

The Matrix gallery(5) of MATLAB

In MATLAB the matrix

$$(1) \quad A = \text{gallery}(5) = \begin{bmatrix} -9 & 11 & -21 & 63 & -252 \\ 70 & -69 & 141 & -421 & 1684 \\ -575 & 575 & -1149 & 3451 & -13801 \\ 3891 & -3891 & 7782 & -23345 & 93365 \\ 1024 & -1024 & 2048 & -6144 & 24572 \end{bmatrix}$$

is provided as an “interesting eigenvalue problem”. According to a remark in MATLAB’s help file the user is invited to “find its *exact* eigenvalues and eigenvectors”.

The naive approach fails, since $\text{eig}(A)$ yields

$$(2) \quad \begin{aligned} \lambda_1 &= -0.03371394752607 + 0.02503344949626i \\ \lambda_2 &= -0.03371394752607 - 0.02503344949626i \\ \lambda_3 &= 0.01337277132340 + 0.03894049608920i \\ \lambda_4 &= 0.01337277132340 - 0.03894049608920i \\ \lambda_5 &= 0.04068235241305 \quad , \end{aligned}$$

which contradicts $\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 = \det(A) = 0$. Somewhat more reliable information is obtained from $\text{poly}(A)$ ’ which yields approximate coefficients of the characteristic polynomial $p(\lambda)$:

$$(3) \quad \begin{aligned} &1.000000000000000 \\ &-0.000000000000770 \\ &0.00000004765977 \\ &-0.00000019062667 \\ &0.00000025761397 \\ &-0.00000012160528 \quad . \end{aligned}$$

This may lead to the conjecture that $p(\lambda) = \lambda^5$ is the characteristic polynomial of A , which would imply that $\lambda = 0$ is a 5-fold eigenvalue, i.e. $\lambda = 0$ has the algebraic multiplicity 5.

According to the theorem of Cayley-Hamilton every matrix satisfies its characteristic equation, which would mean $A^5 = 0$ if the above conjecture is true. In fact we obtain

$$(4.1) \quad A^2 = \begin{bmatrix} 11 & -18 & 39 & -116 & 464 \\ -230 & 301 & -598 & 1799 & -7194 \\ 1717 & -2292 & 4585 & -13750 & 54992 \\ -11674 & 15565 & -31130 & 93391 & -373503 \\ -3072 & 4096 & -8192 & 24576 & -98288 \end{bmatrix} ,$$

$$(4.2) \quad A^3 = \begin{bmatrix} -4 & 8 & -20 & 64 & -255 \\ 243 & -402 & 879 & -2628 & 10509 \\ -1710 & 2852 & -6278 & 18840 & -75323 \\ 11675 & -19458 & 42807 & -128420 & 513437 \\ 3072 & -5120 & 11264 & -33792 & 135104 \end{bmatrix} ,$$

$$(4.3) \quad A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -84 & 168 & -420 & 1344 & -5355 \\ 568 & -1136 & 2840 & -9088 & 36210 \\ -3892 & 7784 & -19460 & 62272 & -248115 \\ -1024 & 2048 & -5120 & 16384 & -65280 \end{bmatrix}$$

and $A^5 = 0$. Since the characteristic polynomial of A has the exact degree 5 the above computations prove that

$$(5) \quad p(\lambda) = \lambda^5$$

is indeed the characteristic polynomial of A , and therefore $\lambda = 0$ is a 5-fold eigenvalue. The result (3) obtained by MATLAB is not that bad after all; it is in error only in the 7th decimal place. However, in this case the actual eigenvalues are extremely sensitive to round-off errors in the computation.

Next we consider the problem of finding eigenvectors of A . We notice that every eigenvector x of A satisfies the homogeneous linear system

$$(6) \quad Ax = 0, \quad x \in \mathbb{R}^5.$$

For solving (6) and related systems the LU factorization $A = LR$ will be used. The statement $[L, R] = \text{lu}(A)$ yields

$$(7) \quad L = \begin{bmatrix} -0.00231303006939 & 1.00000 & 0 & 0 & 0 \\ 0.01799023387304 & 0.50000 & 1.00000 & 0 & 0 \\ -0.14777692109997 & 0 & 0.40000 & 1.00000000000000 & 0 \\ 1.00000000000000 & 0 & 0 & 0 & 0 \\ 0.26317142122848 & 0 & 0 & -0.07843137254902 & 1.00000 \end{bmatrix}$$

$$R = \begin{bmatrix} 3891.00 & -3891.00 & 7782.00 & -23345.0000000000 & 93365.0000000000 \\ 0 & 2.00 & -3.00 & 9.0023130301 & -36.0439475713 \\ 0 & 0 & 2.50 & -5.5191467489 & 22.3637882292 \\ 0 & 0 & 0 & 3.3554356207 & -12.7532767926 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From R we infer $\text{rank}(A) = 4$; therefore there is only one eigenvector, $x^{(1)}$, satisfying $Ax^{(1)} = 0$. Hence the geometric multiplicity of $\lambda = 0$ is 1, and there is only one Jordan block corresponding to $\lambda = 0$.

Instead of the missing eigenvectors there exist 4 linearly independent principal vectors $x^{(k)}$, ($k = 2, 3, 4, 5$) satisfying the conditions

$$(8) \quad Ax^{(k)} = x^{(k-1)}, \quad (k = 1, \dots, 5) \text{ with } x^{(0)} = 0,$$

which, for $k = 1$, include condition (6) for the eigenvector $x^{(1)}$. The vectors $x^{(k)}$ may be computed by backward substitution, beginning with an arbitrary 5-th component $x_5^{(k)}$. If we partition the matrix R as

$$(9) \quad R = \begin{bmatrix} R1 & r \\ 0 & 0 \end{bmatrix},$$

where $R1=R(1:4,1:4)$ is a regular 4×4 matrix and $r=R(1:4,5)$ is a 4-vector, Equ. (8) may be solved by the MATLAB statements

$$c = L \setminus x^{(k-1)}, \quad x^{(k)} = [R1 \setminus (c(1:4) - r * x_5^{(k)}); x_5^{(k)}],$$

to be executed for $k = 1, \dots, 5$. Here $x_5^{(1)} \neq 0$, and, as a check, c_5 vanishes in every step.

It would be desirable to choose the free parameters $x_5^{(k)}$, ($k = 1, \dots, 5$) such that $x^{(k)}$ are vectors with integer elements, if possible. A brute-force approach is to search through the values $x_5^{(k)} = \pm 1, \pm 2, \dots$, until an integer vector $x^{(k)}$ is found. It turns out that with

$$(10) \quad x_5^{(1)} = 256, \quad x_5^{(2)} = -64, \quad x_5^{(3)} = 16, \quad x_5^{(4)} = -4, \quad x_5^{(5)} = 1$$

integer vectors $x^{(k)}$ are indeed produced, and that the values of $x_5^{(k)}$ that result in integer vectors $x^{(k)}$ are determined only modulo 256. A more subtle approach would involve the theory of linear diophantine equations.

By using the values (10) the sequence of integer vectors $x^{(k)}$ may be generated by the short MATLAB program (12) given below. Due to round-off errors the elements of these vectors will not be exact integers, but have to be rounded to the nearest integer vectors. The integer eigenvector and principal vectors are collected as columns of the transformation matrix T . Since $\det(T) = 1$, the inverse $T1$ of T is an integer matrix as well which is recovered by $T1=\text{round}(\text{inv}(T))$. Since A relates to its Jordan normal form J by

$$(11) \quad A = T J T^{-1},$$

J will finally be explicitly computed as $J = T1 * A * T$.

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A = gallery(5)
[L,R] = lu(A); R1 = R(1:4,1:4); r = R(1:4,5); x = zeros(5,1); T = [ ];
for k = 4:-1:0,
(12)      c = L \ x, x5 = (-4)^k; x1 = R1 \ (c(1:4) - x5 * r);
          x = [round(x1); x5]; T = [T, x];
end;
T, T1 = round(inv(T)), J = T1 * A * T

```

The results computed in the last line of the program (12) are

$$(13.1) \quad T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 21 & -3 & 2 & 0 & 0 \\ -142 & 37 & -8 & 3 & 0 \\ 973 & -243 & 61 & -15 & 4 \\ 256 & -64 & 16 & -4 & 1 \end{bmatrix},$$

$$(13.2) \quad T1 = \begin{bmatrix} 1 & -1 & 2 & -6 & 24 \\ 1 & 0 & 0 & 0 & 0 \\ -9 & 11 & -21 & 63 & -252 \\ 11 & -18 & 39 & -116 & 464 \\ -4 & 8 & -20 & 64 & -255 \end{bmatrix},$$

$$(13.3) \quad J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

To find more, possibly all, integer transformation matrices S with the property

$$(14) \quad A = S J S^{-1},$$

assume

$$(15) \quad S = T Q$$

where Q and Q^{-1} are 5×5 matrices with integer elements. Eqs. (14), (15) together with (11) yield the condition

$$(16) \quad J = Q J Q^{-1} \quad \text{or} \quad J Q = Q J,$$

i.e. Q must commute with J . This is achieved by writing Q as a power series in J which, due to $J^5 = 0$, truncates after 5 terms:

$$(17) \quad Q = q_0 I + q_1 J + q_2 J^2 + q_3 J^3 + q_4 J^4.$$

If $q_0 = \pm 1$ and q_1, q_2, q_3, q_4 are integers, the reciprocal series has integer coefficients as well, and both matrices Q and Q^{-1} have integer elements.

The symbolic-algebra package MAPLE contains a procedure `jordan` which is able to directly generate the Jordan normal form of A . The statement `J:= jordan(A, 'S1')` yields J of Equ. (13.3) and

$$(18) \quad \mathbf{S1} = S^{-1} = \begin{bmatrix} 0 & -1 & 7 & -36 & 144171/1024 \\ 0 & -1 & 6 & -28 & 28117/256 \\ 0 & -1 & 5 & -21 & 5291/64 \\ 0 & -1 & 4 & -15 & 949/16 \\ 1 & -2 & 5 & -16 & 255/4 \end{bmatrix},$$

a rational inverse transformation matrix corresponding to

$$(19) \quad Q = \begin{bmatrix} -4 & 11 & -9 & 1 & 1 \\ 0 & -4 & 11 & -9 & 1 \\ 0 & 0 & -4 & 11 & -9 \\ 0 & 0 & 0 & -4 & 11 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix},$$

i.e. $q_0 = -4$, $q_1 = 11$, $q_2 = -9$, $q_3 = 1$, $q_4 = 1$.

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