# Functional Equations related to the Iteration of Functions

by

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**Abstract**

Certain systems of functional equations related to the iteration of functions with a fixed point are considered. We construct smooth solutions in terms of expansions about a fixed point. In a particular example taken from an intuitive geometric situation the solution is obtained explicitly as a convergent Taylor series. Particular attention is given to the question of selecting distinguished solutions from an infinity of possible solutions. This classical topic is presented in a transparent way by consistently using compositional notation. The method described may be applied in similar situations, e.g. for handling iterations arising in discrete dynamical systems.

## 1. Introduction

There are infinitely many smooth – even analytic – functions satisfying the recurrence relation $\Gamma(x+1) = x\,\Gamma(x)$ and the initial condition $\Gamma(1) = 1$, therefore interpolating the set of values $\{(n+1, n!)\}_0^\infty$. There are certain conditions under which Euler's Gamma function is the unique function which does this interpolation – see Davis []. The present paper is similar in spirit, in that at the outset we consider a peculiar sequence of points in the plane that are defined by a certain rigid motion of a rod of unit length, with endpoints on two faces of a cube. This set of points then leads us to a functional recurrence relation, and we then determine conditions under which there is a unique function which interpolates the sequence of points and satisfies our derived recurrence relation.

With reference to Fig. (1) below, consider a unit cube $\mathcal{C} = [0,1]^3$ in the first octant of $\mathbb{R}^3$. Let us denote by $\mathcal{F}_1$ the face of $\mathcal{C}$ in the $(x,y)-$plane, so that $\mathcal{F}_1$ has corners at $(0,0,0), (0,1,0), (1,0,0)$ and $(1,1,0)$, and similarly, let us denote by $\mathcal{F}_2$ the face of $\mathcal{C}$ with corners at $(0,0,0), (0,0,1), (1,0,0)$, and $(1,0,1)$ in the $(x,z)-$plane. Two points, $A_1 = (x,y,0) \in \mathcal{F}_1$ and $A_2 = (x,0,z) \in \mathcal{F}_2$ are connected by a rod of length 1, i.e., $\sqrt{y^2+z^2} = 1$. Suppose now that these two points, $A_1, A_2$ move such that $x$ is monotonically increasing, and such that $A_1 = (x,y,0)$ is constrained to the lens-shaped region, $1 - x \le y \le \sqrt{1-x^2}$. It then turns out that $A_2 = (x,0,z)$ is similarly constrained, i.e., $x \le z \le 1 - \sqrt{1-(1-x)^2}$.

Evidently, the trajectories of $A_1, A_2$ with the properties described above are not unique. For, given any function $y = f(x)$, with $1 - x \le f(x) \le \sqrt{1-x^2}$ in the face $\mathcal{F}_1$, one has $z = g(x)$ in $\mathcal{F}_2$, with $g(x) = \sqrt{1-f(x)^2}$. In particular, if $y = 1 - x$ on $\mathcal{F}_1$, then $z = 1 - \sqrt{1-(1-x)^2}$ on $\mathcal{F}_2$, and similarly, if $y = \sqrt{1-x^2}$ on $\mathcal{F}_1$, then $z = x$ on $\mathcal{F}_2$.

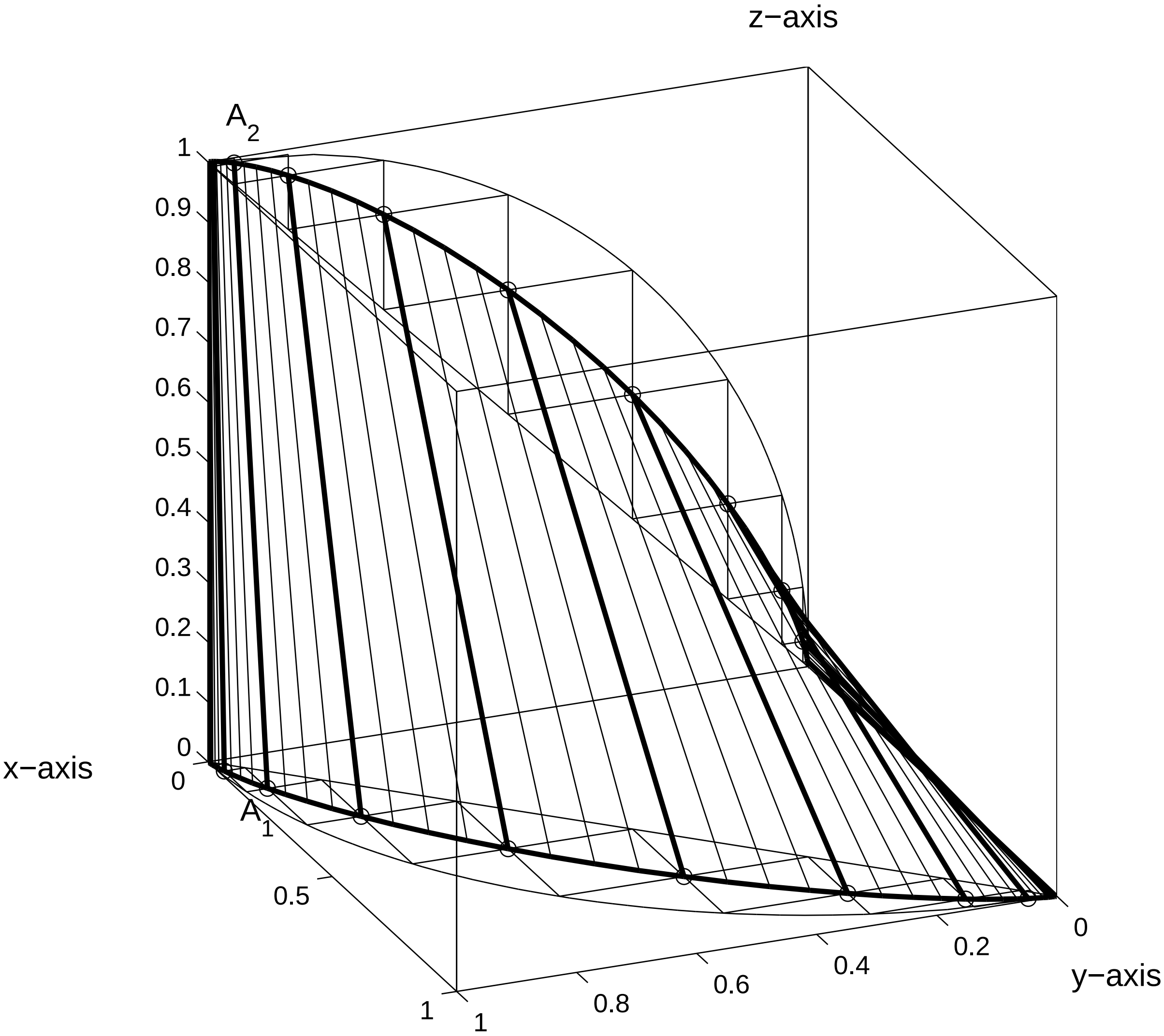


Figure 1: The rod $A_1A_2$ parallel to the $xy$-plane moving in such a way that the endpoints $A_1$ and $A_2$ traverse congruent paths

Next, consider the particular instances of the points $A_1, A_2$ marked by circles in Figure (1). They are constructed in an obvious manner by means of "staircase polygons" starting from the points $A_1 = (\frac{1}{2}, \frac{1}{2}, 0), A_2 = (\frac{1}{2}, 0, \frac{\sqrt{3}}{2})$ or from $A_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), A_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. The question of interpolating these particular points by "nice" congruent curves will be addressed in Section 4, where a set of recurrence relations will be derived first.

Functional equations, being relationships between values of a function at different arguments, often have larger sets of solutions than differential equations which relate function values and derivatives at a single argument. Besides initial conditions additional requirements such as smoothness and monotonicity are needed in order to choose a distinguished particular solution.

The topic of functional equations has been active since the beginnings of calculus, e.g. with Euler's gamma function (see, e.g., [3]), up to the present day. A recent encyclopedia volume by Kuczma et.al. [10] contains an extensive bibliography of more than 800 references. The recent revival of the field is due to the connection between certain functional equations and the modern theory of dynamical systems.

In this paper we will begin with the intuitive geometrical situation drawn in Figure 1, below, which turns out to be intimately connected with the iteration of functions in one variable. In Section 2 the corresponding functional equations are solved by the classical methods of Schröder [11] and Abel [1]. Section 3 is devoted to the involutory case which is picked up in Section 4 by means of the particular example described above. In Sections 4 and 5 this example will be solved completely in terms of convergent power series.

Let $A(x)$, $B(x)$ be two smooth monotonic functions defined on appropriate subintervals of $\mathbb{R}$, and consider their graphs as shown in Figure 2.

We discuss the problem of connecting the points of intersection of two polygons zigzagging between $A$ and $B$ by a simple smooth graph $G$. From Figure 2 we immediately obtain the conditions

$$G(x) = A(y) \tag{1}$$

$$G(y) = B(x)\ . \tag{2}$$

Clearly, the solution is far from unique unless further requirements on $G$ are specified. Consider, e.g., the particular example $A(x) = ax^\alpha$, $B(x) = bx^\beta$ in $x \geq 0$ with $a, b, \alpha, \beta > 0$. Assuming $G(x) = g \cdot x^\gamma$ shows that Eqs. (1) and (2) are satisfied for every $x \geq 0$ if we choose the appropriate mapping $y = Q(x)$ if we choose

$$\gamma = \sqrt{\alpha\beta}\,, \quad g = \left(a^{\sqrt{\beta}} b^{\sqrt{\alpha}}\right)^{1/(\sqrt{\alpha}+\sqrt{\beta})} . \tag{3}$$

Here, this is the simplest possible solution. However, as will be seen later, there exists a continuum of more complicated solutions to the problem.

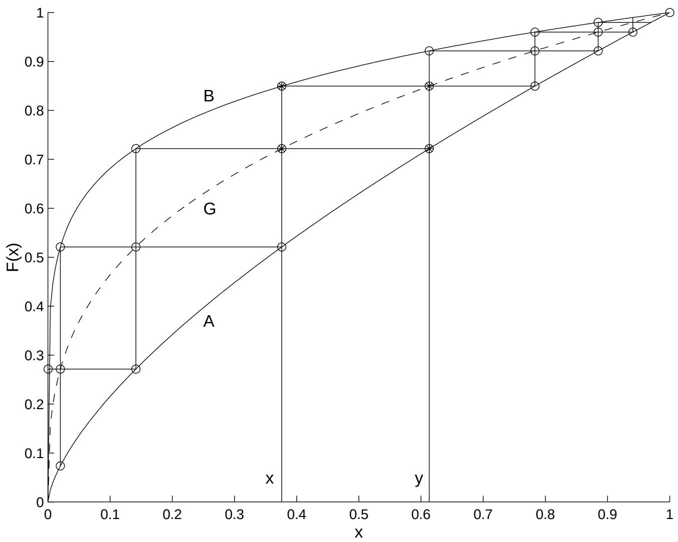


Figure 2: Graphs of functions $A$, $B$ and $G$

In what follows in this paper functions are denoted by capital characters, and composition will be denoted by juxtaposition, e.g.

$$A(B(x)) = (AB)(x) \ .$$

Exponents denote functional iteration, such as $A^n(x) = A(A^{n-1}(x))$, $A^0(x) = x$ ($A^0 = Id$ identity, and $A^{-1}$ accordingly denotes the inverse function of $A$.

## 2. Functional Equations

With the purpose of eliminating $y$ from the system (1), (2) we solve (1) for $y$, assuming that $A^{-1}$ exists:

$$y = Q(x) := A^{-1}(G(x))\,, \quad \text{or} \quad Q = A^{-1}\,G\ . \tag{4}$$

Inserting this into (2) yields

$$G\,A^{-1}\,G = B, \tag{5}$$

and, with the abbreviation

$$R := A^{-1}\,B\ , \tag{6}$$

the simple relation $Q^2 = R$ is obtained. Hence the problem at hand amounts to taking the "compositional square root" $Q$ of the given function $R$ ; then

$$G = AQ = A(A^{-1}\,B)^{1/2}\ . \tag{7}$$

This is a particular case of so-called *fractional iteration* of a function which has a long history dating back at least to 1871 (E. Schröder, [11]). This early work was soon carried on by N. H. Abel [1], G. Königs [8], and E. Kasner [6], and the field has been active up to the present day. Besides the large bibliography in [10] a commented bibliography up to 1964 by Targonski [14] is mentioned. The question of uniqueness and regularity of growth of fractional iterates was discussed, among many others, by Collatz [2], Davis [3], Kuczma [9], and Targonski [15].

We will use the methods of Schröder and Abel in order to construct particular solutions with various properties to the above problem. Using the idea of introducing appropriate "coordinates" [3], we introduce the formal conjugacy of the map $R$ with the shift map $S_p(x) := x + p$ ($p$ is an appropriately chosen shift), by means of an *Abel function* or *logarithm of iteration* [1] $\Phi(x)$ for $R$:

$$\Phi R = S_p \Phi \quad \text{or} \quad \Phi(R(x)) = p + \Phi(x)\ . \tag{8}$$

If both $\Phi$ and $\Phi^{-1}$ exist, we have $R = \Phi^{-1}\,S_p\Phi$, and we immediately verify that

$$Q = \Phi^{-1}\,S_{\frac{p}{2}}\Phi \tag{9}$$

is a solution of Eq. (5).

In order to find more solutions consider the *circle map* $C$ with shift $p$, by definition a monotonic function

$$x \longmapsto C(x) := x + P(x)\ , \tag{10}$$

where $P(x) = P(x + p)$ is a $p$-periodic function. Clearly, we have

$$C\,S_p = S_p\,C\ . \tag{11}$$

**Theorem 1** *If $\Phi$ is an Abel function for $R$ with shift $p$, $\Phi R = S_p\Phi$, and if $C$ is a circle map with the same shift, the function $\widetilde{\Phi} = C\Phi$ is also an Abel function for $R$ with shift $p$.*

**Proof:** $\Phi R = S_p\Phi \Longrightarrow C\Phi R = C\,S_p\Phi \Longrightarrow \widetilde{\Phi}R = S_p\widetilde{\Phi}$. $\square$

Consequently, in Eq. (9) $\Phi(x)$ may be replaced by $\widetilde{\Phi}(x) = \Phi(x) + P(\Phi(x))$, where $P$ is an arbitrary $p$-periodic function with the only restriction that $\widetilde{\Phi}^{-1}$ exists.

In the example of Section 1 we obtain

$$R(x) = \left(\frac{b}{a}\, x^{\beta}\right)^{1/\alpha}$$

and, e.g.,

$$\Phi(x) = \log\log(\mu x),\;\; \mu = \left(\frac{b}{a}\right)^{1/(\beta-\alpha)},\;\; p = \log\left(\frac{\beta}{\alpha}\right). \tag{12}$$

Choosing an arbitrary circle map $C$ with shift $p$, Eqs (4), (9) yield $G = A\Phi^{-1}\, C^{-1}\, S_{\frac{p}{2}}\, C\Phi$ or

$$G(x) = a\left(\frac{1}{\mu}\;\exp\exp\left(C^{-1}\Big(\frac{1}{2}\;\log\frac{\beta}{\alpha} + C\Big(\log\log(\mu x)\Big)\Big)\right)\right)^{\alpha}.$$

With $C = Id$ the simple solution $G(x) = g \cdot x^{\gamma}$ with $g, \gamma$ from (3) is obtained.

## 3. Symmetry

We now consider the case of involutions $A, B$ with $A^2 = B^2 = Id$. Then involutory solutions $G$ of (1), (2) may exist since $G^2 = Id$ is compatible with $B^2 = GA^{-1}\, G^2\, A^{-1}\, G$ $= Id$ (see Eq. (5)). We then have from (7) $R = AB$, $R^{-1} = BA$.

In order to construct an involutory solution we state the following theorem that holds without the symmetry condition on $A$ and $B$. Let $N(x) = -x$ be the "anti-identity"; we have $S_p N = N S_{-p}$.

**Theorem 2** *Let $\Phi$ be an Abel function for $R$ satisfying (8) or $\Phi A^{-1} = S_p\, \Phi B^{-1}$. Then the function $\widetilde{\Phi} := N\Phi A^{-1}$ satisfies $\widetilde{\Phi} A = S_p\, \widetilde{\Phi} B$.*

**Proof:** $\Phi A^{-1} = S_p\, \Phi B^{-1} \Longrightarrow N S_{-p}\, \Phi A^{-1} = N\, \Phi B^{-1} \Longrightarrow S_p \widetilde{\Phi} = \widetilde{\Phi} A B^{-1} \Longrightarrow S_p \widetilde{\Phi} B = \widetilde{\Phi} A$. □

**Corollary.** *If $A$ and $B$ are involutions, $A^{-1} = A$, $B^{-1} = B$, and $\Phi$ satisfies $\Phi A = S_p\, \Phi B$, then $\widetilde{\Phi} := N\, \Phi A^{-1}$ satisfies the same equation, $\widetilde{\Phi} A = S_p\, \widetilde{\Phi} B$.*

Next, we observe that any linear combination $w\Phi + \widetilde{w}\widetilde{\Phi}$ with $w + \widetilde{w} = 1$ satisfies Eq. (8). In particular, if we use $w = \widetilde{w} = \frac{1}{2}$ to define

$$\Psi = \frac{1}{2}\,(\Phi + \widetilde{\Phi}) \quad \text{or} \quad \Psi(x) = \frac{1}{2}\,[\Phi(x) - \Phi(A^{-1}(x)] \tag{13}$$

as our standard Abel function we have

**Theorem 3** *The solution*

$$G = A\,\Psi^{-1} S_{\frac{p}{2}}\,\Psi \tag{14}$$

*generated by the Abel function* $\Psi$ *is involutory.*

**Proof:** We have $\Psi A = N\Psi$. Therefore

$$G^2 = A\Psi^{-1}\,S_{\frac{p}{2}}\,\Psi A\Psi^{-1}\,S_{\frac{p}{2}}\,\Psi = A\Psi^{-1}\,S_{\frac{p}{2}}\,N\,S_{\frac{p}{2}}\,\Psi = A\,\Psi^{-1}\Psi A = Id\,.$$

□

The symmetry of the solution (14) may be displayed by writing (14) as $\Psi\,A^{-1}\,G = S_{\frac{p}{2}}\,\Psi$ or as

$$\Phi(A^{-1}(G)) + \Phi(A^{-1}(x)) = p + \Phi(G) + \Phi(x)\,. \tag{15}$$

This is an implicit equation of the curve $G(x)$ in the $(x, G)$ plane. Obviously, it is invariant if $x$ and $G$ are interchanged (hence involutory), and due to its construction it solves the problem (1), (2). $\Phi$ is an arbitrary Abel function for $R$ satisfying (8), and $p$ is the corresponding shift. More solutions are obtained by using $C\Phi$ instead of $\Phi$ in Eq. (15).

To conclude this section we derive an elegant parameterization of the curve defined by Eq. (15). With the definition

$$D(x) := \frac{1}{p}\left(\Phi(A^{-1}(x)) - \Phi(x)\right) \tag{16}$$

(15) becomes $D(G) + D(x) = 1$. By introducing the parameter $t := D(x)$, or $x = D^{-1}(t)$, where $D^{-1}$ is the inverse of $D$, we obtain the simple parametric form

$$x = D^{-1}(t), \quad G = D^{-1}(1-t), \quad \infty < t < \infty \tag{17}$$

for the curve defined by Eq. (15).

## 4. An Example

Let us now return to the motion of the rod described in the introduction and visualized in Fifure 1. We consider the family $\mathcal{G}$ of paths $y = G(x)$ of the lower endpoint $A_1$, whose $y$-coordinate lies between the curves

$$y = A(x) := 1 - x \;\text{ and }\; y = B(x) := \sqrt{1-x^2}, \quad 0 \le x \le 1. \tag{18}$$

It is clear that the $z$-coordinate of the upper endpoint, $z = A_2(x)$, lies between the trajectories $z = \sqrt{1-(1-x)^2}$ and $z = x$. Upon replacing $x$ by $1 - x$ in the family of

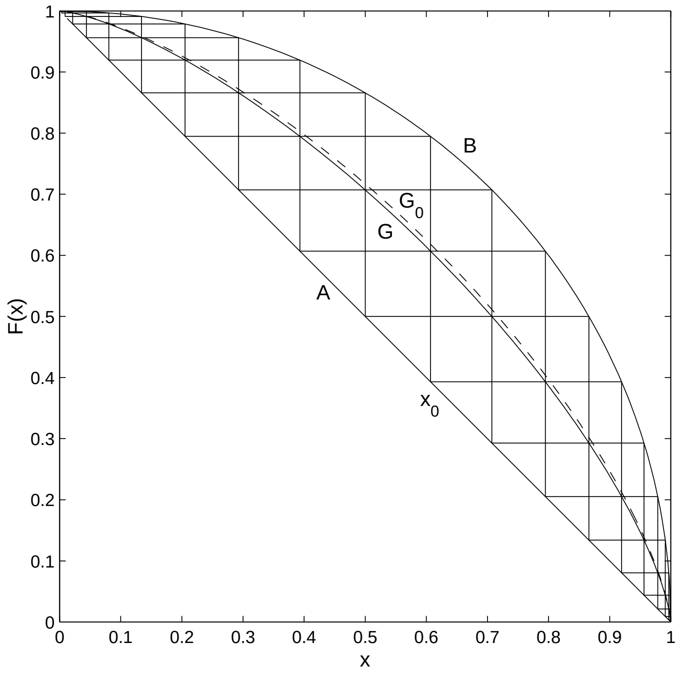


Figure 3: Graphs of the functions $A(x)$, $B(x)$, $G(x)$, $G_0(x)$ together with the four staircase polygons passing through the points $(\frac{1}{2}, \frac{1}{\sqrt{2}})$, or $(x_0, x_0)$, respectively.

trajectories $G(x)$, we may ask: *For what trajectories $G(x)$ do we have $G(1-x) = A_2(x)$?* Equivalently, since $A_2(x) = \sqrt{1 - G^2(x)}$, for what functions $G$ defined on $[0, 1]$, and such that $0 \leq G(x) \leq 1$ do we have [12]

$$G(1-x) = \sqrt{1 - G^2(x)}\ ? \tag{19}$$

Here the exponent 2 denotes the second power. We notice that the functions $A$ and $B$ are involutory, i.e. on the interval $[0, 1]$ they satisfy $A^{-1}(x) = A(x)$ and $B^{-1}(x) = B(x)$, and we furthermore restrict the family $\mathcal{G}$ of functions $G$ such that

$$G^{-1}(x) = G(x) \tag{20}$$

for all of the functions $G \in \mathcal{G}$. We may now ask:

- What are the properties of the family $\mathcal{G}$?
- In what sense is the family $\mathcal{G}$ described as above unique?
- If the family $\mathcal{G}$ consists of more than one member, what other properties reduce $\mathcal{G}$ to a single member $G$?
- Is it possible to construct such a $G \in \mathcal{G}$?

We shall consider these questions in turn, in what follows; see also Figure 3.

We first observe that due to (20), i.e. $G^2 = Id$, all the symmetries of Section 3 apply. Furthermore, Equ. (19), which corresponds to $GA^{-1} = BG$ in the compositional notation of the previous sections, is a direct consequence of (5) and (20). Therefore the theory of Sections 2, 3 holds. Instead of Equ. (8) we use the equivalent form

$$\Phi(1-x) = p + \Phi(\sqrt{1-x^2}) \tag{21}$$

with the goal of choosing an appropriate value of $p$ and a *simple* solution $\Phi(x)$, e.g. a formal ascending series. It is suggested to adopt the ascending series solution $\Phi(x)$ to be defined below as the "best" solution of Equ. (21) in the sense that it shows the most regular growth as $x \to 0$. Other choices may be possible, though. Due to the construction in Section 3, however, the same solution $G$ is obtained from the Abel function based on the most regular behaviour as $x \to 1$. Considerations of this type are important in identifying the gamma function as the "best" solution of its functional equation (see, e.g., [4]); for other examples see Kneser [7], Szekeres [13].

In view of Equ. (12), pointing to a possibly complicated logarithmic singularity at $z = 0$ we introduce

$$\Theta(x) := \exp\,\Phi(x) \tag{22}$$

satisfying the functional equation

$$\Theta(1-x) = e^p \cdot \Theta(\sqrt{1-x^2}) \tag{23}$$

which determines $\Theta(x)$ at most up to an arbitrary factor. The function $\Theta(x)$ satisfies an appropriate Schröder equation [10]. Assuming

$$\Theta(z) = -\log(cz) + O(z)$$

uniquely yields $p = \log 2$, $c = \frac{1}{2}$. In order to avoid logarithmic terms we attempt to find a formal solution $\Theta'(x) = -x^{-1} + O(1)$ of the derivative of (23),

$$\Theta'(1-x) = \frac{2x}{\sqrt{1-x^2}}\,\Theta'(\sqrt{1-x^2})\,. \tag{24}$$

The procedure described below will directly result in a convergent series solving (24). Instead of $z$ we will use the variable $u := (1-x^2)/4 \in [0, \frac{1}{4}]$; therefore

$$\sqrt{1-x^2} = 2\sqrt{u}\,, \quad x = \sqrt{1-4u}\,. \tag{25}$$

The function

$$\vartheta(u) := -2\ \Theta'(2u) \tag{26}$$

then satisfies the functional equation

$$\frac{1}{\sqrt{u}}\,\vartheta(\sqrt{u}) = \frac{1}{\sqrt{1-4u}}\,\vartheta\left(\frac{1-\sqrt{1-4u}}{2}\right), \tag{27}$$

for which a formal solution

$$\vartheta(u) = c_0 u^{-1} + c_1 u + c_2 u^3 + c_3 u^5 + \ldots, \quad c_0 = 1 \tag{28}$$

will be shown to exist.

**Lemma 4** *For every $k \in \mathbb{R}$ the following expansion holds:*

$$f_k(u) := \frac{1}{\sqrt{1-4u}}\left(\frac{1-\sqrt{1-4u}}{2}\right)^k = \sum_{j=0}^{\infty}\binom{k+2j}{j} u^{j+k}\,.$$

**Proof:** We only need the lemma for $k = -1, 0, 1 \ldots$ . The correctness of the expansion is easily seen for $k = -1$ and $k = 0$. Induction with respect to $k$ by using $f_{k+2}(u) - f_{k+1}(u) + u\,f_k(u) = 0$ and the basic relation between the elements of the Pascal triangle establishes Lemma 4 for $k = 1, 2, 3, \ldots$. □

Inserting the expansion (29) into (28) and using Lemma 4 directly yields the recurrence relation

$$c_k = \sum_{j=0}^{[\frac{k}{2}]}\binom{2k-2j-1}{k-1} c_j, \quad (k = 0, 1, \ldots), \ c_0 = 1 \tag{29}$$

for the sequence $c_k$. Its initial elements are

$$c_k = \{1, 1, 4, 13, 49, 181, 685, 2605, 9988, 38479, 148879, 577930, 2249698, \ldots\}\,. \tag{30}$$

Clearly, the coefficients $c_k$ form a monotonically increasing sequence of integers, and the following theorem holds:

**Theorem 5** *The sequence $c_k$ defined by Equ. (31) satisfies $c_k \le 4^k$ for every $k \ge 0$.*

**Proof by induction:** The statement of the theorem is trivially true for $k = 0$. Assume $c_j \le 4^j$ for $j = 0, 1, \dots, k-1$, $(k \ge 1)$. Then (31) implies (with $J := 2j$)

$$c_k \le \sum_{j=0}^{[\frac{k}{2}]} \binom{2k-2j-1}{k-1} 4^j < \sum_{J=-\infty}^{k} \binom{2k-J-1}{k-1} 2^J \, .$$

By introducing the summation index $l := k - J$ instead of $J$ and by using identities between binomial coefficients as well as the binomial expansion the last sum becomes

$$\sum_{J=-\infty}^{k} \binom{2k-J-1}{k-1} 2^J \; = \; 2^k \sum_{l=0}^{\infty} \binom{k+l-1}{l} 2^{-l}$$

$$= \; 2^k \sum_{l=0}^{\infty} \binom{-k}{l} (-\frac{1}{2})^l \; = \; 2^k \, (1 - \frac{1}{2})^{-k} \; = \; 4^k \, ,$$

which establishes the theorem. □

As a consequence of Theorem 5 the series (30) converges for every $u \in \mathbb{C}$ with $|u| < \frac{1}{2}$, and from (26) we obtain

$$\Theta(x) = -c_0 \, \log \frac{x}{2} - \sum_{k=1}^{\infty} \frac{c_k}{2k} \left(\frac{x}{2}\right)^{2k} . \tag{31}$$

## 5. Results

In order to evaluate $\Theta(x)$ for a given $x \in (0, 1)$, we use the functional equation (23) repeatedly before evaluating the series: Let $x_0 := x$ and iterate

$$x_j = R(x_{j-1}) = \frac{x_{j-1}^2}{1 + \sqrt{1 - x_{j-1}^2}} \; , \quad j = 1, 2, \dots, m \tag{32}$$

such that $\Theta(x_m)$ by the appropriately truncated series (31) has sufficient accuracy. Then $\Theta(x) = 2^{-m}\Theta(x_m)$. The graphs of the functions $\Phi(x)$, $\Theta(x)$ and $\exp(-\Theta(x))$ are shown in Figure 4.

In terms of the Schröder function $\Theta$ the equation (15) of the graph $G(x)$ may be written as

$$\Theta(1 - G) \cdot \Theta(1 - x) = 2\Theta(G) \cdot \Theta(x) \, , \tag{33}$$

as is seen by exponentiating Equ. (15). This is one form of the final result of the problem of Section 1 with $A$ and $B$ defined by Equ. (18). In order to solve (33) numerically for

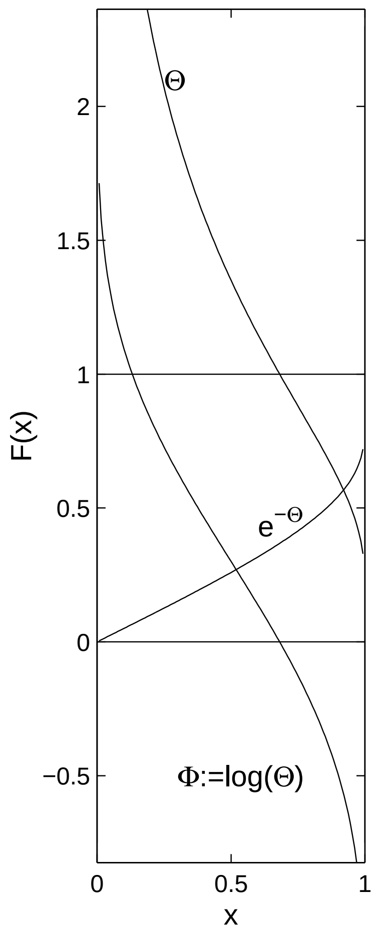


Figure 4: The functions $\Theta(x)$, $\exp(-\Theta(x))$, and $\Phi(x) = \log(\Theta(x)$ for $x \in (0, 1)$

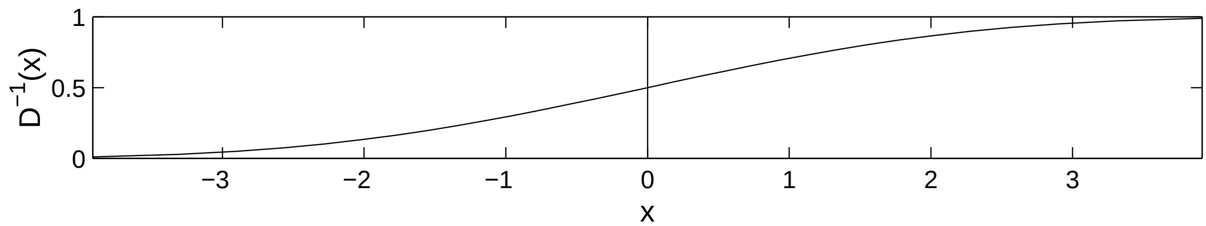


Figure 5: The inverse $F = D^{-1}$ of $D(z) = \log_2(\Theta(1-x)/\Theta(x))$

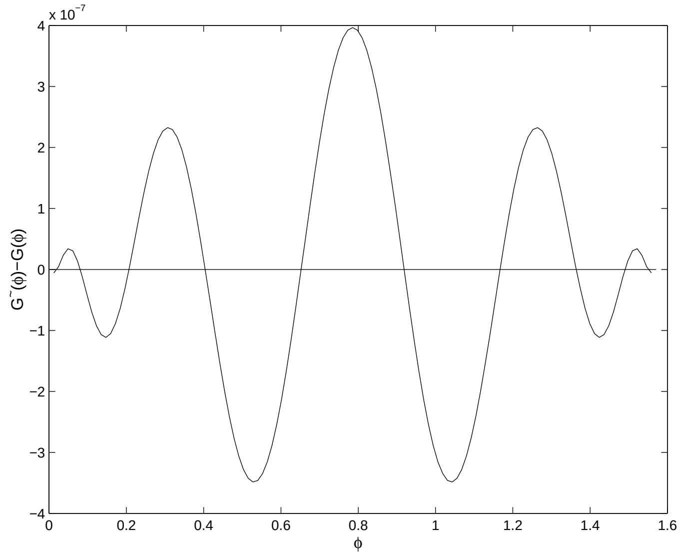


Figure 6: Graph of the difference $\Delta(\varphi) := \widetilde{G}(\varphi) - G(\varphi)$, where $\cos(\varphi) = x$. This graph is nearly symmetric; we have $|\Delta(\varphi) - \Delta(\frac{\pi}{2} - \varphi)| < 1.156 \cdot 10^{-12}$.

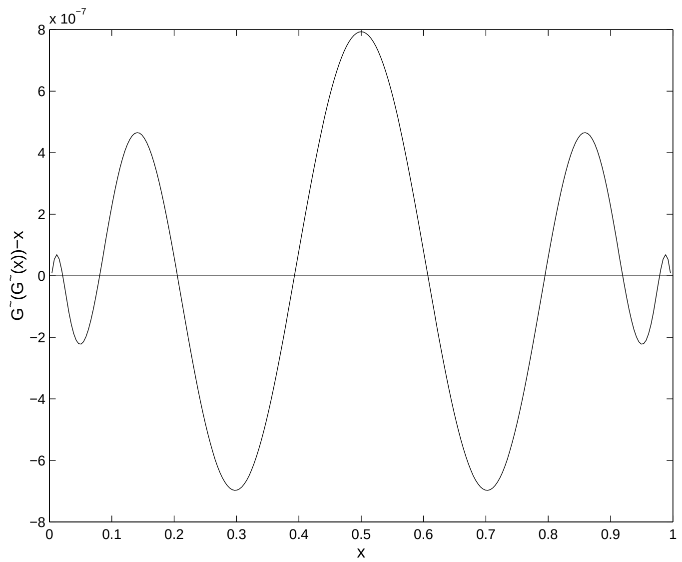


Figure 7: Graph of the function $\widetilde{G}(\widetilde{G}(x)) - x$

$G$ the secant method with the initial guesses $G_0 = G_0(x) := (1 - x^{\sqrt{2}})^{1/\sqrt{2}}$ and a nearby value $G_1$ is recommended, cf. Figure 3.

Another representation of this graph is the parametrization (17) where, according to (16), the function $F(x) = D^{-1}(x)$ is now the inverse of

$$D(x) = \log_2 \frac{\Theta(1-x)}{\Theta(x)}, \tag{34}$$

to be computed, e.g., by solving the equation

$$\Theta(1-F) = 2^x \cdot \Theta(F) \tag{35}$$

for the unknown F (see Figure 5). As a consequence of (17), the fixed point $x_0 = G(x_0)$ of the symmetric solution is now defined by $x_0 = D^{-1}(1/2)$, which results in

$$x_0 = .60694\,81374\,10748\,90686\,44016\,61391\,98879\,45573\,42957. \tag{36}$$

It is interesting to note that the solution $\widetilde{G}$ obtained directly from $\Theta$ by means of Equs (7), (9), (22), i.e.

$$\widetilde{G} = A\,\Theta^{-1} \exp S_{\frac{p}{2}} \log \Theta \quad \text{or} \quad \widetilde{G}(z) = 1 - \Theta^{-1}(\sqrt{2}\,\Theta(x)), \tag{37}$$

passes through the fixed point $x_0$ of $G$ and therefore through all of its forward and backward iterates under the map $A^{-1}B$. $\widetilde{G}$ is not quite symmetric, however; we have $|\widetilde{G}(x) - G(x)| \le 3.9658\,94143 \cdot 10^{-7}$ and $|\widetilde{G}(\widetilde{G}(x)) - x| \le 7.9317\,88286 \cdot 10^{-7}$, see Figure 6 and Figure 7. Since $\widetilde{G}(y)$ deviates from $G(x)$ only by an amount of less than $10^{-6}$, $\widetilde{G}$ is sufficient for graphics purposes.

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