$ are contractible.

Given a loop γ of $\varphi(L)$, we consider the map $g : \mathbb{T}^2 \to M$ defined by $g(t,s) = c(t)(\gamma(s))$. This decomposes into two cylinders: $\{f_t(\gamma(s))\}$ and $\{\varphi_{1-t}(\gamma(s))\}$. Since $f_t(L) \subseteq \mathcal{W}_r(L)$ for all t and L is exact in $\mathcal{W}_r(L)$, the area of the first cylinder is $\lambda_0(\gamma)$. But the area of the second cylinder is the flux of the isotopy $\{\varphi_{1-t}\}$ on γ, which is zero since the isotopy is Hamiltonian. Therefore, g has area $\lambda_0(\gamma)$.

If we are in Case *(a)*, then note that the loop $t \mapsto g(t,0)$ is contractible by the above discussion, so that the area of g is actually the area of a sphere. We then conclude similarly to Theorem 5: pick a base $\{\gamma_i\}$ of $H_1(\varphi(L))^{\text{free}}$, then the corresponding g_i respect

$$\omega(\operatorname{Ker} c_1|_{\pi_2(M)}) \ni |\omega(g_i)| = |\lambda_0(\gamma_i)| \leq r\ell_g^{\min}(\pi_*\gamma_i)$$

by Proposition 41. It thus suffices to take r — and thus δ — small enough so that $r \max_i \ell_g^{\min}(\pi_*\gamma_i)$ is smaller than the positive generator of $\omega(\operatorname{Ker} c_1|_{\pi_2(M)})$. This ensures that $\lambda_0|_{\varphi(L)}$ vanishes, i.e. $\varphi(L)$ is exact in $\mathcal{W}_r(L)$.

If we are in Case *(b)*, then note that for every γ, there is some k such that γ^k is contractible in M. In particular, there is a homotopy α from γ^k to x_0. Then, $(t,s) \mapsto c(t)(\alpha(s))$ defines a homotopy from g^k defined to the loop $t \mapsto \varphi_t(x_0)$, which is contractible by hypothesis. Therefore,

$$\lambda_0(\gamma) = \frac{1}{k}\lambda_0(\gamma^k) = \frac{1}{k}\omega(g^k) = 0$$

for all loops γ, i.e. $\varphi(L)$ is again exact in $\mathcal{W}_r(L)$. □

Appendix A: Lagrangian Klein bottles in cotangent bundles

We now focus our efforts on the case of Conjecture C where L is the Klein bottle K. This case is already covered by Theorem 2, but we give a more direct, stronger proof of it, which is of independent interest. The proof relies on the deep fact that there is no Lagrangian Klein bottle in $\mathbb{C}^2$ [She09, Nem09].

Theorem 45 *Every Lagrangian Klein bottle in T^*K is H-exact. In other words, $c_K(T^*K) = 0$.*

Proof. Let L be a Lagrangian Klein bottle in T^*K. We equip K and the 2-torus $\mathbb{T}^2$ with the flat metric, so that the covering $p : \mathbb{T}^2 \to K$ is a local isometry. By rescaling if necessary, we can suppose that $L \subseteq D_r^*K$ for r arbitrarily small. In particular, we may choose r small enough so that there exists a Weinstein neighbourhood $\Psi : D_r^*\mathbb{T}^2 \to \mathbb{C}^2$ of the standard Clifford torus $S^1 \times S^1$.

Using the flat metric on $\mathbb{T}^2$ and K, the 2:1 covering $p : \mathbb{T}^2 \to K$ lifts to another 2:1 covering $\widetilde{p} : T^*\mathbb{T}^2 \to T^*K$ which is also a local isometry and symplectomorphism. Therefore, $\widetilde{L} := \widetilde{p}^{-1}(L)$ must be a (possibly disconnected) Lagrangian submanifold of $D_r^*\mathbb{T}^2$. Since $\widetilde{p}|_{\widetilde{L}}$ is also a 2:1 covering, $\widetilde{L}$ must either be two disconnected copies of a Klein bottle or a 2-torus. However, if the former was the case, then each connected component of $\Psi(\widetilde{L})$ would be a Lagrangian

Klein bottle in $\mathbb{C}^2$, which does not exist [She09, Nem09]. Therefore, $\widetilde{L}$ must be a 2-torus. In other words, the composition

$$\mathbb{T}^2 \xrightarrow{2:1} L \xhookrightarrow{i} T^*K$$

admits a lift to $T^*\mathbb{T}^2$, but the composition

$$K \xrightarrow{\sim} L \xhookrightarrow{i} T^*K$$

does not.

We now interpret these statements in algebraic terms. To do so, we first look at the fundamental groups $\pi_1(T^*K) = \langle a, b | ab = b^{-1}a \rangle$ and $\pi_1(L) = \langle a', b' | a'b' = (b')^{-1}a' \rangle$. With these presentations, the subgroups associated to the coverings $T^*\mathbb{T}^2 \to T^*K$ and $\mathbb{T}^2 \to L$ are those generated by $\{a^2, b\}$ and $\{(a')^2, b'\}$, respectively. Denote $i_*(a') = a^k b^\ell$ and $i_*(b') = a^m b^n$. Here, we have made use of the presentation above to conclude that any element of $\pi_1(T^*K)$ can be written in that way. Given the lifting criterion for coverings, the fact that the composition $\mathbb{T}^2 \to L \to T^*K$ admits a lift is equivalent to m being even. Indeed, we have that

$$i_* \left((a')^2 \right) = (i_*(a'))^2 = a^{2k} b^{(1+(-1)^k)\ell},$$

so that this element always admits a lift to $T^*\mathbb{T}^2$. In turn, this forces k to be odd, since the composition $K \to L \to T^*K$ does not admit a lift. In particular, k is nonzero. But a generates the free factor and b the torsion factor of $H_1(T^*K; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$ under the Hurewicz morphism (and analogously for a' and b' in $H_1(L; \mathbb{Z})$). Therefore, i induces a monomorphism $i_* : H_1(L; \mathbb{Z})^{\mathrm{free}} \to H_1(T^*K; \mathbb{Z})^{\mathrm{free}}$ between the free part of the homologies. But then $i_* : H_1(L; \mathbb{R}) \to H_1(T^*K; \mathbb{R})$ is also injective. By the long exact sequence in homology, this implies that the boundary map $\partial : H_2(T^*K, L; \mathbb{R}) \to H_1(L; \mathbb{R})$ is zero. Since $\omega_0(H_2(T^*K, L)) = \lambda_0(\partial(H_2(T^*K, L)))$, L must be H-exact. $\square$

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