Reverse Isoperimetric Inequalities for Lagrangian intersection Floer theory

Jean-Philippe Chassé,[1] Jeff Hicks,[2] and Yoon Jae Nick Nho[3]

Abstract

We extend Groman and Solomon's reverse isoperimetric inequality to pseudoholomorphic curves with punctures at the boundary and whose boundary components lie in a collection of Lagrangian submanifolds with intersections locally modelled on $\mathbb{R}^n \cap (\mathbb{R}^k \times \sqrt{-1}\mathbb{R}^{n-k})$ inside $\mathbb{C}^n$. Our construction closely follows the methods used by Duval and Abouzaid and corrects an error appearing in the latter approach.

Contents

I. Introduction

Groman and Solomon's reverse isoperimetric inequality for J-holomorphic curves is an important tool in the study of Floer cohomology of Lagrangian submanifolds. Let (X, ω, J) be a $2n$-dimensional symplectic manifold with a choice of compatible almost complex structure. Given a Lagrangian submanifold $L \subset X$, [GS14, Theorem 1.1] states that there exists a constant A such that, for all J-holomorphic curves $u : (\Sigma, \partial\Sigma) \to (X, L)$ with boundary in L, we have a *reverse isoperimetric inequality*:

$$\mathrm{Length}(u(\partial\Sigma)) \leq A \cdot \mathrm{Area}(u(\Sigma)), \tag{1}$$

where length and area are given by the metric $\omega(\cdot, J\cdot)$. A different proof of this inequality was subsequently given by Duval [Duv16], whose arguments were later adapted to the setting of J-holomorphic polygons with boundary on a configuration of transversely intersecting Lagrangian submanifolds by Abouzaid [Abo21].

An explicit computation of the constant appearing in (1) gives a quantitative bound between the length and area of J-holomorphic curves in terms of the geometry of the Lagrangian L. However, the existence of some constant A bounding the length in terms of area is sufficient

Latest update: August 15, 2023

[1]Partially supported by the Swiss National Science Foundation (200021_204107)

[2]Supported by EPSRC grant EP/V049097/1 (Lagrangians from Algebra and Combinatorics)

[3]Supported by the Cambridge Commonwealth European and International Trust scholarship

for many applications. For example, consider a Liouville domain X and a Lagrangian L that has a cylindrical end. If u is a J-holomorphic curve with boundary on L of bounded energy, then (1) implies that the boundary of u can only travel a fixed distance along the cylindrical end. As a consequence, there is a Gromov-compactness result for curves of this type. Such an idea has been used to ensure the compactness of moduli spaces appearing in the definition of certain quilted Floer cohomology groups [Tor22]. Another application comes from family Floer theory [Abo21], where the convergence of the Floer differential for a nonunitary local system can be proven by showing that the norm of the monodromy of the local system along the boundary of a curve is bounded from above by the perimeter. Similarly, the reverse isoperimetric inequality is useful in adiabatic degeneration situations for multi-graph Lagrangian submanifolds with caustics, where one needs to separate the domain of holomorphic disks into regions that degenerate to Morse flow-trees and regions near the caustics.

In some cases, we can derive tight bounds for the constant A in (1), which endows Floer cohomology with additional structure. For instance, in [Hic19], the second author noticed a relationship between the areas of specific J-holomorphic strips with boundaries on tropical Lagrangian submanifolds and the affine lengths in tropicalization. This observation can be restated in terms of a bound for the constant A in terms of tropical geometry.

When the boundary Lagrangian L is an embedded Lagrangian submanifold, the constant A roughly measures the radius of a standard symplectic neighborhood of L. In this note, we replace L with a collection $\{L_i\}_{i=1}^m$ of Lagrangian submanifolds with pairwise disjoint locally standard intersections (Definition II.1). We also provide a similar description of the constant A in this context.

A reverse isoperimetric inequality for J-holomorphic polygons with boundary on transversely intersecting Lagrangian submanifolds had previously appeared in [Abo21, Appendix A.1]. However, the construction of a weakly plurisubharmonic function in that paper contains an error which we describe in Remark II.6. Therefore, our result also corrects the result appearing there.

Results and strategy of proof

The results that we prove and the method of proof follow closely that of Duval [Duv16]. Let (X, ω, J, g) be a 2n-dimensional almost Kähler manifold. Let S be a Riemann surface with marked boundary points whose boundary arcs $\{C_i\}_{i=1}^m$ are labelled by the collection of embedded Lagrangian submanifolds $\{L_i\}_{i=1}^m$.

Theorem A *Let B be small neighborhood of $\cup_{ij} L_i \cap L_j$, and suppose that the intersections $L_i \cap L_j$ are pairwise disjoint and locally standard (see Definition II.1). There exist constants $K > 0$ and $s_0 > 0$ so that, for any J-holomorphic curve $u : S \to X$ sending the boundary arc C_i of ∂S to L_i, $1 \leq i \leq m$, and for any $s < s_0$, we have that*

$$s \cdot \mathrm{Length}_g(Im(\partial u) \cap B^c) \leq K \cdot \mathrm{Area}_g(Im(u) \cap U_s).$$

Here, $U_s = \bigcup_i N_s(L_i)$, where $N_s(L_i)$ is a tubular neighborhood of L_i of radius s, and

$B^c = X - B$. By modifying the almost complex structure to make transverse intersections locally standard (Proposition II.2), we get the following result.

Corollary B *For any collection of Lagrangian submanifolds $L_1, \ldots, L_m \subset X$ which have pairwise disjoint transverse intersections, there exists a choice of almost complex structure so that a reverse isoperimetric inequality as in (1) holds.*

Remark I.1 As one will see below, given Lagrangian submanifolds $L_1, \ldots, L_m$ with pairwise disjoint transverse intersections and an ω-compatible almost complex structure J', the almost complex structure J satisfying the conclusions of Corollary B can be taken to be C^0-close to J' and equal to J' outside B.

The proof of Theorem A follows the lines of [Duv16], who observes that the square of the distance function $\rho : N_{D_r}(L) \to \mathbb{R}$ can be perturbed to give a strictly plurisubharmonic function $h : N_{D_r}(L) \to \mathbb{R}$ which vanishes on L with weakly plurisubharmonic square root. In that vein, we produce a function $h : U_s \to \mathbb{R}$ which is a small perturbation of $\rho_i : U_s \to \mathbb{R}$ away from a neighborhood of the intersection locus $\cup_{i,j} L_i \cap L_j$ and has weakly plurisubharmonic square root in a neighborhood of the intersection locus. The proof of Theorem A can be broken into three steps:

Section II.a Constructing local models for h near the intersection locus. When L_i and L_j have intersections of the form given by Definition II.1, we show that $\sqrt{\rho_i \rho_j}$ is weakly plurisubharmonic in a neighborhood of the intersection locus.

Section II.b Showing that we can interpolate between the local models near the intersection and the function ρ_i away from the intersection while remaining weakly plurisubharmonic.

Section II.c Modifying Duval's proof to instead use the function $h : U_s \to \mathbb{R}$.

We delay the proofs in Section II.a that the local models of h are plurisubharmonic until Section III to improve readability.

Acknowledgements

Some initial computations for this paper were performed with the aid of Mathematica [WR]. The authors would like to thank M. Abouzaid for encouraging them to write this note. The third author would like to thank his supervisor A. Keating for her useful advice and feedback.

II. The reverse isoperimetric inequality

II.a. The local model near the intersection

We restrict ourselves to Lagrangian submanifolds whose intersections have particularly nice local models.

Definition II.1. We say that the intersection between Lagrangian submanifolds L, L' is locally standard if at every point $x \in L \cap L'$, there exist a chart $U \subset (\mathbb{C}^n, \omega_{\mathbb{C}^n}, J_{\mathbb{C}^n})$, $\phi : U \to X$; and choice of $0 \leq k \leq n$ so that

$$\begin{aligned} &\phi(0) = x && \phi^{-1}(L) = \sqrt{-1}\mathbb{R}^n && \phi^{-1}(L') = \mathbb{R}^{n-k} \times \sqrt{-1}\mathbb{R}^k \\ &\phi^* J = J_{\mathbb{C}^n} && \phi^*\omega = \omega_{\mathbb{C}^n}. \end{aligned}$$

Since intersection points of transversely intersecting Lagrangian submanifolds admit standard neighborhoods, we directly get the following result.

Proposition II.2 *For any pair of transversely intersecting Lagrangian submanifolds L and L′ in (X, ω), there exists a choice of compatible almost complex structure so that the intersection is locally standard.*

Observe that there exist locally standard clean intersections.

EXAMPLE II.3: The following construction comes from [CEL10, Remark, page 9]. Suppose $K = L_0 \cap L_1$ admits a flat metric. Realize the neighbourhood of K as T^*L_0 and L_1 as the conormal NK. Choose a metric g on L such that it is flat in the neighbourhood of K in L_0, makes K totally geodesic, and restricts to a globally flat metric on K. Let J be the almost complex structure on T^*L_0 induced by the connection on T^*L_0 given by g. Taking geodesic normal coordinate sending K to $\mathbb{R}^k \subset \mathbb{R}^n$, we get open charts satisfying the conditions in Definition II.1.

By Bieberbach's theorem, any compact flat Riemannian manifold is a finite quotient of the torus. While this puts a restriction on the topology of the intersection, intersections of this form naturally appear in computations motivated by mirror symmetry.

EXAMPLE II.4: Following the notation from [Hic19]: let $V_1, V_1 \subset Q$ be two tropical subvarieties in an affine manifold Q. Suppose that they intersect cleanly in a collection of points $V_1 \cap V_2 = \{q_1, \ldots, q_k\}$. Whenever V_1, V_2 admit tropical Lagrangian lifts $L_{V_1}, L_{V_2} \subset T^*Q/T^*_{\mathbb{Z}}Q$, then the intersection $L_{V_1} \cap L_{V_2}$ is locally standard and is the union of k disjoint tori of dimension $\dim(Q) - \dim(V_1) - \dim(V_2)$.

For ease of exposition, we will now assume that we are studying J-holomorphic curves with boundary on two Lagrangians L_1, L_2 with locally standard intersections. The local model for this situation is the intersection in $\mathbb{C}^n$ of the Lagrangian planes $L_1 = \{x_i = 0 \mid 1 \leq i \leq n\}$ and $L_2 = \{x_i = 0, y_j = 0 \mid 1 \leq i \leq k, k+1 \leq j \leq n\}$ for some $0 \leq k \leq n$ — the case $k = 0$ corresponds to a transverse intersection.

In what follows, we fix n and k as above and consider the functions

$$\rho_1(x, y) := \sum_{i=1}^{n} x_i^2 \qquad \text{and} \qquad \rho_2(x, y) := \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} y_i^2 \tag{2}$$

on $\mathbb{C}^n = \mathbb{R}^n_x \oplus \sqrt{-1}\mathbb{R}^n_y$. Note that $L_1 = \{\rho_1 = 0\}$ and $L_2 = \{\rho_2 = 0\}$.

Proposition II.5 *The functions $\sqrt{\rho_1\rho_2}$ and $\rho_1\rho_2$ are weakly plurisubharmonic on the standard chart U_x at $x \in L_1 \cap L_2$. Furthermore, outside of some variety V such that $V \cap (L_1 \cup L_2) = L_1 \cap L_2$, $\rho_1\rho_2$ is strictly plurisubharmonic.*

We delay the proof until Section III. We note however that the set V is the precise reason why we need to suppose the existence of standard charts about intersections. Indeed, without it, we do not have an obvious choice of plurisubharmonic function, since $\rho_1\rho_2$ might no longer be — even weakly — plurisubharmonic near V.

REMARK II.6 A different approach to constructing the local model was proposed in [Abo21, Appendix A.1]; unfortunately, this approach contains a gap. The method uses a cutoff function, which is employed to excise a small neighborhood of the intersections before applying the argument from [Duv16]. The proposed local model for plurisubharmonic function is $\rho = \chi(x_1) \cdot |y|^2$, where χ is a cutoff function that is convex and non-negative. Unfortunately, this will usually not be weakly plurisubharmonic. If we restrict to $n = 2$, the determinant of the Levi matrix of ρ is

$$4\chi^2 - (2y_2\chi')^2 + 2|y|^2\chi\chi''$$

Restricting to where $y_1 = 0, y_2 = 1$ we obtain the necessary inequality $2\chi^2 + \chi\chi'' \geq 2(\chi')^2$. Since χ' dominates χ as $x_1 \to 0$, we can simplify to the condition that

$$\chi\chi'' \geq 2(\chi')^2,$$

which is not satisfied, for example, by the standard choice of cutoff function $\exp(-x^{-1})$. In fact, there does not seem to exist a choice of cutoff function which could satisfy this relation.

II.B. INTERPOLATING FROM LOCAL MODEL NEAR INTERSECTIONS TO THE DISTANCE FUNCTION

Let $r_i > 0$ be the radius of a small tubular neighbourhood $N_{r_i}(L_i)$ of L_i. Let $\tilde{\rho}_i$ be the square of the normal distance from L_i — this is well defined on $N_{r_i}(L_i)$ for r_i small enough. By shrinking r_i if necessary, we may assume that $\tilde{\rho}_i$ is (strictly) plurisubharmonic, and that on $U_x \cap N_{r_1}(L_1)$, $\tilde{\rho}_1 = |y|^2$ and similarly for $\tilde{\rho}_2$. Here, for each $x \in L_1 \cap L_2$, U_x denotes the standard neighborhood provided by Definition II.1.

By taking the minimum, we set $r_1 = r_2 = r$. We will also assume that r is less than both half the minimal distance between connected components of the intersection locus, and the Lipschitz constant (which is some $\epsilon > 0$ such that for $x \in L$, $B_\epsilon(x) \cap L$ is contractible, and $d_L(p,q) \leq C d_X(p,q)$ for $p, q \in B_\epsilon(x) \cap L$ for some uniformly finite constant $C > 0$).

The restriction of the functions $\tilde{\rho}_i$ to U_x are the functions ρ_i from (2). Over U_x, $N_r(L_1)$ is given by the set $\{\rho_1 \leq r^2\}$ and $N_r(L_2)$ is given by the set $\{\rho_2 \leq r^2\}$. Let χ be a smooth nondecreasing function such that $\chi(t) = t$ for $0 \leq t \leq \frac{1}{2}$ and $\chi(t) = 1$ for $t \geq 3/4$.

In a local chart $U_x \subset \mathbb{C}^n$ about $x \in L_1 \cap L_2$ as in Definition II.1, define $V_r(x) := (\{\sqrt{\rho_2} < r\} \cup \{\sqrt{\rho_1} < r\}) \cap U_x$.

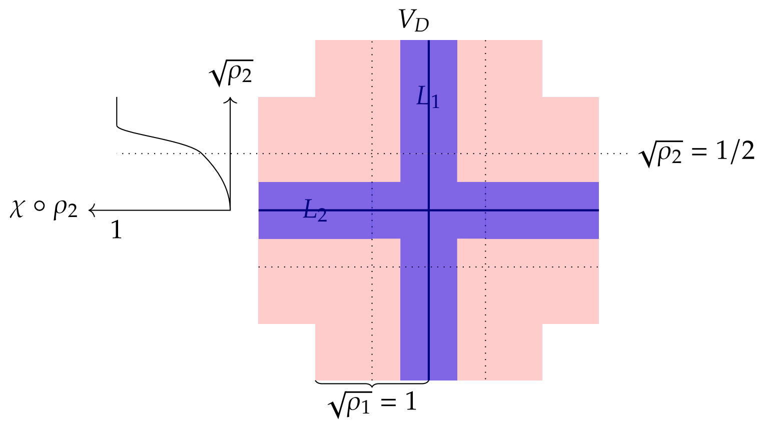

Figure 1. Local model U_x for the intersection between two Lagrangian submanifolds. The red region represents V_1. The blue regions is V_{D_1}, which is divided into three cases by the dashed lines labelling when $\sqrt{\rho_i} = 1/2$.

Proposition II.7 *For $r > 0$ small enough, there exists some $0 < D < 1$ and a nonnegative function $\beta_r : U_x \longrightarrow \mathbb{R}$ whose restriction to $V_{Dr}(x)$ is weakly plurisubharmonic for any x and satisfies*

$$\beta_r = \begin{cases} \rho_1 & \text{on } \{\sqrt{\rho_2} \geq r\} \cap V_{Dr}(x) \\ \rho_2 & \text{on } \{\sqrt{\rho_1} \geq r\} \cap V_{Dr}(x) \end{cases}. \tag{3}$$

Furthermore, β_r vanishes at least up to first order on $\sqrt{-1}\mathbb{R}^n$ and $\mathbb{R}^{n-k} \times \sqrt{-1}\mathbb{R}^k$, and the pseudo-metric obtained from β_r is dominated above by g everywhere and equivalent to g on $\{\sqrt{\rho_1} > r\} \cap V_{Dr}(x)$ and $\{\sqrt{\rho_2} > r\} \cap V_{Dr}(x)$.

PROOF: Observe that the scaling map $(x, y) \longrightarrow \left(\frac{x}{r}, \frac{y}{r}\right)$ sends V_r to V_1 and that V_r contains $V_{r'}$ for $0 < r' < r < 1$. We set $\beta_1(x, y) = \chi(\rho_2)\chi(\rho_1)$ and $\beta_r = r^2\beta_1(\frac{x}{r}, \frac{y}{r})$. We first show that β_1 satisfies Proposition II.7 on a region V_D with $D < 1/2$ by decomposing into three subregions:

- Whenever $\sqrt{\rho_2} < 1/2$ and $\sqrt{\rho_1} < 1/2$, $\beta_1 = \rho_2\rho_1$, which is plurisubharmonic by Proposition II.5.
- Suppose $\sqrt{\rho_1} \geq \frac{1}{2}$ and $\sqrt{\rho_2} < \frac{1}{2}$. Then β_1 has the form $\rho_2\chi(\rho_1)$. So we get

$$\begin{aligned} dd^c\beta_1(\cdot, \sqrt{-1}\cdot) =& \chi(\rho_1)dd^c(\rho_2) + 2\sqrt{\rho_2}\Big(d(\chi(\rho_1)) \wedge d^c\sqrt{\rho_2} + d\sqrt{\rho_2} \wedge d^c(\chi(\rho_1))\Big) \\ & + \rho_2 \cdot dd^c(\chi) \\ =& 2\chi(\rho_1)Id + O(\sqrt{\rho_2}). \end{aligned} \tag{4}$$

  To show that the function β_1 is plurisubharmonic after shrinking $\sqrt{\rho_2}$, we need to show that the form (4) is non-negative.

But since $\chi(\rho_1) \geq \frac{1}{4}$ for $\sqrt{\rho_1} \geq \frac{1}{2}$, we can choose D sufficiently small so that (4) is positive definitive when $\sqrt{\rho_2} < D$.

- The argument is exactly the same for $\sqrt{\rho_2} \geq \frac{1}{2}$ and $\sqrt{\rho_1} < \frac{1}{2}$.

So we have shown that β_1 is plurisubharmonic on V_D for some $0 < D < 1/2$. Furthermore, note that for $\sqrt{\rho_1} \geq 1$, $\beta_1 = \rho_2$ as desired. Likewise, $\beta_1 = \rho_1$ when $\sqrt{\rho_2} \geq 1$.

Now observe that β_r is plurisubharmonic in the intermediate region on V_{Dr} since

$$\beta_r = r^2 \beta_1(z/r)$$

and so

$$\frac{\partial^2 \beta_r}{\partial z_i \partial \bar{z}_j}(z) = \frac{\partial^2 \beta_1}{\partial z_i \partial \bar{z}_j}(z/r) \tag{5}$$

because the r^{-1}-scaling factor cancels out the contribution of r^2. Furthermore, for $\sqrt{\rho_1} \geq r$ and $\sqrt{\rho_2} < \frac{Dr}{2} < \frac{r}{4}$, we have that $\beta_r = \rho_2$. We likewise have $\beta_r = \rho_1$ for $\sqrt{\rho_2} \geq r$ and $\sqrt{\rho_1} < \frac{Dr}{2}$. The vanishing of β_r along $\sqrt{-1}\mathbb{R}^n$ and $\mathbb{R}^{n-k} \times \sqrt{-1}\mathbb{R}^k$ is unchanged by scaling.

Finally, the comparisons with g follow from (4) and (5). For β_1, the uniform metric comparison is obvious, but (5) tells us that the metric coming from β_r has the same components as β_1. Therefore, the metric induced by β_r must be also equivalent to g on $\{\sqrt{\rho_1} > r\} \cap V_{Dr}(x)$ and $\{\sqrt{\rho_2} > r\} \cap V_{Dr}(x)$. It follows that the metric equivalence constant can be chosen independently of r. This finishes the proof. ◻

Choose a finite cover of $L_1 \cap L_2$ by sets $U_{x'}$ given by Definition II.1. Pick D sufficiently small so that the construction of Proposition II.7 on $U_{x'}$ yields functions $\beta_{r,x'}$ which agree on the overlaps of $U_{x'} \cap U_{x''}$. By shrinking $D > 0$ again, this gives us a nonnegative function $\tilde{\beta} : U \to \mathbb{R}$ defined by

$$\tilde{\beta}(x) = \begin{cases} \beta_{r,x'}(x) & \text{if } x \in U_{x'} \\ \rho_i(x) & \text{if } x \in N_{Dr}(L_i) \setminus B_r \end{cases},$$

where $U := \bigcup_i N_{Dr}(L_i)$ and $B_r = \bigcup_{x \in L_1 \cap L_2} \{x' \in U_x \mid \max\{\sqrt{\rho_1}, \sqrt{\rho_2}\} < \frac{r}{2}\}$ is a neighborhood of the intersection locus. The function $\tilde{\beta}$ satisfies the following properties.

- $\sqrt{\tilde{\beta}} = \frac{\sqrt{\rho_1}\sqrt{\rho_2}}{r}$ on B_{Dr}. In particular, $\sqrt{\tilde{\beta}}$ is weakly plurisubharmonic near the clean intersection.
- On $N_{Dr}(L_i) \setminus B_r$, we have $\tilde{\beta} = \rho_i$.

We now modify $\tilde{\beta}$ so that it has weakly plurisubharmonic square root everywhere.

Proposition II.8 *There exist some $D > 0$, constants $C_1, C_2, A_1 > 0$, and a nonnegative function $h : U \to \mathbb{R}$ such that the following holds*:

(1) *h vanishes on $L_1 \cup L_2$;*

(2) *$\sqrt{h}$ is weakly plurisubharmonic on U, and h is strictly plurisubharmonic on $U \setminus B_r$;*

(3) *the pseudometric $k = dd^c h(\cdot, \sqrt{-1}\cdot)$ is dominated by $C_1 g$;*

(4) *on $U \setminus B_r$, the pseudometric k is metric-equivalent to g with $C_2^{-1} g \leq k \leq C_2 g$;*

(5) *$A_1\sqrt{h} \geq |\nabla h|$ outside B_r.*

The strategy of Duval tells us how to perform this modification over the region $U \setminus B_r$, and we already have weak plurisubharmonicity of $\tilde{\beta}$ over B_{Dr}. To handle the remaining region, i.e. $B_r \setminus B_{Dr}$, we need the following estimate in the local model.

Lemma II.9 *There exist constants $C_0, C_1', C_2' > 0$ and $D > 0$, independent of r, such that, for every $x \in L_1 \cap L_2$, $\sqrt{\beta_r} + C_0 r^{-1}\beta_r$ is weakly plurisubharmonic on each $V_{Dr}(x)$ for $r > 0$ small enough, the pseudometric induced by $(\sqrt{\beta_r} + C_0 r^{-1}\beta_r)^2$ is C_1'-dominated by g on $V_r(x)$, and C_2'-equivalent to g on $V_r(x) \setminus B_r$.*

PROOF: For $\sqrt{\rho_2} < \frac{r}{2}$ and $\sqrt{\rho_1} < \frac{r}{2}$, the function $\sqrt{\beta_r}$ is just equal to $r^{-1}\sqrt{\rho_2}\sqrt{\rho_1}$ which is plurisubharmonic by Proposition II.5. For $\sqrt{\rho_1} < \frac{Dr}{2}$ and $\sqrt{\rho_2} > r$, $\sqrt{\beta_r} = \sqrt{\rho_2}$ which is plurisubharmonic. For $\frac{r}{2} < \sqrt{\rho_2} < r$ and $\sqrt{\rho_1} < \frac{Dr}{2}$, the function $\sqrt{\beta_r}$ has the form

$$\sqrt{\beta_r} = \sqrt{\rho_1}\sqrt{\chi\left(\frac{\rho_2}{r^2}\right)},$$

so that

$$dd^c\sqrt{\beta_r} = \sqrt{\chi}\cdot dd^c\sqrt{\rho_1} + d\sqrt{\rho_1}\wedge d^c\sqrt{\chi} + d\sqrt{\chi}\wedge d^c\sqrt{\rho_1} + \sqrt{\rho_1}dd^c\sqrt{\chi}.$$

We split the rest of the proof into two parts. We first show that there exists some C_0' possibly dependent on r^{-1} but *independent* of $\sqrt{\rho_1}$ such that $dd^c\sqrt{\beta_r} + C_0' dd^c\beta_r$ is positive-semi-definitive. Then we show that such a C_0' can be chosen to be of form $C_0 r^{-1}$ where $C_0 > 0$ is some constant.

We first note that the only term that might become unbounded as $\sqrt{\rho_1} \to 0$ is $\sqrt{\chi}dd^c\sqrt{\rho_1}$ since its expression can contain negative powers of ρ_1. However, it is known that the form $dd^c\sqrt{\rho_1}(\cdot, \sqrt{-1}\cdot) \leq 2\sqrt{\chi}dd^c\sqrt{\rho_1}(\cdot, \sqrt{-1}\cdot)$ is positive semidefinite. So the only term that contains negative powers of $\sqrt{\rho_1}$ must already be positive semi-definite. Furthermore, the last three terms may be negative, but they do not contain negative powers of $\sqrt{\rho_1}$. Therefore, their negative contribution may be canceled out by adding some multiple of $dd^c\beta_r$.

In other words, for C_0' large enough, there exists a choice of $D > 0$ so that both terms on the right hand side of

$$\begin{aligned} dd^c\sqrt{\beta_r} + C_0' dd^c\beta_r =& \sqrt{\chi}\cdot dd^c\sqrt{\rho_1} \\ &+ \left(C_0' dd^c\beta_r + d\sqrt{\rho_1}\wedge d^c\sqrt{\chi} + d\sqrt{\chi}\wedge d^c\sqrt{\rho_1} + \sqrt{\rho_1}dd^c\sqrt{\chi}\right). \end{aligned}$$

are positive semidefinite on V_{Dr}. This proves the first part of the claim.

We are now left with showing the r-dependence of C_0'. Note that each differentiation of $\sqrt{\chi}$ gives a r^{-1} contribution since $\sqrt{\rho_2}$ is of the order of r in this region. Since $\sqrt{\rho_1} < \frac{Dr}{2}$ and $|d\sqrt{\rho_1}|$ is uniformly bounded in r, this means that the terms

$$d\sqrt{\rho_1} \wedge d^c\sqrt{\chi} + d\sqrt{\chi} \wedge d^c\sqrt{\rho_1} + \sqrt{\rho_1}dd^c\sqrt{\chi}$$

behave like r^{-1} with regard to the radius r. Therefore, we may take C_0' of the form $\frac{C_0}{r}$ for $C_0 \geq 0$ independent of r. Hence, it follows that $\sqrt{\beta_r} + C_0 r^{-1}\beta_r$ is weakly plurisubharmonic for C_0 independent of r.

We now show that the resulting pseudometric is dominated by $C_1'g$ on U as we vary r. Observe first that when we square the function $\sqrt{\beta_r} + C_0 r^{-1}\beta_r$, we obtain

$$\rho_1\chi\left(\frac{\rho_2}{r^2}\right) + C_0^2 r^{-2}\rho_1^2\chi\left(\frac{\rho_2}{r^2}\right)^2 + C_0 r^{-1}\chi^{3/2}\left(\frac{\rho_2}{r^2}\right)\sqrt{\rho_1}^3. \tag{6}$$

To show that C_1' can be chosen independent of negative powers of r, we show that the second derivatives of terms in (6) are of size $O(r^0)$. Observe as above that each differentiation introduces a r^{-1}-contribution. For instance, the term $\rho_1\chi\left(\frac{\rho_2}{r^2}\right)$ is of size $O(r^2)$ but differentiation of the ρ_1 term reduces the size to $O(r)$, and the differentiation of the term $\chi\left(\frac{\rho_2}{r^2}\right)$ gives $\chi'\left(\frac{\rho_2}{r^2}\right)\cdot\frac{\sqrt{\rho_2}}{r^2}$ which is of size $O(r^{-1})$. Continuing in this manner, it follows that the second derivatives of each of the terms in (6) are of size $O(r^0)$. This shows the claim on metric domination in the intermediate region. Near the intersection itself, the same argument can be applied to show that the pseudometric is dominated by some $C_1'g$.

For equivalence with $C_2'g$ on $U \setminus B_r$, observe that for $\sqrt{\rho_2} > r$ (6) becomes

$$\rho_1 + C_0^2 r^{-2}\rho_1^2 + C_0 r^{-1}\sqrt{\rho_1}^3.$$

The argument above tells us that each differentiation introduces a r^{-1} contribution but since the expression is of size $O(r^2)$, it follows that the size of the second derivatives is again, $O(r^0)$. The crucial ingredient now is that $\chi > 1/2$ in the intermediate region $\frac{1}{2} < \sqrt{\rho_2} < 1$. This implies that the Hessian of $(\sqrt{\beta_r} + C_0 r^{-1}\beta_r)^2$ is of the form $Id + O(r^0)$, and we can make the $O(r^0)$ contribution small enough by taking $\sqrt{\rho_1} < \delta r$ for some small constant $\delta > 0$. We see that we have neither lost the metric domination nor the metric equivalence for say, $\sqrt{\rho_2} > r$. We then just need to take $D \leq \delta$. □

We now finally have every tool necessary to prove Proposition II.8.

Proof of Proposition II.8: Following Duval's approach [Duv16], we prove that there is some constant C_0, here independent of r, such that $h = \left(\sqrt{\tilde{\beta}} + C_0 r^{-1}\tilde{\beta}\right)^2$ respects conditions (1)–(5) and that the constants in II.8 are also independent of r. The only subtlety comes from checking the r-independence; the rest of the proof is the same.

Lemma II.9 implies that conditions (1)–(5) already hold in the neighbourhood $V_r = \cup_{x\in L_1\cap L_2} V_r(x)$ of the intersection locus — Item (5) being vacuously true there. Therefore, it suffices to show these conditions hold on $N_r(L_i) \setminus B_r$, where $\tilde{\beta} = \rho_i$. These proofs are essentially those of Duval [Duv16], but we need to be especially careful, so as to make sure that the metric equivalence constants do not get a r^{-1}-contribution.

We follow the proof of Duval closely. We first start by taking good coordinates in a Weinstein neighborhood of a Lagrangian $L = L_i$ constructed as follows. Given $x \in L$ and a small neighbourhood V of x in M, take $W = V \cap L$. By making V smaller if necessary, we can take W to be a geodesic normal coordinate chart $\phi : W \to \mathbb{R}^n$ which extends to a relative chart $\psi : (V, W) \to (\mathbb{C}^n, \mathbb{R}^n)$ via

$$\psi^{-1}(x+iy) = \exp_{\phi^{-1}(x)}\left(J(\phi^{-1}(x))d\phi_x^{-1}(y)\right).$$

We have made use of the fact that L is Lagrangian, so that J sends its tangent bundle to its normal one. Along W, this identifies J with J_0, and the C^k-norms of ψ depends only on J and W. The distance function to L is of the form $|y| + O(|y|^2)$ and the almost complex structure, of the form $J = J_0 + O(|y|)$. The scheme then follows the logic of the proof of Proposition II.7, but we give here some details.

We first check Item (2). The Taylor expansion of $\sqrt{\tilde{\beta}} + C_0 r^{-1}\tilde{\beta}$ is

$$\sqrt{h} = |y| + Cr^{-1}|y|^2 + C(x)O(|y|^2) + r^{-1}O(|y|^3). \tag{7}$$

Now, we already know that $|y|$ is weakly plurisubharmonic, and since $C_0 r^{-1}$ is already quite large and $dd^c(Cr^{-1}|y|^2)(\cdot, \sqrt{-1}\cdot) = 2Cr^{-1}Id$, the possible negativity coming from the remaining two terms of (7) can be controlled. This shows the first part of Item (2).

Now we show the second part of Item (2) and Item (3). The Taylor expansion of $\left(\sqrt{\tilde{\beta}} + C_0\tilde{\beta}\right)^2$ gives

$$h = |y|^2 + C_0^2 r^{-2}|y|^4 + r^{-1}|y|^3 + r^{-1}O(|y|^3) + r^{-2}O(|y|^6).$$

As before, differentiating once can create at most an r^{-1} contribution; since all the terms are of $O(|y|^2)$, by differentiating twice, we reduce to terms of size $O(r^0)$. So shrinking to $|y| < \delta r$ for some small $\delta > 0$, we see that

$$dd^c h(\cdot, \sqrt{-1}\cdot) = 2 \cdot Id + O(\delta^2, r^0),$$

again since $dd^c|y|^2(\cdot, \sqrt{-1}\cdot) = 2\cdot Id$. This shows both (strict) plurisubharmonicity and metric equivalence to g.

We now show the final condition: Item (5). Using that $d|y|^4 = 4|y|^2 y_i dy_i$, and that $d|y|^3 = 3y_i|y|dy_i$, we obtain

$$|d(r^{-2}|y|^4 + r^{-1}|y|^3)| \le C|y|.$$

But $d(|y|^2) = 2y_i dy_i$ and so it follows that

$$(\nabla h) \le C'|y|.$$

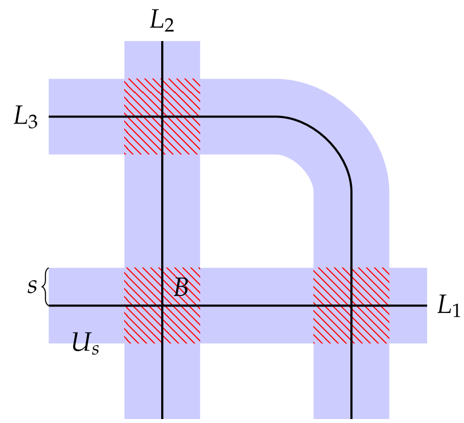

Figure 2. The neighborhood U_s is highlighted in blue, while the region B (which is excluded in computing the length) has red hash lines.

for some constant $C' > 0$. But for this local form, we can find some constant $C'' > 0$ such that $|y| \leq C''\sqrt{h}$. So we have proved Item (5). This finishes the proof of Proposition II.8.

□

REMARK II.10 In view of the above, we see that the L_i's in Theorem A can be allowed to be immersed as long as the self-intersection locus is disjoint from $\cup_{ij} L_i \cap L_j$ and also locally standard. We then only need to modify $\tilde{\beta}$ on $N_s(L_i)$ away from $\cup_{ij} L_i \cap L_j$ to be $\tilde{\rho}_i$ away from the self-intersection locus and of the form $\sqrt{\rho_1}\sqrt{\rho_2}$ inside its standard charts. The set B will then need to be a neighborhood of the entire intersection locus, not just $\cup_{ij} L_i \cap L_j$.

Likewise, we can allow $L_i = L_{i+1} = L$ with the corresponding marked point of ∂S being sent to a self-intersection point. For example, this implies that Theorem A gives an estimate for teardrops.

II.c. Proof of Theorem A

Let $U_s = \cup_i N_s(L_i)$ and $U_s^h := \{h \leq s^2\}$ for $s < Dr$.

Corollary II.11 *There exists some $K > 0$ such that*

$$\frac{K}{s} \operatorname{Area}_g(C \cap U_s) \geq \operatorname{Length}_g(\partial C \cap B^c) \tag{8}$$

PROOF: First, observe that there exists some $l_1 > 0$ such that U_s contains $U^h_{l_1 s}$. By metric domination, we then see that

$$C_1 \operatorname{Area}_g(C \cap U_s) \geq \operatorname{Area}_h(C \cap U^h_{l_1 s}).$$

But just as in [Duv16], we can show that the function $\frac{1}{t}\operatorname{Area}_h(C\cap U_t^h)$ is monotone increasing. We can also show that $\frac{1}{s}\operatorname{Area}_h(C\cap U_{l_1 s}^h)$ is bounded below by $C_3\operatorname{Length}_g(\partial C\cap B^c)$ for some constant $C_3>0$. Indeed, for all $t\leq l_1 s$,

$$\begin{aligned}\frac{1}{t}\int_{C\cap U_t^h} dd^c h &= \frac{1}{t^2}\int_{C\cap U_t^h} t\, dd^c h\\ &\geq \frac{1}{t^2}\int_{C\cap U_t^h\cap B^c} t\, dd^c h \geq \frac{A^{-1}}{t^2}\int_{C\cap U_t^h\cap B^c} |\nabla h| dd^c h\\ &\geq \frac{C_3}{t^2}\int_0^{t^2}\operatorname{Length}_h(C\cap\{h=\tau\}\cap B^c)d\tau. \end{aligned}\tag{9}$$

The constant A is chosen so that $|\nabla h|\leq A\cdot t$ in U_t^h. To pass from the first line to the second line, use the plurisubharmonicity of h, and for the second inequality on the second line, use the final condition in Proposition II.8. Finally, to pass from the second line to the third line, use the coarea formula. Note that the limit of (9) as $t\to 0$ is $C_3\operatorname{Length}_h(\partial C\cap K^c)$. Using the C_2-metric equivalence of $dd^c h(\cdot,\sqrt{\cdot})$ and g to pass to the g-length, we get (8). □

III. PROOF OF PROPOSITION II.5

We give here the full computations necessary to the proof of Proposition II.5 in the general clean case. To retrieve the transverse case, i.e. the case $k=0$, one can ignore the computations for $i\leq k$ and set $\delta_{i\leq k}=0$ and $\delta_{i>k}=1$ in the notation below.

To reduce the number of subscripts in this section, we adopt the notation that $\alpha:=\rho_1,\beta:=\rho_2$. We will need the following two technical lemmata, which we will prove later on. To enunciate them, we introduce the following notation:

$$\delta_{i\leq k}=\begin{cases}1 & \text{if } i\leq k\\ 0 & \text{if } i>k\end{cases} \quad\text{and}\quad \delta_{i>k}=\begin{cases}1 & \text{if } i>k\\ 0 & \text{if } i\leq k\end{cases}.$$

Lemma III.1 *Set* $v_0:=\sum_i(\alpha y_i\delta_{i>k}\frac{\partial}{\partial x_i}+(\beta-\alpha\delta_{i\leq k})x_i\frac{\partial}{\partial y_i})$ *and* $w_0:=-J_{\mathbb{C}^n}v_0$*, where* $J_{\mathbb{C}^n}$ *is the* $(2n\times 2n)$*-matrix representing multiplication by* $\sqrt{-1}$ *in* $\mathbb{C}^n=\mathbb{R}^{2n}$*. The matrix* M_0 *representing the form* $dd^c\sqrt{\alpha\beta}(\cdot,\sqrt{-1}\cdot)$ *in the standard basis* $\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial y_n}\}$ *is equal to*

(*a*) $\dfrac{2\sum_{i\leq k}x_i^2}{\sqrt{\alpha\beta}}Id$ *on* $\operatorname{span}_{\mathbb{R}}\{v_0,w_0\}$;

(*b*) $\dfrac{\alpha+\beta}{\sqrt{\alpha\beta}}Id$ *on the orthogonal complement* $\operatorname{span}_{\mathbb{R}}\{v_0,w_0\}^\perp$

outside of $L_1\cup L_2$*. Here, we take the convention that* $\sum_{i\leq k}x_i^2=0$ *if* $k=0$*. In particular,* $\sqrt{\alpha\beta}M_0$ *is a multiple of the identity matrix precisely on* $L_1\cup L_2\cup S_0$*, where* $S_0:=\{x_i=y_i=0|i>k\}$*.*

Lemma III.2 *Set* $v_1 := \sum_i (\alpha y_i \delta_{i>k} \frac{\partial}{\partial x_i} - (\beta + \alpha\delta_{i\le k}) x_i \frac{\partial}{\partial y_i})$ *and* $w_1 := J_{\mathbb{C}^n} v_1$*. The matrix* M_1 *representing the form* $(d\sqrt{\alpha\beta} \wedge d^c\sqrt{\alpha\beta})(\cdot, \sqrt{-1}\cdot)$ *in the standard basis* $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\}$ *is equal to*

(*a*) $\left(2\sum_{i\le k} x_i^2 + \alpha + \beta\right) Id$ *on* $\mathrm{span}_{\mathbb{R}}\{v_1, w_1\}$*;*

(*b*) 0 *on the orthogonal complement* $\mathrm{span}_{\mathbb{R}}\{v_1, w_1\}^\perp$.

In particular, M_1 *is* 0 *precisely on* $L_1 \cup L_2$*.*

Proof of Proposition II.5: Since weak plurisubharmonicity of a function f is equivalent to the positive semidefinitiveness of the form $dd^c f(\cdot, \sqrt{-1}\cdot)$, weak plurisubharmonicity of $\sqrt{\alpha\beta}$ follows directly from Lemma III.1.

For the weak plurisubharmonicity of $\alpha\beta$, note that

$$dd^c\alpha\beta = 2\sqrt{\alpha\beta}\left(dd^c\sqrt{\alpha\beta}\right) + 2\left(d\sqrt{\alpha\beta} \wedge d^c\sqrt{\alpha\beta}\right) \tag{10}$$

outside of $L_1 \cup L_2$. One can also directly check that the formula holds also on $L_1 \cup L_2$ by taking limits. Since the sum of positive semidefinite matrices is still positive semidefinite, weak plurisubharmonicity of $\alpha\beta$ then follows from Lemmata III.1 and III.2.

It also follows from (10) that there exists a vector $v \in T_{(x,y)}\mathbb{R}^{2n} = \mathbb{R}^{2n}$ such that $dd^c\alpha\beta(v, \sqrt{-1}v) = 0$ if and only if the kernels of $\sqrt{\alpha\beta}M_0$ and M_1 intersect nontrivially. In view of Lemmata III.1 and III.2, this means one of two things:

(a) either $(x, y) \notin L_1 \cup L_2 \cup S_0$, $x_i = 0$ for all $i \le k$, and $\mathrm{span}_{\mathbb{R}}\{v_0, w_0\} \cap \mathrm{span}_{\mathbb{R}}\{v_1, w_1\}^\perp \neq \{0\}$;

(b) $(x, y) \in L_1 \cup L_2 \cup S_0$ and $\alpha + \beta = 0$.

In option (b), note that $(x, y) \in L_1 \cup L_2$ and $\alpha + \beta = 0$ is equivalent to $(x, y) \in L_1 \cap L_2$. But $(x, y) \in S_0$ and $\alpha + \beta = 0$ is also equivalent to $(x, y) \in \{x_i = y_j = 0 | 1 \le i \le n, j > k\} = L_1 \cap L_2$. Therefore, option (b) reduces to $(x, y) \in L_1 \cap L_2$.

Suppose now that we are in option (a). Since $\mathrm{span}_{\mathbb{R}}\{v_i, w_i\}$ is a 1-dimensional complex subspace, $\mathrm{span}_{\mathbb{R}}\{v_0, w_0\} \cap \mathrm{span}_{\mathbb{R}}\{v_1, w_1\}^\perp \neq \{0\}$ is equivalent to $v_0 \perp v_1$, $v_0 \perp w_1$, and $v_0, v_1 \neq 0$. However, the fact that $(x, y) \notin L_1 \cup L_2 \cup S_0$ ensures precisely that the last condition is automatically satisfied. Therefore, option (a) reduces to

$$\begin{cases} x_i = 0 & \forall i \le k \\ \sum_{i>k} x_i y_i = 0 \\ \alpha = \beta \end{cases}$$

since $\alpha, \beta \neq 0$ here.

Noting that points respecting option (b) also respect these equations, we thus get that the set where $dd^c\alpha\beta$ is degenerate, i.e. where $\alpha\beta$ is not strictly plurisubharmonic, is the variety

$$V = \left\{ x_i = 0,\ \sum_{j>k} x_j y_j = 0,\ \alpha = \beta \,\middle|\, i \leq k \right\}.$$

□

REMARK III.3 When $n-k \leq 1$, we simply have that $V = L_1 \cap L_2$. However, when $n-k \geq 2$, V will be a bigger $(k+2)$-codimensional variety of $\mathbb{R}^{2n}$. For example, when $n-k=2$, it is the union of the two n-planes $\{x_i = 0, x_n = \pm y_{n-1}, y_n = \mp x_{n-1} | i \leq k\}$.

PROOF OF LEMMA III.1: The bilinear form $dd^c\sqrt{\alpha\beta}(\cdot, \sqrt{-1}\cdot)$ can be computed in coordinates to be

$$\begin{aligned}
&\left(\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}}\right)\sum_{i=1}^{n}(dx_i \otimes dx_i + dy_i \otimes dy_i) \\
&\quad + \sum_{i,j=1}^{n}\left[\left(\frac{\delta_{i\leq k} + \delta_{j\leq k}}{\sqrt{\alpha\beta}} - \sqrt{\frac{\alpha}{\beta^3}}\delta_{i\leq k}\delta_{j\leq k}\sqrt{\frac{\beta}{\alpha^3}}\right)x_i x_j - \sqrt{\frac{\alpha}{\beta^3}}y_i y_j \delta_{i>k}\delta_{j>k}\right] \\
&\qquad \times (dx_i \otimes dx_j + dy_i \otimes dy_j) \\
&\quad + \sum_{i,j=1}^{n}\left[\left(\frac{1}{\sqrt{\alpha\beta}} - \sqrt{\frac{\alpha}{\beta^3}}\delta_{i\leq k}\right)x_i y_j \delta_{j>k} - \left(\frac{1}{\sqrt{\alpha\beta}} - \sqrt{\frac{\alpha}{\beta^3}}\delta_{j\leq k}\right)x_j y_i \delta_{i>k}\right] \\
&\qquad \times (dx_i \otimes dy_j + dy_j \otimes dx_i).
\end{aligned} \tag{11}$$

Putting the expression for v_0 in (11) gives, for $j \leq k$, that

$$dd^c\sqrt{\alpha\beta}\left(v_0, \sqrt{-1}\frac{\partial}{\partial x_j}\right) = \sum_{i>k}\left(\sqrt{\frac{\alpha}{\beta}} - \sqrt{\frac{\beta}{\alpha}}\right)x_i x_j y_i + \sum_{i>k}\left(\sqrt{\frac{\beta}{\alpha}} - \sqrt{\frac{\alpha}{\beta}}\right)x_i x_j y_i = 0$$

and

$$
\begin{aligned}
dd^c\sqrt{\alpha\beta}\left(v_0, \sqrt{-1}\frac{\partial}{\partial y_j}\right) &= \left(\sqrt{\frac{\beta^3}{\alpha}} - \sqrt{\frac{\alpha^3}{\beta}}\right) x_j \\
&\quad + \sum_{i\le k}\left(3\sqrt{\frac{\beta}{\alpha}} - 3\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\alpha^3}{\beta^3}} - \sqrt{\frac{\beta^3}{\alpha^3}}\right) x_i^2 x_j \\
&\quad + \sum_{i>k}\left[\left(\sqrt{\frac{\beta}{\alpha}} - \sqrt{\frac{\beta^3}{\alpha^3}}\right) x_i^2 x_j - \left(\sqrt{\frac{\alpha}{\beta}} - \sqrt{\frac{\alpha^3}{\beta^3}}\right) x_j y_i^2\right] \\
&= 2\left(\sqrt{\frac{\beta}{\alpha}} - \sqrt{\frac{\alpha}{\beta}}\right) x_j \sum_{i\le k} x_i^2 + \left(\sqrt{\frac{\beta^3}{\alpha}} - \sqrt{\frac{\alpha^3}{\beta}}\right) x_j \\
&\quad + \left(\sqrt{\frac{\beta}{\alpha}} - \sqrt{\frac{\beta^3}{\alpha^3}}\right) \alpha x_j - \left(\sqrt{\frac{\alpha}{\beta}} - \sqrt{\frac{\alpha^3}{\beta^3}}\right) \beta x_j \\
&= \frac{2\sum_{i\le k} x_i^2}{\sqrt{\alpha\beta}} b_j.
\end{aligned}
$$

Likewise, for $j > k$, one gets from (11) that

$$
\begin{aligned}
dd^c\sqrt{\alpha\beta}\left(v_0, \sqrt{-1}\frac{\partial}{\partial x_j}\right) &= \left(\sqrt{\frac{\alpha^3}{\beta}} + \sqrt{\alpha\beta}\right) y_j - \sum_{i\le k}\left(\sqrt{\frac{\beta}{\alpha}} - 2\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\alpha^3}{\beta^3}}\right) x_i^2 y_j \\
&\quad - \sum_{i>k}\left[\sqrt{\frac{\beta}{\alpha}} x_i x_j y_i + \sqrt{\frac{\alpha^3}{\beta^3}} y_i^2 y_j + \sqrt{\frac{\beta}{\alpha}} x_i^2 y_j - \sqrt{\frac{\beta}{\alpha}} x_i x_j y_i\right] \\
&= 2\sqrt{\frac{\alpha}{\beta}} y_j \sum_{i\le k} x_i^2 - \sqrt{\frac{\beta}{\alpha}} \alpha y_j - \sqrt{\frac{\alpha^3}{\beta^3}} \beta y_j + \sqrt{\frac{\alpha^3}{\beta}} y_j + \sqrt{\alpha\beta} y_j \\
&= \frac{2\sum_{i\le k} x_i^2}{\sqrt{\alpha\beta}} a_j
\end{aligned}
$$

and

$$\begin{aligned} dd^c\sqrt{\alpha\beta}\left(v_0,\sqrt{-1}\frac{\partial}{\partial y_j}\right) &= \left(\sqrt{\alpha\beta}+\sqrt{\frac{\beta^3}{\alpha}}\right)x_j + \sum_{i\le k}\left(\sqrt{\frac{\beta}{\alpha}}-\sqrt{\frac{\alpha}{\beta}}-\sqrt{\frac{\beta^3}{\alpha^3}}+\sqrt{\frac{\beta}{\alpha}}\right)x_i^2x_j \\ &\quad - \sum_{i>k}\left[\sqrt{\frac{\beta^3}{\alpha^3}}x_i^2x_j+\sqrt{\frac{\alpha}{\beta}}x_iy_iy_j-\sqrt{\frac{\alpha}{\beta}}x_iy_iy_j+\sqrt{\frac{\alpha}{\beta}}x_jy_i^2\right] \\ &= 2\sqrt{\frac{\beta}{\alpha}}x_j\sum_{i\le k}x_i^2+\left(\sqrt{\alpha\beta}+\sqrt{\frac{\beta^3}{\alpha}}\right)x_j-\sqrt{\frac{\alpha}{\beta}}\beta x_j-\sqrt{\frac{\beta^3}{\alpha^3}}\alpha x_j \\ &= \frac{2\sum_{i\le k}x_i^2}{\sqrt{\alpha\beta}}b_j. \end{aligned}$$

Therefore, when nonzero, v_0 is an eigenvector of M_0 with associated eigenvalue $\frac{2\sum_{i\le k}x_i^2}{\sqrt{\alpha\beta}}$. But note that M_0 commutes with J_0, the $2n\times 2n$ matrix representing multiplication by i, since $dd^c\sqrt{\alpha\beta}\left(\cdot,\sqrt{-1}\cdot\right)$ is a $\mathbb{C}$-sesquilinear form. Therefore, $w_0 := -J_0v_0 = \sum_i((\beta-\alpha\delta_{i\le k})x_i\frac{\partial}{\partial x_i} - \alpha y_i\delta_{i>k}\frac{\partial}{\partial y_i})$ must also be an eigenvector with the same eigenvalue.

Suppose now that $v=\sum_j(a_j\frac{\partial}{\partial x_j}+b_j\frac{\partial}{\partial y_j})$ is orthogonal to both v_0 and w_0, i.e.

$$\sum_{i>k}(\alpha y_i\alpha_i+\beta x_ib_i)+(\beta-\alpha)\sum_{i\le k}x_ib_i=0 \tag{12}$$

and

$$(\beta-\alpha)\sum_{i\le k}x_ia_i+\sum_{i>k}(\beta x_ia_i-\alpha y_ib_i)=0. \tag{13}$$

Denote by m the matrix representing the form $dd^c\sqrt{\alpha\beta}\left(\cdot,\sqrt{-1}\cdot\right)-(\sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}})(\cdot,\cdot)$, where $(\cdot,\cdot)$ is the usual inner product. Using again (11), we get for $j\le k$ that

$$\begin{aligned} \left(mv,\frac{\partial}{\partial x_j}\right) &= \sum_{i\le k}\left(\frac{2}{\sqrt{\alpha\beta}}-\sqrt{\frac{\alpha}{\beta^3}}-\sqrt{\frac{\beta}{\alpha^3}}\right)x_ix_ja_i+\sum_{i>k}\left(\frac{1}{\sqrt{\alpha\beta}}-\sqrt{\frac{\beta}{\alpha^3}}\right)x_ix_ja_i \\ &\quad+\sum_{i>k}\left(\frac{1}{\sqrt{\alpha\beta}}-\sqrt{\frac{\alpha}{\beta^3}}\right)x_jy_ib_i \\ &= \sum_{i\le k}\left(\frac{2}{\sqrt{\alpha\beta}}-\sqrt{\frac{\alpha}{\beta^3}}-\sqrt{\frac{\beta}{\alpha^3}}\right)x_ix_ja_i-\sum_{i\le k}(\beta-\alpha)\left(\frac{1}{\sqrt{\alpha\beta^3}}-\frac{1}{\sqrt{\alpha^3\beta}}\right)x_ix_ja_i \\ &= 0, \end{aligned}$$

where we have used (13) to get the second equality. We analogously get $(mv, \frac{\partial}{\partial y_j}) = 0$ from (12). For $j > k$, we instead have

$$\begin{aligned}
\left(mv, \frac{\partial}{\partial x_j}\right) &= \sum_{i\le k}\left(\frac{1}{\sqrt{\alpha\beta}} - \sqrt{\frac{\beta}{\alpha^3}}\right)x_i x_j a_i - \sum_{i>k}\left(\sqrt{\frac{\beta}{\alpha^3}}x_i x_j + \sqrt{\frac{\alpha}{\beta^3}}y_i y_j\right)a_i \\
&\quad - \sum_{i\le k}\left(\frac{1}{\sqrt{\alpha\beta}} - \sqrt{\frac{\beta}{\alpha^3}}\right)x_i y_j b_i - \sum_{i>k}\left(\frac{x_i y_j}{\sqrt{\alpha\beta}} - \frac{x_j y_i}{\sqrt{\alpha\beta}}\right)b_i \\
&= \sum_{i\le k}\left[\left(\frac{1}{\sqrt{\alpha\beta}} - \sqrt{\frac{\beta}{\alpha^3}}\right)(x_i x_j a_i - x_i y_j b_i) + \frac{\beta-\alpha}{\sqrt{\alpha^3\beta}}x_i x_j a_i + \frac{\beta-\alpha}{\sqrt{\alpha\beta^3}}x_i y_j b_i\right] \\
&= 0,
\end{aligned}$$

where we get the second equality using both (12) and (13). We get that $(mv, \frac{\partial}{\partial y_j}) = 0$ similarly.

In other words, when restricted to the orthogonal complement of $\mathrm{span}_{\mathbb{R}}\{v_0, w_0\}$, the form $dd^c\sqrt{\alpha\beta}\left(\cdot, \sqrt{-1}\cdot\right)$ is just $(\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}})(\cdot,\cdot)$. This proves the first part of the lemma.

For the second part, note that there are two ways in which $\sqrt{\alpha\beta}M_0$ becomes a multiple of the identity: either both possible eigenvalues become the same, or $v_0 = 0$. The first situation happens precisely on S_0, while the second one happens precisely on $L_1 \cup L_2 \cup (S_0 \cap \{\alpha = \beta\})$. The union of these spaces is of course $L_1 \cup L_2 \cup S_0$. □

Proof of Lemma III.2: The proof follows the same structure as that of Lemma III.1; we give here the details. The bilinear form $d\sqrt{\alpha\beta} \wedge d^c\sqrt{\alpha\beta}(\cdot, \sqrt{-1}\cdot)$ can be computed in coordinates to be

$$\begin{aligned}
&\sum_{i,j=1}^{n}\left[\left(\frac{\beta}{\alpha} + \delta_{i\le k} + \delta_{j\le k} + \frac{\alpha}{\beta}\delta_{i\le k}\delta_{j\le k}\right)x_i x_j + \frac{\alpha}{\beta}y_i y_j \delta_{i>k}\delta_{j>k}\right](dx_i \otimes dx_j + dy_i \otimes dy_j) \qquad (14)\\
&+ \sum_{i,j=1}^{n}\left[\left(1 + \frac{\alpha}{\beta}\delta_{i\le k}\right)x_i y_j \delta_{j>k} - \left(1 + \frac{\alpha}{\beta}\delta_{j\le k}\right)x_j y_i \delta_{i>k}\right](dx_i \otimes dy_j + dy_j \otimes dx_i).
\end{aligned}$$

Putting the expression for v_1 in (14) gives, for $j \le k$, that

$$d\sqrt{\alpha\beta} \wedge d^c\sqrt{\alpha\beta}\left(v_1, \sqrt{-1}\frac{\partial}{\partial x_j}\right) = \sum_{i>k}\left(\frac{\beta}{\alpha} + 1\right)\alpha x_i x_j y_i - \sum_{i>k}\left(1 + \frac{\alpha}{\beta}\right)\beta x_i x_j y_i = 0$$

and

$$
\begin{aligned}
d\sqrt{\alpha\beta} \wedge d^c\sqrt{\alpha\beta}\left(v_0, \sqrt{-1}\frac{\partial}{\partial y_j}\right) &= -\sum_{i\le k}\left(\frac{\beta}{\alpha}+2+\frac{\alpha}{\beta}\right)(\alpha+\beta)x_i^2x_j - \sum_{i>k}\left(\frac{\beta}{\alpha}+1\right)\beta x_i^2 x_j \\
&\quad - \sum_{i>k}\left(1+\frac{\alpha}{\beta}\right)\alpha x_j y_i^2 \\
&= -2(\alpha+\beta)\sum_{i\le k} x_i^2 x_j - \left(\beta+\frac{\beta^2}{\alpha}\right)\alpha x_j - \left(\frac{\alpha^2}{\beta}+\alpha\right)\beta x_j \\
&= \left(2\sum_{i\le k} x_i^2 + \alpha + \beta\right) b_j.
\end{aligned}
$$

Likewise, for $j > k$, one gets from (14) that

$$
\begin{aligned}
d\sqrt{\alpha\beta} \wedge d^c\sqrt{\alpha\beta}\left(v_1, \sqrt{-1}\frac{\partial}{\partial x_j}\right) &= \sum_{i>k}\left[\frac{\beta}{\alpha}\alpha x_i x_j y_i + \frac{\alpha}{\beta}\alpha y_i^2 y_j + \beta x_i^2 y_j - \beta x_i y_j y_i\right] \\
&\quad + \sum_{i\le k}\left(1+\frac{\alpha}{\beta}\right)(\alpha+\beta)x_i^2 y_j \\
&= 2\alpha\sum_{i\le k} x_i^2 y_j + \alpha\beta y_j + \alpha^2 y_j \\
&= \left(2\sum_{i\le k} x_i^2 + \alpha + \beta\right) a_j
\end{aligned}
$$

and

$$
\begin{aligned}
d\sqrt{\alpha\beta} \wedge d^c\sqrt{\alpha\beta}\left(v_1, \sqrt{-1}\frac{\partial}{\partial y_j}\right) &= -\sum_{i\le k}\left(\frac{\beta}{\alpha}+1\right)(\alpha+\beta)x_i^2 x_j \\
&\quad - \sum_{i>k}\left[\frac{\beta}{\alpha}\beta x_i^2 x_j + \frac{\alpha}{\beta}\beta x_i y_i y_j - \alpha x_i y_i y_j + \alpha x_j y_i^2\right] \\
&= -2\beta\sum_{i\le k} x_i^2 x_j - \frac{\beta^2}{\alpha}\alpha x_j - \alpha\beta x_j \\
&= \left(2\sum_{i\le k} x_i^2 + \alpha + \beta\right) b_j.
\end{aligned}
$$

Therefore, when nonzero, v_1 is an eigenvector of M_1 with associated eigenvalue $(2\sum_{i\le k} x_i^2 + \alpha+\beta)$. But note that M_1 commutes with J_0 since $d\sqrt{\alpha\beta}\wedge d^c\sqrt{\alpha\beta}\left(\cdot, \sqrt{-1}\cdot\right)$ is a $\mathbb{C}$-sesquilinear form. Therefore, $w_1 := J_0 v_1 = \sum_i((\beta+\alpha\delta_{i\le k})x_i\frac{\partial}{\partial x_i} + \alpha y_i \delta_{i>k}\frac{\partial}{\partial y_i})$ must also be an eigenvector with the same eigenvalue.

Suppose now that $v = \sum_j (a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j})$ is orthogonal to both v_1 and w_1, i.e.

$$\sum_{i>k} (\alpha y_i \alpha_i - \beta x_i b_i) - (\alpha+\beta) \sum_{i \le k} x_i b_i = 0 \tag{15}$$

and

$$(\alpha+\beta) \sum_{i\le k} x_i a_i + \sum_{i>k} (\beta x_i a_i + \alpha y_i b_i) = 0. \tag{16}$$

Using again (14), we get for $j \le k$ that

$$\begin{aligned} d\sqrt{\alpha\beta} \wedge d^c \sqrt{\alpha\beta} \left(v, \sqrt{-1}\frac{\partial}{\partial x_j}\right) &= \sum_{i\le k} \left(\frac{\beta}{\alpha} + 2 + \frac{\alpha}{\beta}\right) x_i x_j a_i + \sum_{i>k} \left(\frac{\beta}{\alpha} + 1\right) x_i x_j a_i \\ &\quad + \sum_{i>k} \left(1 + \frac{\alpha}{\beta}\right) x_j y_i b_i \\ &= \sum_{i\le k} \left(\frac{\beta}{\alpha} + 2 + \frac{\alpha}{\beta}\right) x_i x_j a_i - \sum_{i\le k} (\alpha+\beta) \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) x_i x_j a_i \\ &= 0, \end{aligned}$$

where we have used (16) to get the second equality. We analogously get $d\sqrt{\alpha\beta} \wedge d^c \sqrt{\alpha\beta}(v, \frac{\partial}{\partial y_j}) = 0$ from (15). For $j > k$, we instead have

$$\begin{aligned} d\sqrt{\alpha\beta} \wedge d^c \sqrt{\alpha\beta} \left(v, \sqrt{-1}\frac{\partial}{\partial x_j}\right) &= \sum_{i\le k} \left(\frac{\beta}{\alpha} + 1\right) x_i x_j a_i + \sum_{i>k} \left[\frac{\beta}{\alpha} x_i x_j + \frac{\alpha}{\beta} y_i y_j\right] a_i \\ &\quad - \sum_{i\le k} \left(1 + \frac{\alpha}{\beta}\right) x_i y_j b_i - \sum_{i>k} \left[x_i y_j - \frac{\alpha}{\beta} x_j y_i\right] b_i \\ &= 0, \end{aligned}$$

where we get the second equality using both (15) and (16). We get that $d\sqrt{\alpha\beta} \wedge d^c \sqrt{\alpha\beta}(v, \sqrt{-1}\frac{\partial}{\partial y_j}) = 0$ similarly.

In other words, when restricted to the orthogonal complement of $\mathrm{span}_{\mathbb{R}}\{v_1, w_1\}$, the form $d\sqrt{\alpha\beta} \wedge d^c \sqrt{\alpha\beta}(\cdot, \sqrt{-1}\cdot)$ is just 0. This proves the first part of the lemma.

For the second part, note that there are two ways in which M_1 becomes the 0 matrix: either both possible eigenvalues become the same, or $v_1 = 0$. The first situation happens precisely on $L_1 \cap L_2$, while the second one happens precisely on $L_1 \cup L_2$. The union of these spaces is of course $L_1 \cup L_2$. $\square$

REFERENCES

[Abo21] M. Abouzaid. "Homological mirror symmetry without correction." *Journal of the American Mathematical Society* 34.4 (2021), pp. 1059–1173.

[CEL10] K. Cieliebak, T. Ekholm, and J. Latschev. "Compactness for holomorphic curves with switching Lagrangian boundary conditions." *Journal of Symplectic Geometry* 8.3 (2010), pp. 267–298.

[Duv16] J. Duval. "On a result by Y. Groman and JP Solomon." *Mathematische Annalen* 364 (2016), pp. 1361–1363.

[GS14] Y. Groman and J. P. Solomon. "A reverse isoperimetric inequality for J-holomorphic curves." *Geometric and Functional Analysis* 24 (2014), pp. 1448–1515.

[Hic19] J. S. Hicks. *Tropical Lagrangians and homological mirror symmetry*. University of California, Berkeley, 2019.

[Tor22] B. C. Torricelli. *Projective twists and the Hopf correspondence*. University of Cambridge, 2022.

[WR] W. R. Inc. *Mathematica, Version 13.2*. Champaign, IL, 2022.

J.-P. Chassé, DEPARTMENT OF MATHEMATICS, ETH ZÜRICH

E-mail address: `jeanphilippe.chasse@math.ethz.ch`

J. Hicks, SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH

E-mail address: `jeff.hicks@ed.ac.uk`

Y.J. Nho , DPMMS, UNIVERSITY OF CAMBRIDGE

E-mail address: `yjn23@dpmms.cam.ac.uk`