Finite Groups in Physics

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The Schrödinger Equation

In nonrelativistic quantum mechanics (QM), the behavior of a particle in some potential $V : \mathbb{R}^2 \to \mathbb{C}$ is described by its *wavefunction*, i.e. a (usually) smooth function $\Psi : \mathbb{R}^2 \to \mathbb{C}$ such that

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x,t)\Psi(x,t) = i\frac{\partial}{\partial t}\Psi(x,t)$$

and $||\Psi||_{L^2} = 1$.

When $V(x,t) = V(x)$, we get, by letting $\Psi(x,t) = \psi(x)\varphi(t)$,

$$H(x)\psi(x) := -\frac{1}{2}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

for some constant E: the *energy* of the wavefunction. $H(x)$ is then called the *Hamiltonian* of the system.

Finding Analytical Solutions

Even though the time-independant Schrödinger equation (TISE) is linear homogeneous, which guarantees a unique solution if the potential V is continous, an actual analytical solution is often impossible to find.

In fact, it is as hard to solve that equation than any other second-order linear homogeneous equation. Indeed, it is easy to prove that any equation of the latter form can be put in the form of the TISE by some change of variable.

If you ever taken or will take a QM course, you have learned or will learn that the only analytically solvable systems are variations of the harmonic oscillator ($V(x) = x^2/2$) or the Coulomb potential in 3D ($V(r) = 1/r$, where $r \in \mathbb{R}^3$ and we have replaced the derivative in the TISE by the Laplacian).

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Basic Definitions

A *linear representation* of a finite group G in V, where V is a n-dimensional vector space over $\mathbb{C}$, is a group homomorphism $\rho : G \to \mathbf{GL}(V)$. Then, n is the *degree* of the representation.

Two representations ρ and ρ' are said to be *isomorphic* if there exists a linear isomorphism $\tau : V \to V'$ such that $\tau \circ \rho(s) = \rho'(s) \circ \tau$, $\forall s \in G$.

Furthermore, a representation ρ is said to be *irreducible* if when $V = W \oplus W'$, where $\rho^W : G \to \mathbf{GL}(W)$ and $\rho^{W'} : G \to \mathbf{GL}(W')$ are subrepresentations (defined in the most intuitive sense), then $W = 0$ or $W' = 0$.

Examples

(1) If $\rho : G \to \mathbb{C}$ is a representation of degree 1, $\forall s \in G$ let $n \in \mathbb{N}$ such that $s^n = 1$. Then,

$$\rho(s)^n = \rho(s^n) = \rho(1) = 1$$

Thus, $\rho(s)$ is a root of unity $\forall s \in G$. Furthermore, every degree 1 representation is irreducible.

(2) The homomorphism given by $\rho(s) := 1 \in \mathbb{C}^* \ \forall s \in G$ is called the *trivial representation*.

(3) Let V be the free vector space over $\mathbb{C}$ generated by the elements of G. Then, there is a natural homomorphism $\rho : G \to \mathbf{GL}(V)$ given by $\rho(s)(t) := s \cdot t$ and forcing linearity on the second argument. This is called the *regular representation* of G.

Basic Results

Proposition

Every linear representation of a finite group is isomorphic to a unitary one, i.e. there is always a base on V such that $\rho(s)$ is representated by a unitary matrix $\forall s \in G$.

Theorem

Every linear representation is isomorphic to the direct sum of irreducible representations, i.e. if $\rho : G \to \mathbf{GL}(V)$ is a representation, then there exists $\{W_i\}_{i \in I}$ such that $V = \oplus_{i \in I} W_i$ and $\rho^{W_i} : G \to GL(W_i)$ is an irreducible subrepresentation $\forall i \in I$.

More Results

Even though there are an infinite number of representations, the number, up to isomorphism, of irreducible representation of G, denoted N_I, is equal to the number of conjugacy class of G. In fact,

$$\sum_{i=1}^{N_I} n_i^2 = |G|$$

where n_i is the degree of the i-th irreducible representation.

From that, we get that abelian groups can only have irreducible representations of degree 1, e.g. $\mathbb{Z}_2$ has 2 non-isomorphic such representations:

$$\rho_1(0) = \rho_1(1) = 1 \quad \text{and} \quad \rho_2(0) = -\rho_2(1) = 1$$

A Non-Abelian Example

For an easy non-abelian example, we can look at S_3, the symmetry group on the set of 3 elements. Since S_3 has 3 conjugacy classes (identity, 2-cycles and 3-cycles), the previous formula forces the degree of the 3 corresponding representations to be 1, 1 and 2. In fact, a bit more work gives us

$$\rho_1(s) = 1 \quad \forall s \in D_3$$

$$\rho_2(s) = \begin{cases} 1 & \text{for the identity and 3-cycles} \\ -1 & \text{for 2-cycles} \end{cases}$$

$$\rho_3(\begin{pmatrix}1 & 2\end{pmatrix}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_3(\begin{pmatrix}1 & 2 & 3\end{pmatrix}) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad \ldots$$

Action on Hilbert Spaces

Let $\mathcal{H}$ be a finite dimensional subspace of L^2 with orthonormal basis $\{\psi_\alpha\}$, and let G be a finite group acting on $\mathcal{H}$ by change of coordinates, i.e. $\forall s \in G$, there is a linear isomorphism $T_s : \mathcal{H} \to \mathcal{H}$ given by

$$T_s \psi_\alpha(x) := \psi_\alpha(s^{-1} \cdot x) = \sum_\beta (\rho(s))_{\beta\alpha} \psi_\beta(x)$$

where $\cdot$ denotes some action of G on $\mathbb{R}$. The second egality stems from the fact that $\{\psi_\alpha\}$ forms a basis for $\mathcal{H}$.

It is easy to check the matrices $(\rho(s))_{\beta\alpha}$ define a representation of G on $\mathcal{H}$; that we can take to be unitary by the previous theorem.

The Symmetry Group of an Hamiltonian

Now suppose the ψ_α's are the eigenfunctions of some Hamiltonian H and that $G = G_H$ is the symmetry group of H, i.e. G_H is the subgroup of $\mathbf{GL}(\mathcal{H})$ composed of all the elements s such that

$$T_s\left(H(x)\psi(x)\right) = H(x)T_s\psi(x)$$

Thus, if ψ_α is an eigenfunction of H with energy E_α, then so is $T_s\psi_\alpha$, $\forall s \in G_H$. Therefore, a n-dimensional eigenspace of H leads to a represention of degree n of G_H on $\mathcal{H}$.

In fact, when they are no so-called "*accidental degeneracies*", that representation is irreducible. Then, an eigenspace of H **is the same thing** as an irreducible represention of G_H, and we can use representation theory to study physical system!

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Symmetric Potential

Let's go back to the TISE! Suppose you have a Hamiltonian H such that the only symmetry of the potential is

$$V(-x) = V(x)$$

e.g. $V(x) = x^2/2$. Then, the symmetry group of H is isomorphic to $\mathbb{Z}_2$.

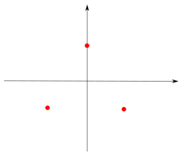
But as we've seen, they are only two irreducible representions of this group, and they are both of degree 1. Therefore, the eigenvalues are **not degenerate** and

$$\psi(-x) = \psi(r \cdot x) = T_r\psi(x) = \pm\psi(x)$$

Thus, eigenfunctions of H are either **even or odd**!

Three Electric Charges

Suppose you have three electric charges of the same charge placed in the plane as the vertices of an equilateral triangle (I have attempted to illustrate it on the picture below).



Then the symmetry group of the associated Hamiltonian is isomorphic to S_3, and as we have seen, there can now be degenerate eigenvalues of H. However, by the same reasoning as before, non-degenerate eigenfunctions are either **even or odd** relatively to reflections, which correspond to 2-cycles in the isomorphism.

References

[1] Serre, J.-P. *Linear Representations of Finite Groups.* No. 42 in Graduate Texts in Mathematics. Springer, 1977.

[2] MacKenzie, R. *Introduction to Group Theory in Quantum Mechanics.* (To Be Published).