Beyond the symplectically aspherical setting

An unofficial appendix to *Morse Theory and Floer homology* by M. Audin and M. Damian
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These are the notes for the final lecture of the course Introduction to Floer theory (401-3584-25L) *that I gave in the Spring 2025 semester at ETH Zürich. The course was based on Audin and Damian's book [AD14], and the purpose of the last lecture was to give a taste to the students of what lies beyond. On top of that book, these notes are also intellectually indepted to Chapter 12 of McDuff and Salamon's modern classic on J-holomorphic curves [MS12].*

Through discussions with some of the students attending that last lecture, I was motivated to write these notes down for posterity. Nonetheless, to ease my work, I will make only minimal effort to recall concepts or notations that have been introduced in Audin and Damian's book — the perplexed reader may use the index there to look up what I am talking about. I hope that I will be forgiven for my simplifications and probable errors.

1. Additional structures

We start this appendix by introducing some additional structures on Floer homology that had not appeared in the course before this point. Note that we stay, for this section, in the setting of the book: spheres have vanishing area and Chern number. These structures are interesting in their own right but will take an additional meaning once we remove these hypotheses.

1.1. Product

Since we always assume that M is a closed ($2n$-dimensional symplectic) manifold, there is a natural product structure on singular homology with $\mathbb{Z}_2$ coefficients:

$$\begin{aligned} \odot \colon H_i(M;\mathbb{Z}_2) \otimes H_j(M;\mathbb{Z}_2) &\longrightarrow H_{i+j-2n}(M;\mathbb{Z}_2) \\ \alpha \otimes \beta &\longmapsto \mathrm{PD}^{-1}\left(\mathrm{PD}(\alpha) \cup \mathrm{PD}(\beta)\right) \end{aligned},$$

where $\mathrm{PD} : H_i(M;\mathbb{Z}_2) \to H^{2n-i}(M;\mathbb{Z}_2)$ is the isomorphism given by Poincaré duality and $\cup : H^i(M;\mathbb{Z}_2) \otimes H^j(M;\mathbb{Z}_2) \to H^{i+j}(M;\mathbb{Z}_2)$ is the cup product.

This is usually called the ***intersection product*** because, if α is represented by a submanifold A, β is represented by a submanifold B, and A and B intersect transversally, then $\alpha \odot \beta$ is represented by the submanifold $A \cap B$.

There is an interpretation of the intersection product in purely Morse theoretical terms [Fuk96]. Broadly, it is done so: for critical points x and y of a Morse function f, the coefficient of $x \odot y$ in the critical point z is given by counting configurations of the following form:

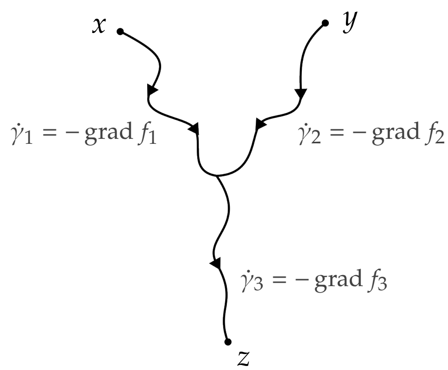


Figure 1: The Y-shaped configurations contributing to the Morse product.

Here, we take generic choices of f_i that agree with f near its critical points, and the point where all three paths intersect is ***not*** a critical point. The usual approach shows that this is well-defined. Looking at possible breakings ensures that the resulting product on the Morse complex respects the Leibniz rule and thus descends to homology. However, showing that this is equal to the intersection product requires more work.

There is then a natural question: can we do the same for Floer homology? But of course!

Definition 1.1.1 *Let $S := S^2 \setminus \{0, 1, \infty\}$, and let $\xi_0, \xi_1 : (-\infty, 0] \times S^1 \to S$ and $\xi_\infty : [0, +\infty) \times S^1 \to S$ be conformal charts in a neighbourhood of 0, 1, and ∞, respectively. Take a nondegenerate Hamiltonian $H : [0,1] \times M \to \mathbb{R}$ and a compatible almost complex structure $J \in \mathcal{J}_c(\omega)$, and consider $\mathcal{H} : S \times M \to \mathbb{R}$ and $\mathcal{J} : S \to \mathcal{J}_c(\omega)$ such that $\mathcal{H}_{\xi_i(s,t)} = H_t$ and $\mathcal{J}_{\xi_i(s,t)} = J$ for all above charts ξ_i and all $(s,t) \in \mathbb{R} \times S^1$. Finally, let x, y, and z be contractible 1-periodic orbits of the above Hamiltonian H.*

The moduli space of **pair of pants** *with asymptotics x, y, and z is given by*

$$\begin{aligned}\mathcal{M}(x, y, z; \mathcal{J}, \mathcal{H}) := \Big\{u : S \to M \,\Big|\, & du_z + \mathcal{J}_z \circ du_z \circ j + (\operatorname{grad} \mathcal{H}_z)_{u(z)} = 0, \\ & \lim_{s \to -\infty} u(\xi_0(s)) = x, \ \lim_{s \to -\infty} u(\xi_1(s)) = y, \\ & \lim_{s \to +\infty} u(\xi_\infty(s)) = z, \ u \text{ contractible}\Big\}\end{aligned}$$

where j is the unique complex structure on S^2.

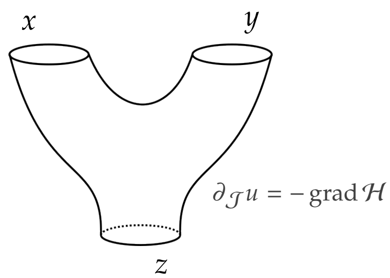


Figure 2: A pair of pants.

Remarks 1.1.2. A direct computation allows one to verify that the equation appearing in the definition of $\mathscr{M}(x, y, z; \mathcal{J}, \mathcal{H})$ reduces to the Floer equation for (J, H) in the charts ξ_i. In particular, the asymptotic conditions appearing in the definition are the usual ones, and all results on the asymptotics of Floer cylinders carry over to pairs of pants.

Using similar techniques to what we have done so far, we get the following.

Theorem 1.1.3 ([PSS96]) *For $\mathcal{H}$ and $\mathcal{J}$ generic, $\mathscr{M}(x, y, z; \mathcal{J}, \mathcal{H})$ is a smooth manifold of dimension $\mu(x) + \mu(y) - \mu(z) - n$. If the dimension is 0, then it is compact. If the dimension is 1, it admits a compactification as a manifold with boundary*

$$\left(\bigsqcup_{\mu(x)-\mu(w)=1} \mathscr{M}(x, w; J, H) \times \mathscr{M}(w, y, z; \mathcal{J}, \mathcal{H})\right) \sqcup \left(\bigsqcup_{\mu(y)-\mu(w)=1} \mathscr{M}(y, w; J, H) \times \mathscr{M}(x, w, z; \mathcal{J}, \mathcal{H})\right)$$
$$\sqcup \left(\bigsqcup_{\mu(w)-\mu(z)=1} \mathscr{M}(x, y, w; \mathcal{J}, \mathcal{H}) \times \mathscr{M}(w, z; J, H)\right)$$

.

The statement on 0-dimensional moduli spaces allows us to define a product via

$$m(x, y) := \sum_{\mu(z)=\mu(x)+\mu(y)-n} (\#_2 \mathscr{M}(x, y, z)) \cdot z,$$

while the statement on 1-dimensional moduli spaces ensures that this product respects the Leibniz rule with respect to the Floer differential (of (J, H)). Therefore, the product descends to homology. In fact, we have a much stronger statement:

Theorem 1.1.4 *For (H, J) generic and (f, g) Morse-Smale, the isomorphism of groups $\Psi : HF_\bullet(H, J) \to HM_{\bullet+n}(f, g)$ (obtained from Chapters 10 and 11 of [AD14]) is in fact an isomorphism of rings, where HM has the intersection product.*

We could go on like this and recreate higher operations on singular homology, e.g. the Massey product, via Floer homology — in the symplectically aspherical setting, that is. However, to keep things relatively short, we will instead move on to another structure.

1.2. Filtration

Although introducing the pair-of-pants product has required some extra analysis, there are some additional structures where only a bit of algebra is required to get us there. This is the case for the theory of ***filtered Floer homology***.

Spectral invariants

The basic observation at the core of filtered Floer homology is that, for every $\lambda \in \mathbb{R}$, the (graded) vector subspace

$$CF_\bullet^{\leq\lambda}(H,J) := \mathbb{Z}_2 \left\langle \{x \in \mathrm{Crit}(\mathscr{A}_H) \mid \mathscr{A}_H(x) \leq \lambda\} \right\rangle$$

of the Floer complex of (H,J) is actually a subcomplex. Indeed, this is ensured by the fact that the action is nonincreasing along its negative gradient trajectories. Note that we take the convention that $\mathbb{Z}_2\langle\varnothing\rangle = 0$.

In particular, we can define the filtered homology $HF_\bullet^{\leq\lambda}(H,J)$ as the homology of the complex $(CF_\bullet^{\leq\lambda}(H,J), \partial_{H,J})$. This is well defined whenever nonfiltered Floer homology is. This observation has been central to the (sub)field of quantitative symplectic topology, as it allows for the definition of various invariants.

Definition 1.2.1 ([Vit92, Oh05a]) *The* **spectral invariant** *associated to a homology class $\alpha \in HM_\bullet(M;\mathbb{Z}_2)$ is given by*

$$c(\alpha;H) := \inf\left\{\lambda \,\middle|\, \alpha \in \mathrm{Im}\left(\Psi : HF_\bullet^{\leq\lambda}(H,J) \to HM_{\bullet+n}(M;\mathbb{Z}_2)\right)\right\}.$$

In other words, $c(\alpha;H)$ is the lowest action level at which (the preimage by Ψ of) α can be written as a linear combination of orbits of at most that action.

Remarks 1.2.2. As the notation suggests, spectral invariants do not depend on the choice of almost complex structure. However, they very much depend on the choice of Hamiltonian. In fact, it is this dependence that is at the heart of all applications.

Spectral invariants enjoy many properties, e.g. under perturbations of H or product of classes in Morse homology. We refer the interested reader to [Oh05a] for more details, but we briefly note the following surprising consequence of these properties.

Theorem 1.2.3 ([Vit92, Oh05b]) *The quantity*

$$\gamma(H) := c([M];H) - c([pt];H)$$

only depends on φ_H^1. It defines a bi-invariant metric, called the **spectral metric**, *on the group of Hamiltonian diffeomorphisms of M.*

Persitence modules

If spectral invariants capture how Morse homology classes fit in Floer homology, persistence modules capture all Floer homology classes, even those that eventually do not appear in the full, unfiltered homology. This tool — originating from applied mathematics — allows one to extract additional invariants beyond spectral invariants, e.g. the boundary depth of Floer homology, which have been of great use in contemporary research.

One can see Floer persistence modules as following from the following observation: Not only do we have a homology for each $\lambda \in \mathbb{R}$, but for a pair $\lambda \leq \mu$, the inclusion $CF_\bullet^{\leq\lambda} \subseteq CF_\bullet^{\leq\mu}$ induces a morphism $HF_\bullet^{\leq\lambda} \to HF_\bullet^{\leq\mu}$. This defines a direct system over $\mathbb{R}$ — whose limit is the unfiltered Floer homology. As mentionned above, this system contains more information than the spectral invariants; it can also be also be conveniently encoded into a nice multiset known as a ***barcode***.

Because this could — and has been — a subject for a course of its own, we will not dwell much more on the subject. We refer the interested reader to Polterovich, Rosen, Samvelyan, and Zhang's introductory book on persistence modules [PRSZ20], which also includes some applications to symplectic topology.

1.3. Coefficients

To close off this section, we finally address a question that has been staring at us this entire time: Why are we working with $\mathbb{Z}_2$ coefficients? Replacing $\mathbb{Z}_2$ by any other field $\mathbb{K}$ of characteristic 2 is trivial. However, when going to other characteristics, we run into a very important problem: not all manifolds are $\mathbb{K}$-orientable and, even when an orientation exists, there are no canonical choice for one.

As it turns out, all the moduli spaces that we are concerned with are indeed orientable — though this is not the case in all variations of Floer homology, e.g. Lagrangian Floer homology. It is the second problem that troubles us. Indeed, how does one pick orientations on all moduli spaces in such a manner that it is compatible with breaking and between different choices of data? Although it is not always the most technically complex modification of Floer homology, doing things properly is famously headache-inducing. Therefore, we will not go further than this and simply mention that it is possible to work over $\mathbb{K} = \mathbb{Q}$, for example — Appendix A.2 of [MS12] explains how to adapt Fredholm theory to that context. Going to integer coefficients is also possible, but one needs to be even more careful.

2. Spheres

We now move to the main subject of this appendix: how to deal with spheres with nonvanishing area or Chern number?

2.1. Nonvanishing Chern class

A remark in class was already made to the effect of removing this assumption. Indeed, it is only used in the definition of the Maslov index (in fact, only in· Section 7.1.a of [AD14]). More precisely, it ensures that two choices of capping of the same orbit induce homotopic paths in $\mathrm{Symp}(n)$ and thus, in turn, the same Maslov index.

In general, when we have $\langle c_1(M), \pi_2(M)\rangle = N\mathbb{Z}$ for $N > 0$, this can no longer be ensured. However, if u and u' are cappings of the same orbit x, we can always write $u' = u\#(\overline{u}\#u')$, i.e. u' is obtained from u by gluing to it the sphere $\overline{u}\#u'$. Therefore, the Maslov index of an orbit only depends on the choice of capping up to recapping by a sphere. Moreover, it follows from the definition of the Maslov index — and some computations — that recapping by a sphere changes the Maslov class by twice its Chern number. Therefore, the Maslov index of an orbit is well defined up to twice the minimum Chern number of M. In other words, Floer homology is only $\mathbb{Z}_{2N}$-graded in general.

Remarks 2.1.1. Even when $CF_\bullet$ is $\mathbb{Z}$-graded, i.e. graded in the usual sense, we can see it as being $\mathbb{Z}_{2N}$-graded by taking

$$CF_k^{\mathbb{Z}_{2N}} = \bigoplus_{i\equiv k \mod 2N} CF_i^{\mathbb{Z}},$$

where the exponents indicate what is the implied grading. This will be useful below.

2.2. Nonvanishing area

The problems arising when $\langle \omega, \pi_2(M)\rangle \neq 0$ are ***much*** more complicated to resolve. The point of this subsection is to explain those problems and offer paths toward their solution. At the end, we also explain how the additional structures of Section 1 interact with spheres of nonvanishing area.

As mentioned in prior lectures, the action functional is now only defined on

$$\widetilde{\mathscr{L}M} := \{(x,v) \in \mathscr{L}M \times C^\infty(\mathbb{D}, M) \mid v|_{S^1} = x\}/\!\sim,$$

where $(x,v) \sim (y,w)$ if and only if $x = y$, $\omega(v\#\overline{w}) = 0$, and $c_1(v\#\overline{w}) = 0$. Note that under the assumptions of [AD14], i.e. symplectical asphericity, $\widetilde{\mathscr{L}M} = \mathscr{L}M$.

Remarks 2.2.1. Consider

$$\Gamma := \frac{\pi_2(M)}{\operatorname{Ker}\omega \cap \operatorname{Ker} c_1}.$$

Note that it acts on $\widetilde{\mathscr{L}M}$ via recappings: $A \cdot [x,v] = [x, A\#v]$. In fact, we even have that $\mathscr{L}M = \widetilde{\mathscr{L}M}/\Gamma$. In particular, there is a cover $p : \widetilde{\mathscr{L}M} \to \mathscr{L}M$ with fiber Γ.

But recall from Exercise 26 of Chapter 6 of [AD14] that there is always a closed 1-form α_H on $\mathscr{L}M$ that is locally the differential of the action functional. The cover p is precisely the minimal one such that $p^*\alpha_H$ is exact, i.e. such that there is some $\mathscr{A} : \widetilde{\mathscr{L}M} \to \mathbb{R}$ with the property that $d\mathscr{A} = p^*\alpha_H$.

We now explain the main problems encountered when trying to define Floer homology without any assumptions on ω (or c_1).

The algebraic problem. By analogy to our previous efforts, we consider

$$\mathscr{M}([x,v],[y,w];J,H) := \Big\{ u : \mathbb{R}\times S^1 \to M \;\Big|\; \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \left(\operatorname{grad} H\right)_u = 0, \lim_{s\to-\infty} u(s) = x,\ \lim_{s\to+\infty} u(s) = y, [y,w] = [x, v\#u] \Big\}.$$

The last compatibility condition should be thought of as the equivalent in the general setting to the one that u be contractible in the symplectically aspherical case.

Suppose that, magically, all moduli spaces between capped orbits of Maslov indices differing by at most 2 are smooth manifolds of the right dimension with appropriate compactifications. Then, we would be tempted to define

$$CF_k(H,J) \text{ “=” } \mathbb{Z}_2 \left\langle \{[x,v] \in \operatorname{Crit}(\mathscr{A}_H) \mid \mu(x,v) = k \mod 2N\} \right\rangle$$

and

$$\partial[x,v] = \sum_{\mu(x,v)-\mu(y,w)=1} \left(\#\mathcal{L}([x,v],[y,w])\right) \cdot [y,w],$$

where $\mathcal{L}$ denotes the quotient of $\mathscr{M}$ under the natural $\mathbb{R}$-action.

However, there is one big problem with this approach: the sum appearing in the definition of the differential ∂ might be infinite, but $CF_k(H,J)$ only contains finite sums.

One would be tempted to simply allow all formal series in the Floer complex to avoid this problem. However, the result has quite poor algebraic properties; see Remark 2.2.3 below. Therefore, a less drastic approach is required.

Solution: Define the Floer complex as the following space of formal series.

$$CF_k(H,J) := \left\{ \sum_{\substack{[x,v]\in\operatorname{Crit}(\mathscr{A}_H)\\ \mu(x,v)=k \mod 2N}} \lambda_{[x,v]}[x,v] \;\middle|\; \#\left\{[x,v] \;\middle|\; \lambda_{[x,v]} \neq 0,\ \mathscr{A}_H(x,v) \leq c\right\} < \infty,\ \forall c \in \mathbb{R} \right\}$$

Here, $\lambda_{[x,v]}$ are coefficients in $\mathbb{Z}_2$.

That this space is indeed closed under the differential follows from the following compactness result.

Theorem 2.2.2 (Gromov–Floer compactness) *Let* $C > 0$. *The space*

$$\{u \in \mathscr{M}(H, J) \mid E(u) \leq C\}$$

is compact in the C^∞_{loc} *topology.*

The proof is essentially that of Theorem 6.5.4 of [AD14]. The reason for the weaker result is that, without the symplectical asphericity assumption, there is no longer a uniform bound on the gradient of Floer cylinders (Proposition 6.6.2 of [AD14]).

Remarks 2.2.3. As a vector space over $\mathbb{Z}_2$, $CF_k(H, J)$ is, in general, infinite-dimensional. However, it has dimension #Crit(α_H), which is finite for H nondegenerate, over the ***Novikov field*** of M:

$$\Lambda_\omega := \left\{ \sum_{A \in \Gamma} \lambda_A q^A \,\middle|\, \#\{A \mid \lambda_A \neq 0,\ \omega(A) \leq c\} < \infty,\ \forall c > 0 \right\}.$$

Here, the action of Λ_ω on $CF_\bullet(H, J)$ is defined via $q^A \cdot [x, v] := [x, A\#v]$. Finite dimensionality is the main algebraic argument for using this definition of the Floer complex rather than allowing every formal series.

Alternatively, this allows us to see $CF_k(H, J)$ as the free Λ_ω-vector space generated by Crit(α_H). This also explains the $\mathbb{Z}_{2N}$-grading: even though we could make sense of a $\mathbb{Z}$-grading over $\mathbb{Z}_2$, we can only talk of a $\mathbb{Z}_{2N}$-grading over Λ_ω.

Finally, we note that this approach was already implicit in Floer's original papers but was spelled out by Hofer and Salamon [HS95]. The name comes from Novikov's work [Nov81] on a version of Morse theory for closed, nonexact 1-forms on a manifold, which also leads to a version of the above field. In general, there are many versions of "the" Novikov field, each better suited to one setting.

The transversality problem. In general, we cannot ensure that $\mathscr{M}([x, v], [y, w])$ is indeed a manifold of the right dimension, no matter how much we perturb (H, J).

Solution: The simplest solution to this problem is to restrict the type of symplectic manifolds that we study. This is the approach that we have adopted so far and that Floer [Flo89] also did; we will see below this allows us to deal with ***monotone*** symplectic manifolds.

If we want to study more general symplectic manifolds, however, we need to develop some theory of a "generalized manifold", where things like counting in

0-dimensional objects still makes sense. A first step in that direction is the notion of ***pseudocycles***, as it appears in McDuff and Salamon's book [MS12], which is a natural continuation of what we have done so far. This allows us to deal with all ***semisimple*** symplectic manifolds, which include all symplectic manifolds of dimension at most six.

However, to truly deal with general symplectic manifold, a fundamental break from everything that we have done so far is needed. Although we will not give a complete bibliography here, we should note that there are three important approaches to do it: the Kuranishi structures of Fukaya, Oh, Ohta, and Ono [FOOO09], the theory of polyfolds of Hofer, Wysocki, and Zehnder [HWZ07, HWZ09b, HWZ09a], and the implicit atlases of Pardon [Par16]. It is important to note that the completeness, rigorousness, and acceptance of each approach vary; it is best to talk to a professional before delving into these methods.

The compactness problem. Even if $\mathscr{M}([x,v],[y,w])$ is a manifold of the right dimension, we could have a problem with its compactification.

In fact, that problem was already present in the proof of Proposition 6.6.2 of [AD14]: along a sequence of Floer cylinders, the gradient at a point could explode, which leads to the existence of a J-holomorphic sphere in the limit. Without any symplectic asphericity assumption, there is no way to avoid this phenomenon. Despite this, there is still a version of Gromov–Floer compactness, where (portions of) cylinders can now be reparametrized to converge to spheres.

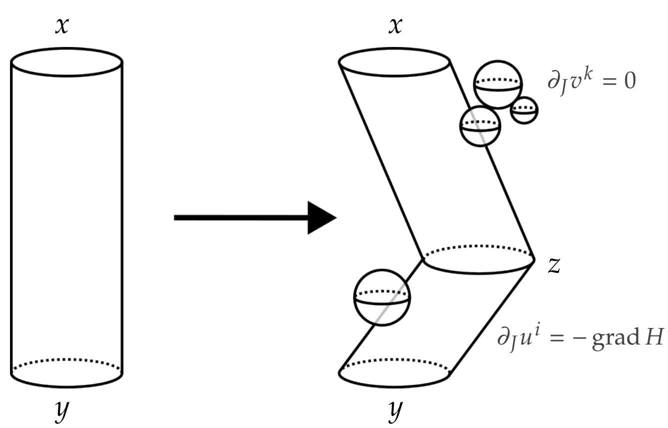


Figure 3: A typical limit with both cylinders u^i and spheres v^k.

Solution: In general, the compactification $\overline{\mathcal{L}}([x,v],[y,w])$ will now include ***stable maps***. Broadly speaking, these are broken Floer trajectories, together with a finite number of J-holomorphic spheres, fitting together appropriately.

To still be able to conclude that $\partial^2 = 0$ however, we then need to show that either these new contributions cancel each other out, or that they somehow do not matter for the relevant counts, e.g. they appear in strata of codimension at least two. To do the latter, we need to use one of the technologies mentioned in the above paragraph.

The isomorphism problem. In view of Section 2.1 and Remark 2.2.3, the best that we can hope for is an isomorphism

$$HF_k(H,J) \cong \bigoplus_{i \equiv k \mod 2N} HM_{i+n}(f,g) \otimes \Lambda_\omega$$

for every $k \in \mathbb{Z}_{2N}$ and any (H,J) and (f,g) generic.

We could try a similar approach to what has been done when H is a C^2-small Morse function and J is generic. However, even when M is a nice symplectic manifold, some heavy machinery is required to make things work. In some sense, this should not be too surprising: the isomorphism with Morse homology was probably one of the least direct proofs that we have done.

Solution: We build an isomorphism — without any smallness or Morse hypotheses on H — by counting "half cylinders". This approach was initiated by Piunikhin, Salamon, and Schwarz [PSS96], and thus, the resulting map is called the ***PSS isomorphism***.

More precisely, we fix a nonincreasing function $\beta : \mathbb{R} \to [0,1]$ such that $\beta(s) = 1$ if $s \le 0$ and $\beta(s) = 0$ if $s \ge 1$. For $[x,v] \in \mathrm{Crit}(\mathscr{A}_H)$ and $y \in \mathrm{Crit}(f)$, we consider the moduli space

$$\begin{aligned}\mathscr{M}^{PSS}([x,v],y;J,H,f,g) := \Big\{(u,\gamma) \in C^\infty(\mathbb{R}\times S^1, M) \times C^\infty([0,+\infty), M) \,\Big|\, & \\ \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \beta\left(\operatorname{grad} H\right)_u = 0,\ \frac{d\gamma}{ds} + (\operatorname{grad} f)_\gamma = 0 & \\ \lim_{s\to-\infty} u(s) = x,\ \lim_{s\to+\infty} u(s) = \gamma(0),\ \lim_{s\to+\infty} \gamma(s) = y & \\ [x,v] = [x,u]\Big\}. &\end{aligned}$$

Here, we can see u as a capping of x because of the asymptotic $\lim_{s\to+\infty} u(s) = \gamma(0)$. In other words, elements of $\mathscr{M}^{PSS}$ are pairs (u,γ), where u is a disk solving some interpolation between the Floer and Cauchy–Riemann equations and γ is a negative gradient trajectory of the Morse function f on M.

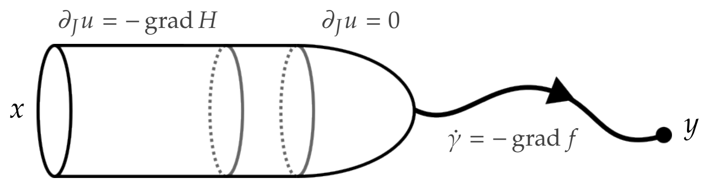


Figure 4: Visualization of an element of $\mathscr{M}^{PSS}([x,v],y;J,H,f,g)$.

We can then define a morphism $\Psi^{PSS} : CF_\bullet(H,J) \to CM_{\bullet+n}(f,g)$ via

$$\Psi^{PSS}(x,v) := \sum_{\mu(x,v)=|y|-n} \left(\#_2 \mathscr{M}^{PSS}([x,v],y)\right) \cdot y.$$

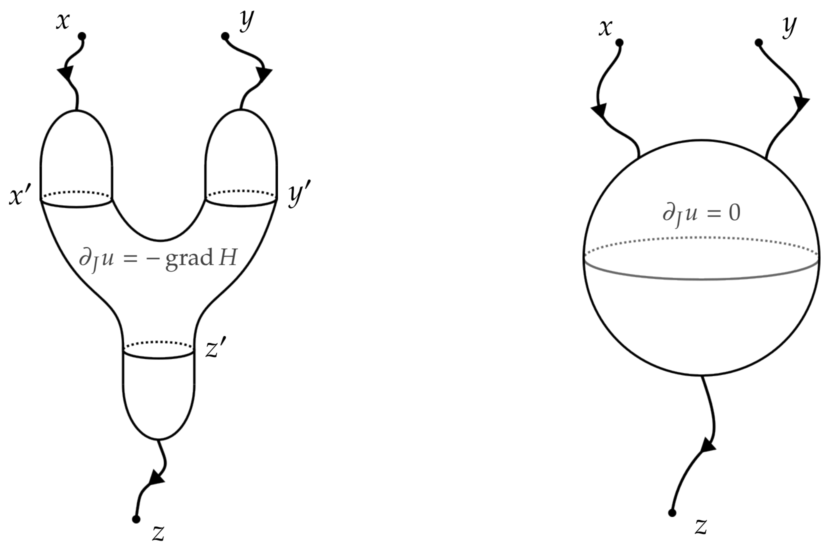
By reversing the roles of $[x,v]$ and y, we can similarly define a morphism $\Phi_{PSS} : CM_\bullet(f,g) \to CF_{\bullet-n}(H,J)$. Looking at 0-dimensional moduli spaces $\mathscr{M}^{PSS}$ and $\mathscr{M}_{PSS}$ ensure that these maps are well defined, while looking at (the compactification) of 1-dimensional ones ensures that they are chain morphisms. One then needs to consider composite moduli spaces — where the analysis notably complexifies — to conclude that $\Psi^{PSS} \circ \Phi_{PSS}$ and $\Phi_{PSS} \circ \Psi^{PSS}$ are chain homotopic to the identity. This finally ensures that Ψ^{PSS} and Φ_{PSS} induce inverse isomorphisms on homology.

Remarks 2.2.4. Even though the isomorphism to Morse homology becomes increasingly more complicated to construct as we relax the hypotheses on M, we can build the continuation maps $CF_\bullet(H^a,J^a) \to CF_\bullet(H^b,J^b)$ in a similar manner to that of Chapter 11 of [AD14]. Therefore, proving the invariance of Floer homology does not require the use of the PSS isomorphism.

Additional structures

We end this subsection by exploring the interaction between the structures of Section 1 with spheres of nonvanishing area and Chern number.

Product. The PSS isomorphism to Morse homology is only an isomorphism of groups. Indeed, in the presence of nontrivial elements in Γ, a pair of pants might not degenerate to a Y-shaped Morse diagram like that in Figure 1. More precisely, we can construct a moduli space of configurations consisting of three Morse flow trajectories with endpoints on a sphere solving some perturbed, parametrized Floer equation. Then, looking at its compactification, we can show that this forms a cobordism between the moduli space appearing in the formula for $\Psi^{PSS} \circ m \circ (\Phi_{PSS} \otimes \Phi_{PSS})$ and a moduli space of Y-shaped Morse diagram, but where the triple intersection has been replaced by a J-holomorphic sphere — see Figure 5 below. Therefore, counting 0-dimensional moduli spaces associated to either configuration gives the same product.



(a) The configuration counted by the composition of the pair-of-pants product with PSS isomorphisms.

(b) The Y-shaped configuration with a J-holomorphic sphere in the middle giving the same count.

Figure 5: Configurations appearing on each end of the moduli space.

Intuitively, if A is the homology class of the sphere in Figure 5b, then such a configuration should contribute to the term $z \otimes q^A$ in $CM_\bullet(f, g) \otimes \Lambda_\omega$. But that term does not appear in the usual intersection product. Note however that if $A = 0$, then we do recover the previous product.

Therefore, the appropriate product on $CM_\bullet(f, g) \otimes \Lambda_\omega$ should be a deformation of the usual intersection product, taking into account these extra configurations. More precisely, we take

$$q(x, y) := \sum_{A \in \Gamma} GW_A(x, y, z)\, z \otimes q^A$$

and call it the ***quantum product***. Here, $GW_A(x, y, z)$ are called ***Gromov–Witten invariants***; they count moduli spaces of J-holomorphic spheres with 3 marked points laying in the un/stable manifolds of x, y, and z. The resulting homology is denoted $QH_\bullet(M)$ and is called the ***quantum homology***. As we shall see below, there are times were this is fundamentally different from the usual homology.

Filtration. Using the PSS isomorphism, we can define spectral invariants similarly to what was done in Section 1.2. The main difference is that the class α in $c(\alpha; H)$ is seen as an element of quantum homology, not Morse homology.

Defining persistence modules is however a bigger problem. Essentially, for the algebra to work, we need to work on a finite-dimensional vector space, and thus over the Novikov field. This causes problems since, when acting by an element of Λ_ω, the action of an orbit shifts. This means that we can only define when a class "appears" up to a certain translation. Under the hypothesis that $\omega(\pi_2(M))$ be discrete, Usher and Zhang [UZ16] have nonetheless managed to define an analogue to the Floer persistence module and have extracted from it many results.

2.3. The monotone case

We close this appendix by dealing with one case beyond the symplectically aspherical world where we can handle spheres of nonvanishing area and Chern number without heavy machinery.

Definition 2.3.1 *A symplectic manifold* (M, ω) *is called* **monotone** *if there exists* $\tau > 0$ *such that*

$$\omega = \tau c_1 \qquad \text{over } \pi_2(M),$$

i.e. the area of a sphere is positively proportional to its Chern number.

The main example of a monotone symplectic manifold is $\mathbb{C}P^n$ with the Fubini-Study form. In general, (partial) flag manifolds can be equipped with monotone symplectic forms.

We now explain how this case interacts with the various problems presented in Section 2.2.

Algebra. Note that, in a monotone symplectic manifold, the element that $A \in \pi_2(M)$ represents in the group Γ (see Remark 2.2.1) can be identified by its Chern number. Therefore, as soon as there is a sphere of nonvanishing area, the Novikov field of M (see Remark 2.2.3) is the field of Laurent series in one variable:

$$\Lambda_\omega \cong \mathbb{Z}_2[q^{-1}, q]].$$

Here, q has degree $|q| = -2N$, where we recall that N is the minimal Chern number of M. In other words, we can identify $CF_\bullet(H, J)$ with the free $\mathbb{Z}_2[q^{-1}, q]]$-vector space generated by $\mathrm{Crit}(\alpha_H)$.

Transversality. To not extend an already long note, we will simply note that transversality works in much the same way as what we have done before — without any of the heavy machinery mentioned above. The main difference is the perturbation scheme: Instead of starting with a pair (H, J) and taking small perturbations of H to achieve transversality, we may need to perturb both H and J to achieve it.

Compactness. Consider a sequence $\{u_k\} \subseteq \mathscr{M}([x,v],[y,w])$, where $\mu(x,v) - \mu(y,w) = 1$. By Gromov–Floer compactness, this sequence of cylinders cannot break in the limit but could *a priori* bubble off. Suppose that there are some non-constant bubbles $v_i : S^2 \to M$ in the limit, and denote by u the limit cylinder.

Gromov–Floer compactness then also gives that the capping $w = v\#u_k$ of y must be homotopic relative boundary to the connected sum of $v\#u$ with all spheres v_i. Therefore, u is in $\mathscr{M}([x,v],[y,(-\sum_i [v_i])\#w])$. But recall from Section 2.1 that recapping by a sphere changes the Maslov index by (minus) twice its Chern number. Therefore,

$$\mu(x,v) - \mu(y,(-\sum_i [v_i])\#w) = 1 - 2\sum_i c_1(v_i) < 0,$$

since $c_1(v_i) \geq 1$ for any sphere with $\omega(v_i) > 0$ on a monotone symplectic manifold. Indeed, recall that nonconstant J-holomorphic spheres have positive area.

But this is of course a contradiction, since $\mu(x,v) - \mu(y,(-\sum_i [v_i])\#w)$ is the dimension of $\mathscr{M}([x,v],[y,(-\sum_i [v_i])\#w])$; the above inequality implies that it must be empty.

We can similarly argue that sequences of cylinders with $\mu(x,v) - \mu(y,w) = 2$ cannot bubble off — we still have that the only nonempty (parametrized) moduli spaces of dimension 0 are the constant ones. Therefore, ∂ is well defined, and $\partial^2 = 0$.

Isomorphism. The PSS isomorphism can be shown to be well-defined and indeed an isomorphism of groups without the use of heavy machinery. However, it will not be an isomorphism of rings in general. To showcase this, we explore the example of the complex projective space.

Example (11.1.12 of [MS12][1])*:* From standard algebraic topology, we know that the homological ring of $\mathbb{C}P^n$ is given by

$$H_{2n-\bullet}(\mathbb{C}P^n;\Lambda_\omega) \cong \frac{\mathbb{Z}[p]}{(p^{n+1}=0)} \otimes_{\mathbb{Z}} \mathbb{Z}_2[q^{-1},q]] = \frac{\mathbb{Z}_2[p,q^{-1},q]]}{(p^{n+1}=0)}.$$

Note that, because of the twist in the grading, $|p| = 2$ and $|q| = -2(n+1)$ here.

However, spheres of nonvanishing area do contribute nontrivially to the quantum product. This exceptionally can be computed somewhat easily because the standard complex structure on $\mathbb{C}P^n$ is regular (see, for example, Proposition 7.4.3 of [MS12]), so that everything follows from complex algebraic geometry. We will skip these computations, but we note that the only relevant relation· is $[\mathbb{C}P^{n-1}]^{n+1} = [\mathbb{C}P^n] \otimes q$,

[1] They work with the cohomological convention and over $\mathbb{Z}[q]$, instead of $\mathbb{Z}_2[q^{-1},q]]$, but that does not affect the relevent computations.

where $[\mathbb{C}P^{n-1}]$ is the class of a hyperplane in $\mathbb{C}P^n$. In other words, we instead have an isomorphism

$$QH_{2n-\bullet}(\mathbb{C}P^n) \cong \frac{\mathbb{Z}_2[p, q^{-1}, q]]}{(p^{n+1} = q)},$$

where the equality in the quotient makes sense in the $\mathbb{Z}_{2(n+1)}$-grading of quantum homology.

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