Metric geometry and geometrically bounded Lagrangian submanifolds

(work in progress by)

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Outline

Plan

Objective: Study the space of all (say exact) Lagrangians $\mathscr{L}$ of some (say Liouville) symplectic manifold M under some natural metric d.

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Idea: Restric ourselves to a smaller subspace $\mathscr{L}_k$ of Lagrangians with "geometry bounded by k", where $k > 0$.

Plan

A metric theorem

Theorem (A)

The space $(\mathscr{L}_k, d)$ is totally bounded, i.e. for each $\varepsilon > 0$, $\mathscr{L}_k$ can be covered by finitely many ε-balls.

A corollary from Theorem A

Corollary

The full space $(\mathscr{L}, d)$ is separable, i.e. it has a countable dense subset.

Indeed, totally bounded spaces are separable, and $\mathscr{L} = \cup_{k\in\mathbb{N}}\mathscr{L}_k$.

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Remarks (Humilière, Shelukhin)

When all Lagrangians in $\mathscr{L}$ are Hamiltonian isotopic and $d \leq d_H$, then this is some folkloric result.

A symplectic theorem

Theorem (B)

The space $(\mathscr{L}_k, d)$ is contains only finitely many Hamiltonian isotopy classes. Furthermore, there is some $A = A(k) > 0$ such that

$$d(L, L') \geq A$$

whenever L and L' are *not* Hamiltonian isotopic.

Some corollaries from Theorem B

Corollary

The full space $(\mathscr{L}, d)$ has at most countably many Hamiltonian isotopy classes.

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(i) $L_0 = L$ and $L_1 = L'$;

(ii) $L_t \in \mathscr{L}_k$ for (almost) all $t \in [0, 1]$,

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then L and L' are Hamiltonian isotopic.

Some corollaries from Theorem B

Corollary

Let ψ be an exact symplectomorphism, i.e. $\psi^*\lambda = \lambda + dF$ for some $F : M \to \mathbb{R}$, and let $L \in \mathscr{L}_k$ be such that $\psi^i(L) \in \mathscr{L}_k$ for all i.

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In particular, if $L_1, \dots, L_N$ (split-)generates the (closed exact) Fukaya category of M and $\psi^i(L_j) \in \mathscr{L}_k$ for all i, j, then

$$h_{\mathrm{cat}}(\psi) = 0.$$

Some last remarks

Remarks

There are Liouville manifolds with countably many Hamiltonian isotopy classes of exact Lagrangians.

We can take $M = T^*N$ and $d = \gamma$ to get some result towards the nearby Lagrangian conjecture.

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Remarks

All the above results also applies to monotone Lagrangians in closed manifolds with some extra topological conditions, e.g. $H^1(M;\mathbb{R}) = 0$ or $H^1(L;\mathbb{R}) = 0$.

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The Lagrangian Hofer metric

Definition (Hofer,'90; Chekanov,'00)

The **Lagrangian Hofer metric** is given by

$$d_H(L, L') := \inf_{\substack{H \in C_c^\infty([0,1] \times M) \\ L' = \varphi_1^H(L)}} \int_0^1 \left(\max_{x \in M} H(t,x) - \min_{x \in M} H(t,x) \right) dt.$$

Here, $\inf \varnothing = +\infty$.

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Idea: Given two Hamiltonian isotopic Lagrangians L and L', $d_H(L, L')$ is the least amount of "energy" needed to send L to L'.

Chekanov-type metrics between Lagrangians

More generally, we will be working with a **Chekanov-type** metric d, i.e. essentially one of the following

- d_H: Lagrangian Hofer metric;
- γ: spectral metric;
- $\hat{d}_S^{\mathscr{F},\mathscr{F}'}$: shadow metric associated to nice families $\mathscr{F}$ and $\mathscr{F}'$;
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- ... and many variations on these themes.

The key property is that, for any $x \in L \cup L'$, there exists a J-holomorphic polygon $u : S_r \to M$ with boundary along Lagrangians in $\{L, L'\} \cup \mathscr{F} \cap \mathscr{F}'$ passing through x such that

$$\omega(u) \leq d(L, L').$$

The second fundamental form

Definition

The **second fundamental form** B_L of a submanifold L of a Riemannian manifold (M, g) is given fiberwise by

$$\begin{aligned} (B_L)_x \colon T_xL \otimes T_xL \otimes (T_xL)^\perp &\longrightarrow \mathbb{R} \\ (X, Y, N) &\longmapsto g(\nabla_X Y, N). \end{aligned}$$

Its **norm** is then defined to be

$$||B_L|| := \sup_{x \in L} |(B_L)_x|.$$

The tameness condition

Definition (Sikorav,'94; Groman–Solomon,'14)

Let L be a submanifold of (M, g), and let $\varepsilon \in (0, 1)$. We say that L is **strongly ε-tame** if

$$\frac{d(x, y)}{\min\{1, d^L(x, y)\}} > \varepsilon \qquad \forall x \neq y \in L,$$

where d is the distance function on M induced by g, and d^L is the distance function on L induced by $g|_L$.

The Hausdorff metric

Definition

Let A and B be closed subsets of (M, g). Consider the quantity

$$s(A, B) := \sup_{x \in A} d(x, B) = \sup_{x \in A} \inf_{y \in B} d(x, y).$$

The **Hausdorff metric** between closed subsets is defined as

$$\delta_H(A, B) := \max \{ s(A, B),\ s(B, A) \}.$$

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The inequality $\delta_H(A, B) < \varepsilon$ means that A is in a ε-neighborhood of B and vice-versa.

Some useful notation

- $(M, d\lambda, J)$ is a Liouville manifold with a compatible a.c.s. which is convex at infinity.

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- d is a Chekanov-type metric on $\mathscr{L}$ which is bounded from above by d_H.

Plan

From d to δ_H

Theorem (C., 2023)

There exist constants $C_1, R_1 > 0$ with the following property. For all $L, L' \in \mathscr{L}_k$ such that $d(L, L') < R_1$, we have that

$$\delta_H(L, L') \leq C_1 \sqrt{d(L, L')}.$$

From δ_H to d

Theorem (C., 2024)

For all L in $\mathscr{L}_k$, there exist constants $C_2, R_2 > 0$ with the following property. Whenever $L' \in \mathscr{L}_k$ is such that $\delta_H(L, L') < R_2$, there exists a C^2-small function $f : L \to \mathbb{R}$ such that $L' = \operatorname{graph} df$ in a Weinstein neighborhood of L, and

$$d(L, L') \leq C_2 \delta_H(L, L').$$

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$$d(L, L') \leq C_2 \delta_H(L, L').$$

Moreover, if a sequence $\{L_i\} \subseteq \mathscr{L}_k$ has Hausdorff limit N, then N is an embedded C^1-Lagrangian submanifold, and there exist diffeomorphisms $f_i : N \xrightarrow{\sim} L_i$ for i large such that $f_i \to \mathbb{1}$ in the C^1-topology.

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Main technical result

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Proposition

On $\mathscr{L}_k$, d and δ_H have the same Cauchy sequences. Furthermore, two Cauchy sequences are equivalent in d if and only if they are in δ_H.

Proof of Theorem A

(1) By the proposition, $(\mathscr{L}_k, d)$ and $(\mathscr{L}_k, \delta_H)$ have homeomorphic completions.

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(1) By the proposition, $(\mathscr{L}_k, d)$ and $(\mathscr{L}_k, \delta_H)$ have homeomorphic completions.

(2) The metric completion of $(\mathscr{L}_k, \delta_H)$ is compact, as it is a closed subspace of the space of all closed subsets of the compact W_k.

(3) Thus, $(\mathscr{L}_k, d)$ is precompact in its completion, which is equivalent to being totally bounded in complete metric spaces.

□

Proof of Theorem B

(1) Suppose there exist infinitely many Hamiltonian isotopy classes in $\mathscr{L}_k$, and let $\{L_i\} \subseteq \mathscr{L}_k$ be such that L_i and L_j are not Hamiltonian isotopic in $i \neq j$.

Proof of Theorem B

(1) Suppose there exist infinitely many Hamiltonian isotopy classes in $\mathscr{L}_k$, and let $\{L_i\} \subseteq \mathscr{L}_k$ be such that L_i and L_j are not Hamiltonian isotopic in $i \neq j$.

(2) Since $(\mathscr{L}_k, d_H)$ is precompact, there is a converging subsequence, still denoted $\{L_i\}$.

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(3) But then L_i and L_{i+1} must be Hamiltonian isotopic for i large. Hence, we have a contradiction.

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The proof that $d(L, L') \geq A$ whenever L and L' are not Hamiltonian isotopic follows a similar logic. □

Proof of technical proposition

One side is evident, since we always have $\delta_H \leq C\sqrt{d}$. We give the idea on how to prove the other side. Let $\{L_i\}$ thus be δ_H-Cauchy.

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(1) Fix $\varepsilon > 0$. We take Hamiltonian perturbations L_i' of L_i such that

   (i) $d_H(L_i, L_i') \leq \varepsilon$;

   (ii) $L_i' \xrightarrow{\delta_H} L'$ is smooth.

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  (i) $d_H(L_i, L_i') \leq \varepsilon$;
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(2) Since $d(L_i', L') \leq C_2(L')\delta_H(L_i', L')$, $\{L_i'\}$ is d-Cauchy. In particular,

$$d(L_i', L_j') \leq \varepsilon$$

for i, j large.

Proof of technical proposition

(3) We thus have

$$d(L_i, L_j) \leq d_H(L_i, L_i') + d(L_i', L_j') + d_H(L_j', L_j) \leq 3\varepsilon$$

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Proof of technical proposition

(3) We thus have

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The statement on equivalences follows essentially from the same proof. □

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Other results

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(2) Everything above also holds if $\mathscr{L}$ is the space of graphs of Hamiltonian diffeomorphisms of some monotone symplectic manifold M. From this, we can exclude something similar to Ostrover's example in the corresponding $\mathscr{L}_k$.

Thank you for your attention!

I will be happy to answer your questions.