# The impact of metric constraints on the behavior of shadow metrics

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# Broad picture

Biran, Cornea and Shelukhin introduced metrics on large collections $\mathscr{L}^\star(M)$ of Lagrangians, and we would like to understand them.

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**Upshot**: We will get some sort of dichotomy!

# Plan

1. Some metrics on $\mathscr{L}^\star(M)$
   - Lagrangian Hofer metric
   - Shadow metrics
2. A conjecture of Cornea
   - Statement of the conjecture
   - Idea of a proof
   - The reverse isoperimetric inequality

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  (a) $(\star = \mathbf{we})$: $\omega = 0$ on $H_2^D(M, L)$;
  (b) $(\star = \mathbf{m}(\rho, \mathbf{d}))$: $\omega = \rho\mu$ on $H_2^D(M, L)$, $N_L \geq 2$ and $d_L = \mathbf{d}$,

  for $\rho > 0$ and $\mathbf{d} \in \mathbb{Z}_2$.

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**Remarks**

Most of what follows also works when $M$ is a Liouville manifold, with some additional details. In that case, we can also take $\star = e$, i.e. $\omega|_L = d\lambda$.

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### Definition (Chekanov, 2000)

The **Lagrangian Hofer metric** on the Hamiltonian orbit of a Lagrangian submanifold $L_0$ is defined by

$$d_H(L, L') := \inf_{\substack{\varphi \in \mathrm{Ham}(M) \\ \varphi(L) = L'}} ||\varphi||_H,$$

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**Q**: But what does it mean to converge in $d_H$?

**A**: Well, it's complicated...

# Some bad behaviour in $(\mathscr{L}^{we}(\mathbb{T}^2), d_H)$

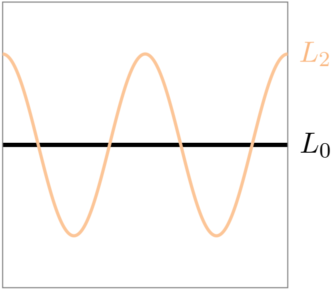
Take $H_n(x,y) := \frac{1}{n}\sin(nx)$ on $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$. These functions generate Hamiltonian flows

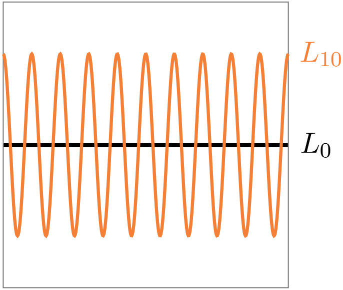
$$f_n^t(x,y) = (x, y + t\cos(nx)).$$

Therefore, if we take

$$L_0 := \{y = 0\} \quad \text{and} \quad L_n := f_n^1(L_0) = \{(x, \cos(nx))\},$$

we will get $d_H(L_0, L_n) = \frac{2}{n} \xrightarrow{n\to\infty} 0$, even though the $L_n$'s get quite messy.



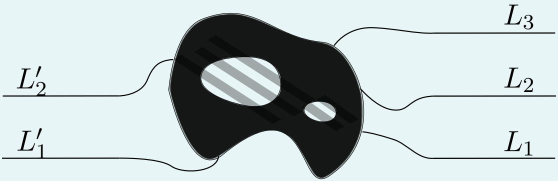

Note that we have $d(L_0, L_n) \equiv 1$, where $d$ is the Hausdorff distance induced by the flat metric.

**Definition (Arnol'd, 1980; Biran-Cornea, 2013; Cornea-Shelukhin, 2019)**

Let $L_1, \ldots, L_k, L'_1, \ldots, L'_{k'} \in \mathscr{L}^\star(M)$. A **Lagrangian cobordism** between $(L_1, \ldots, L_k)$ and $(L'_1, \ldots, L'_{k'})$, denoted

$$V : (L_1, \ldots, L_k) \rightsquigarrow (L'_1, \ldots, L'_{k'}),$$

is a Lagrangian $V \in \mathscr{L}^\star_{\mathrm{cob}}(M \times \mathbb{C})$ whose projection onto $\mathbb{C}$ looks like the following




The **shadow** $\mathcal{S}(V)$ of $V$ is the area of the black and grey regions.

### Definition (Cornea-Shelukhin, 2019; Biran-Cornea-Shelukhin, 2018)

The **shadow pseudometric** on $\mathscr{L}^\star(M)$ associated to a family $\mathscr{F} \subseteq \mathscr{L}^\star(M)$ is defined by

$$d^{\mathscr{F}}(L, L') := \frac{s^{\mathscr{F}}(L, L') + s^{\mathscr{F}}(L', L)}{2},$$

where

$$s^{\mathscr{F}}(L, L') := \inf\{\mathcal{S}(V) | V : L \rightsquigarrow (F_1, \dots, L', \dots, F_k), k \geq 0, F_i \in \mathscr{F}\}.$$

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### Remarks

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## Remarks

- $d^{\mathscr{F}} \leq d_H$, for any family $\mathscr{F}$.
- Shadow pseudometrics are a special case of what are called **weighted fragmentation pseudometric** on $\mathrm{DFuk}^\star(M)$.

### Theorem (Biran-Cornea-Shelukhin, 2018)

If $\mathscr{F}, \mathscr{F}' \subseteq \mathscr{L}^\star(M)$ are such that

$$\left(\overline{\bigcup_{F\in\mathscr{F}} F}\right) \cap \left(\overline{\bigcup_{F'\in\mathscr{F}'} F'}\right)$$

is totally disconnected, then

$$\hat{d}^{\mathscr{F},\mathscr{F}'} := \frac{d^{\mathscr{F}} + d^{\mathscr{F}'}}{2}$$

is a metric on $\mathscr{L}^\star(M)$.

In what follows, we fix families $\mathscr{F}$ and $\mathscr{F}'$ having these properties.

# Plan

# Some metric constraints on Lagrangians

We restrict our attention from $\mathscr{L}^\star(M)$ to a subspace of Lagrangians behaving well metrically.

Fix a Riemannian metric $g$ on $M$ and, for a submanifold $L$ of $M$, denote by $B_L$ its second fundamental form. For $\Lambda_0, \Lambda_1 \geq 0$, we then define subspaces

$$\mathscr{L}^\star_{\Lambda_0}(M,g) := \{L \in \mathscr{L}^\star(M) \ \mid \ ||B_L|| \leq \Lambda_0\},$$
$$\mathscr{L}^\star_{(\Lambda_0,\Lambda_1)}(M,g) := \{L \in \mathscr{L}^\star(M) \ \mid \ ||B_L|| \leq \Lambda_0, ||\nabla B_L|| \leq \Lambda_1\}.$$

### Conjecture (Cornea, 2018)

Fix a compatible almost complex structure $J$. The topology induced by the Hausdorff metric on $\mathscr{L}^\star_{\Lambda_0} = \mathscr{L}^\star_{\Lambda_0}(M, g_J)$ is stronger than the one induced by $\hat{d}^{\mathscr{F},\mathscr{F}'}$.

In other words, if $L_n \xrightarrow{n\to\infty} L_0$ in $\hat{d}^{\mathscr{F},\mathscr{F}'}$, then $L_n \xrightarrow{n\to\infty} L_0$ in the Hausdorff metric $d$ induced by $g_J$.

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### Theorem (C. – in progress)

The corresponding conjecture on $\mathscr{L}^\star_{(\Lambda_0,\Lambda_1)}$ holds.

It is not too hard to convince oneself that the conjecture in the space $\mathscr{L}^\star_{(\Lambda_0,\Lambda_1)}$ would follow from the following result.

### Proposition

Let $\{L_n\}_{n\geq 1} \subseteq \mathscr{L}^\star_{(\Lambda_0,\Lambda_1)}$ be such that $L_n \xrightarrow{n\to\infty} L_0 \in \mathscr{L}^\star_{(\Lambda_0,\Lambda_1)}$ in $d^{\mathscr{F}}$. Then, for all sequences $\{x_n\}_{n\geq 1} \subseteq M$ such that $x_n \in L_n$ and $x_n \xrightarrow{n\to\infty} x_0 \in M$, we have that

$$x_0 \in L_0 \cup \left(\overline{\bigcup_{F\in\mathscr{F}} F}\right).$$

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(1) Find $\{f_n\}_{n \geq 1} \subseteq \mathrm{Ham}(M)$ such that

(i) $f_n(L_n) \xrightarrow{n \to \infty} L_0$ in $d^{\mathscr{F}}$;

(ii) $f_n(L_n) \in \mathscr{L}^\star_{(\Lambda_0', \Lambda_1')}$, $\forall n \geq 1$, for some $\Lambda_i' \geq \Lambda_i$;

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(2) (Biran-Cornea-Shelukhin) For each $n \geq 1$, there exists a $J$-holomorphic polygon $u_n : S_{r_n} \to M$ such that
  (i) $u_n$ passes through $x_0$;
  (ii) $u_n$ has boundary in $\{f_n(L_n)\} \cup \{L_0\} \cup \mathscr{F}$;
  (iii) $\omega(u_n) \leq 2d^{\mathscr{F}}(f_n(L_n), L_0)$.

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Therefore, we get a contradiction if we can find a constant $D = D(g_J, \Lambda_0', \Lambda_1', d(x_0, L_0 \cup (\bigcup_{F\in\mathscr{F}} F))) > 0$ such that

$$\omega(u_n) \geq D, \qquad \forall n \geq 1.$$

**Theorem (Groman-Solomon; Duval; Abouzaid; C. – in progress)**

Let $u : S_r \to M$ be a $J$-holomorphic polygon with boundaries in Lagrangians $L_1, \ldots, L_m$. Suppose that $L_i \in \mathscr{L}^\star_{(\Lambda_0,\Lambda_1)}$ for some $i \in \{1, \ldots, m\}$ such that $L_i \cap (\cup_{j\neq i} L_j) \neq L_i$. Then, for any $x_0 \in L_i - (\cup_{j\neq i} L_j)$ and any $\varepsilon > 0$, there exists a constant

$$A = A\left(g_J, \Lambda_0, \Lambda_1, d(x_0, \cup_{j\neq i} L_j), \varepsilon\right) > 0$$

such that

$$\ell_{L_i \cap B(\varepsilon)}(\partial u) \leq A\omega(u),$$

where $B(\varepsilon) := B_{d(x_0,\cup_{j\neq i} L_j)-\varepsilon}(x_0)$ is the metric ball of radius $d(x_0, \cup_{j\neq i} L_j) - \varepsilon$ centered at $x_0$.

# Duval's proof

## Definition

A function $f : U \subseteq M \to [0, +\infty)$ is **psh** (resp. **weakly psh**) if

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If there exists $f$ on a tubular neighborhood of size $r$ of $L$ such that

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then

$$\ell^{dd^J f}(\partial u) \leq \frac{a}{r}(dd^J f)(u).$$

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- But $\operatorname{Hess}\sqrt{\rho}$ solves a Ricatti-type equation, and thus related quantitites respect comparison theorems! One also get comparison results for $\operatorname{Hess}\rho$, since

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  Furthermore, $|\mathrm{grad}\sqrt{\rho}| \equiv 1$.
- We can then take $f = (\sqrt{\rho} + B\rho)^2$ for suitable $B > 0$.

Thank you for your attention!
Got any (more) question?