Convergence & Riemannian bounds on Lagrangian submanifolds

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Setup

Problematic: Suppose that d is some symplectically significant metric between Lagrangians, e.g. d_H, γ, or $d_{\mathcal{S}}^{\mathscr{F},\mathscr{F}'}$. If $\{L_n \subseteq M\}$ converges to L_0 in d, how does L_n relate to L_0 for n large?

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Idea: If $L_n \to L_0$ in Hausdorff metric, this implies some properties for L_n when n is large. Maybe we can ensure that this is the case.

A problem with that idea

Take $H_n(x,y) := \frac{1}{n}\sin(nx)$ on $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$. These functions generate Hamiltonian flows

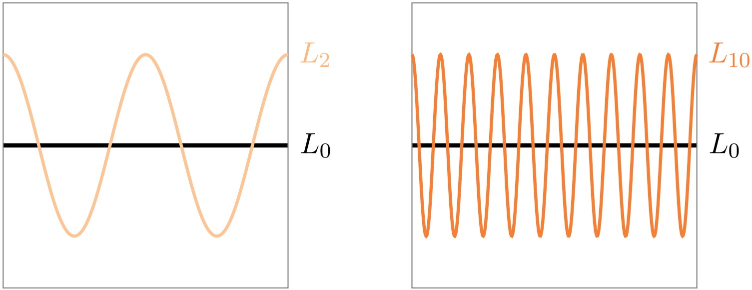
$$f_n^t(x,y) = (x, y + t\cos(nx)).$$

Therefore, if we take

$$L_0 := \{y = 0\} \quad \text{and} \quad L_n := f_n^1(L_0) = \{(x, \cos(nx))\},$$

we will get $d_H(L_0, L_n) = \frac{2}{n} \xrightarrow{n\to\infty} 0$, even though the L_n's get quite messy.

A problem with that idea



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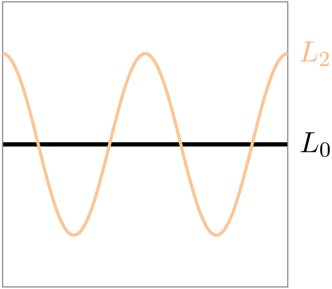

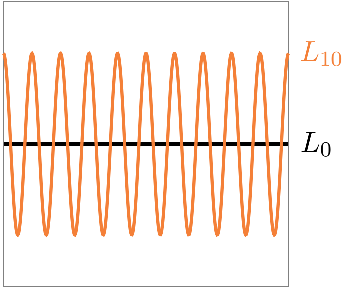

Counterpoint: What if we only look at Lagrangians with bounded curvature?

Outline

Plan

Some notation

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  (c) $(\star = \mathbf{m}(\rho, \mathbf{d}))$: $\omega = \rho\mu$ on $\pi_2(M, L)$, $N_L \geq 2$ and $d_L = \mathbf{d}$,

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  for $\rho > 0$ and $\mathbf{d} \in \mathbb{Z}_2$.
- $\mathscr{F}, \mathscr{F}' \subseteq \mathscr{L}^\star(M)$ s.t. $(\cup_{F\in\mathscr{F}} F) \cap (\cup_{F'\in\mathscr{F}'} F')$ is discrete.

J-adapted metrics on $\mathscr{L}^\star(M)$

A J-adapted pseudometric $d^{\mathscr{F}}$ will be one of the following

- d_H: Lagrangian Hofer metric;
- γ: spectral norm;
- $d_S^{\mathscr{F}}$: shadow pseudometric associated to $\mathscr{F}$;
- $D^{\mathscr{F}}$: (some) weighted fragmentation pseudometrics;
- ... and many variations on these themes.

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The key property is that, for any $x \in L \cup L'$, there exists a J-holomorphic polygon $u : S_r \to M$ with boundary along Lagrangians in $\{L, L'\} \cup \mathscr{F}$ passing through x such that

$$\omega(u) \leq d^{\mathscr{F}}(L, L').$$

The second fundamental form

We fix the Riemannian metric $g = g_J := \omega(\cdot, J\cdot)$. Let ∇ denote its Levi-Civita connection.

Definition

The *second fundamental form* B_L of a submanifold L of M is given by

$$\begin{aligned} (B_L)_x \colon T_xL \otimes T_xL \otimes (T_xL)^\perp &\longrightarrow \mathbb{R} \\ (X, Y, N) &\longmapsto g(\nabla_X Y, N). \end{aligned}$$

Its *norm* is then defined to be

$$||B_L|| := \sup_{x \in L} |(B_L)_x|.$$

The tameness condition

Definition (Sikorav, 1994; Groman–Solomon, 2014)

Let L be a submanifold of M, and let $\varepsilon \in (0,1]$. We say that L is ε-*tame* if

$$\frac{d_M(x,y)}{\min\{1, d_L(x,y)\}} \geq \varepsilon \qquad \forall x \neq y \in L,$$

where d_M is the distance function on M induced by g, and d_L is the distance function on L induced by $g|_L$.

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For $\Lambda \geq 0$ and $\varepsilon \in (0,1]$, we consider

$$\mathscr{L}^\star_\Lambda(M) := \{L \in \mathscr{L}^\star(M) | ||B_L|| \leq \Lambda\}$$

$$\mathscr{L}^\star_{\Lambda,\varepsilon}(M) := \{L \in \mathscr{L}^\star_\Lambda(M) | L \text{ is } \varepsilon\text{-tame}\}.$$

Plan

1. Definitions
   - Symplectic topology
   - Riemannian geometry

2. A conjecture of Cornea
   - Statement of the conjecture
   - Idea of the proof

A conjecture

Conjecture (Cornea, 2018)

Let $\hat{d}^{\mathscr{F},\mathscr{F}'}$ be a J-adapted metric. Take $\{L_n\} \subseteq \mathscr{L}^{\star}_{\Lambda}(M)$ for some fixed $\Lambda \geq 0$. If $L_n \xrightarrow{n\to\infty} L_0$ in $\hat{d}^{\mathscr{F},\mathscr{F}'}$, then $L_n \xrightarrow{n\to\infty} L_0$ in the Hausdorff metric δ induced by g.

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The corresponding conjecture on $\mathscr{L}^\star_{\Lambda,\varepsilon}(M)$ holds. Furthermore, if $\dim M = 2$, the conjecture holds as stated.

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Remarks

The condition that $\{L_n\} \subseteq \mathscr{L}^\star_\Lambda(M)$ for a fixed Λ depends on J, but the condition that $\{L_n\} \subseteq \mathscr{L}^\star_\Lambda(M)$ for *some* Λ does not.

A corollary

Theorem (Perelman's stability theorem, 1991)

Let $\{X_n\}$ be a sequence of compact n-dimensional Alexandrov spaces of curvature bounded from below by κ. If $X_n \xrightarrow{n\to\infty} X_0$ in Gromov-Hausdorff metric, then X_n is homeomorphic to X_0 for n large.

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Corollary (C., 2021)

If $\{L_n\} \subseteq \mathscr{L}^{\star}_{\Lambda,\varepsilon}(M)$ converges in some J-adapted metric to L_0 embedded, then L_n is homeomorphic to L_0 for n large.

1) The key property

By the key property, for any $x \in L_0 - (L_n \cup (\cup F))$ and $x' \in L_n - (L_0 \cup (\cup F))$, we get J-holomorphic polygons u and u' passing through x and x', respectively — modulo arbitrarily small perturbations such that

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We have a similar statement for $d^{\mathscr{F}'}(L_n, L_0)$.

2) The monotonicity lemma

Proposition

Consider a nonconstant J-holomorphic curve $u : (\Sigma, \partial\Sigma) \to (B(x,r), \partial B(x,r) \cup L)$ for some $x \in L$ and $r \leq \delta_0$ such that $x \in u(\Sigma)$. Then,

$$\omega(u) \geq Cr^2,$$

where $\delta_0 = \delta_0(M, \Lambda) > 0$ and $C = C(M, \varepsilon) > 0$.

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where $\delta_0 = \delta_0(M, \Lambda) > 0$ and $C = C(M, \varepsilon) > 0$.

This allows to get a lower bound on $\omega(u)$ and $\omega(u')$ in terms of M, Λ, ε, and the distances $d_M(x, L_n \cup (\cup F))$ and $d_M(x', L_0 \cup (\cup F))$.

3) The condition on $(\overline{\cup F}) \cap (\overline{\cup F'})$

Using the fact that $(\overline{\cup F}) \cap (\overline{\cup F'})$ is discrete, it is possible to turn the dependence on the different distances onto one on the Hausdorff distance $\delta_H(L_n, L_0)$.

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Remarks

Only Step 2 changes when $\dim M = 2$: we then prove that curves have a "nice" osculating disk and use an absolute version of the monotonicity lemma on it.

Thank you for your attention!