Analytic Continuation of the Theodorus Spiral

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Abstract

The remarkable classical pattern of the Theodorus spiral, or square root spiral, can intuitively be supplemented by a closely related spiral asymptotic to it. A "nice" analytic interpolating curve of the Theodorus spiral was constructed by P.J. Davis (1993) as an infinite product satisfying the same functional equation as the discrete points. We consider the analytic continuation of the Davis solution and show that it contains the supplementing spiral as a discrete subset. We also discuss efficient evaluation algorithms for the analytic functions involved.

1 Introduction

Summary of the history according to [4].

2 Two Discrete Spirals

The classical spiral of Theodorus is constructed from the sequence of right triangles with sides $(\sqrt{n}, 1, \sqrt{n+1})$, $n = 1, 2, \ldots$. As it was done by several authors, e.g. [4], [5], the complex plane $\mathbb{C}$ with origin O will be used as a convenient coordinate system. The triangles are arranged such that the first one has a cathetus on the real axis, and all of them have the origin O as a common vertex. Two neighbouring triangles must have a common side, i.e. the hypotenuse of one triangle becomes a cathetus of the next one (see Figure 1). The free catheti of length 1 form a polygon; the set of its vertices, described by their complex coordinates $F_0 = 0$, $F_1 = 1,\ F_2,\ F_3, \dots,$ will be referred to as the *outer* discrete spiral of Theodorus.

In addition to this classical pattern we also consider the mirror image of each triangle with respect to its hypotenuse. The new vertices – together with the origin – may also be connected by a polygon as shown in Figure 1; the new set of points $G_0 = 0,\ G_1 = i,\ G_2,\ G_3, \dots$ will be referred to as the *inner* discrete spiral of Theodorus.

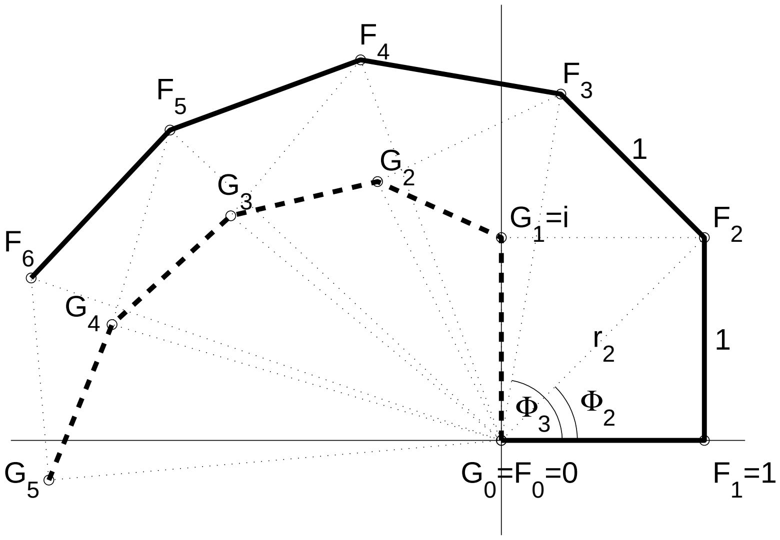


Figure 1: The outer (solid) and the inner (dashed) Theodorus spiral.

By using polar coordinates r_n, Φ_n instead of the complex coordinate F_n,

$$F_n = r_n\, e^{i\,\Phi_n}\,, \quad r_n = |F_n|\,, \quad \Phi_n = \arg F_n\,, \quad n = 1, 2, \dots\,, \tag{1}$$

the geometric relations of Figure 1 may be expressed as

$$r_n = \sqrt{n}\,, \quad \Phi_{n+1} - \Phi_n = \arctan\left(\frac{1}{r_n}\right), \quad \Phi_1 = 0\,, \quad n \in \mathbb{N}\,. \tag{2}$$

As a consequence of the identity $\exp\big(i\ \arctan(1/\sqrt{n})\big) = (\sqrt{n}+i)/\sqrt{n+1}$, F_n satisfies the functional equation

$$F_{n+1} = F_n \cdot \left(1 + \frac{i}{r_n}\right), \quad F_1 = 1\,, \quad n \in \mathbb{N}\,. \tag{3}$$

From (2), (3) we obtain the cumulative sum and product

$$\Phi_n = \sum_{k=1}^{n-1} \arctan\left(\frac{1}{\sqrt{k}}\right), \quad F_n = \prod_{k=1}^{n-1}\left(1 + \frac{i}{\sqrt{k}}\right), \quad n \in \mathbb{N}\,. \tag{4}$$

The geometry of Figure 1 also yields the relations determining the complex coordinates of the inner Theodorus spiral:

$$G_n \cdot \left(1 - \frac{i}{r_n}\right) = F_{n+1} = F_n \cdot \left(1 + \frac{i}{r_n}\right), \quad r_n = \sqrt{n}\,, \quad F_1 = 1\,. \tag{5}$$

Therefore the points G_n of the inner spiral may be defined by

$$\boxed{G_n = F_n \, \frac{r_n + i}{r_n - i}}\,, \quad n \in \mathbb{N}\,. \tag{6}$$

Note that Equs. (2), (3), (5) are meaningful also for $n = 0$,

$$r_0 = 0, \quad \Phi_0 = -\frac{\pi}{2}, \quad F_0 = 0, \quad G_0 = 0\,,$$

if the function $\arctan(1/r)$ is defined by means of the principal branch $\mathrm{Arctan}(r)$ (with branch cuts on the imaginary axis from i to $i\,\infty$ and from $-i\,\infty$ to $-i$) as

$$\arctan\left(\frac{1}{r}\right) = \frac{\pi}{2} - \mathrm{Arctan}(r)\,,\ r \in \mathbb{R}\,. \tag{7}$$

3 An Interpolating Curve

Inspired by Euler's construction of the gamma function, P.J. Davis (1993) suggested the infinite product

$$F_n = \prod_{k=1}^{\infty} \frac{1 + \dfrac{i}{\sqrt{k}}}{1 + \dfrac{i}{\sqrt{k-1+n}}} \tag{8}$$

as a smooth interpolant of the sequence F_n of Equ. (4), valid for positive n, $n \in \mathbb{R}_+$. The infinite product (8) converges absolutely for every fixed $n \in \mathbb{R}_+$ since the logarithms of the factors are of the order $O(k^{-3/2})$ as $k \to \infty$. Therefore, F_n is an analytic function of n, at least for $n \in \mathbb{R}_+$. Clearly we have $F_1 = 1$, and for $n \in \mathbb{N}$ the definition (8) satisfies the functional equation (3) owing to the "telescoping" property of the product (8). A simple argument shows that $|F_n|^2 = n$ holds for $n \in \mathbb{R}_+$.

3.1 Analytic Continuation

In view of (2), (3), (5), we will adopt the variable $r \in \mathbb{R}$ with the property

$$\boxed{r^2 = n \geq 0\,,\ \ r \in \mathbb{R}} \tag{9}$$

as a new parameter for describing the spirals, and we use the corresponding lower-case characters for denoting functions expressed in terms of r:

$$F_n = F(r^2) = f(r)\,, \quad G_n = G(r^2) = g(r)\,, \quad \Phi_n = \Phi(r^2) = \varphi(r)\,. \tag{10}$$

Now the functional equation (3) becomes

$$f(\sqrt{r^2+1}) = f(r)\ \big(1 + \frac{i}{r}\big)\,. \tag{11}$$

Substituting $n = r^2 \geq 0$ in (8) reveals that in the factors of the infinite product (8) with $k > 1$ the expression $k - 1 + r^2 > 0$ never reaches the branch point 0 of the square root function. Therefore those square roots always remain on their principal branch, enter with the positive sign. In contrast, the factor with $k = 1$ passes right though the branch point at $r = 0$. It is therefore natural to allow for negative values of r as well and change the sheet of the Riemann surface of the first factor when passing through $r = 0$. We therefore may rewrite the Davis function (8) as

$$f(r) = \frac{1+i}{1+\dfrac{i}{r}} \cdot \prod_{k=2}^{\infty} \frac{1+\dfrac{i}{\sqrt{k}}}{1+\dfrac{i}{\sqrt{k-1+r^2}}}\,, \qquad r \in \mathbb{R}\,. \tag{12}$$

The point $r = 0$ is a regular point of the analytic function f with $f(0) = 0$, hence (12) constitutes the analytic continuation of f through $r = 0$ to negative values $r < 0$. The function f is analytic on $\mathbb{R}$, but has branch points at $r = \pm\, i\,\sqrt{n}$, $n \in \mathbb{N}$. A good way of defining a principal branch is introducing branch cuts on the imaginary axis from i to $i\,\infty$ and from $-i\,\infty$ to $-i$. Equ. (12) implies

$$f(-r)\ \big(1 - \frac{i}{r}\big) = f(r)\ \big(1 + \frac{i}{r}\big)\,; \tag{13}$$

comparison with (5) yields

$$g(r) = f(-r)\,, \quad r \in \mathbb{R}\,. \tag{14}$$

By using the particular values $r = \sqrt{n},\ n \in \mathbb{N}$ it is seen that the analytic continuation of Davis' interpolating curve of the outer discrete Theodorus spiral passes through all points of the inner discrete spiral.

Theorem 1. Equ. (12) defines a single analytic curve connecting all points of the outer and the inner discrete Theodorus spiral. □

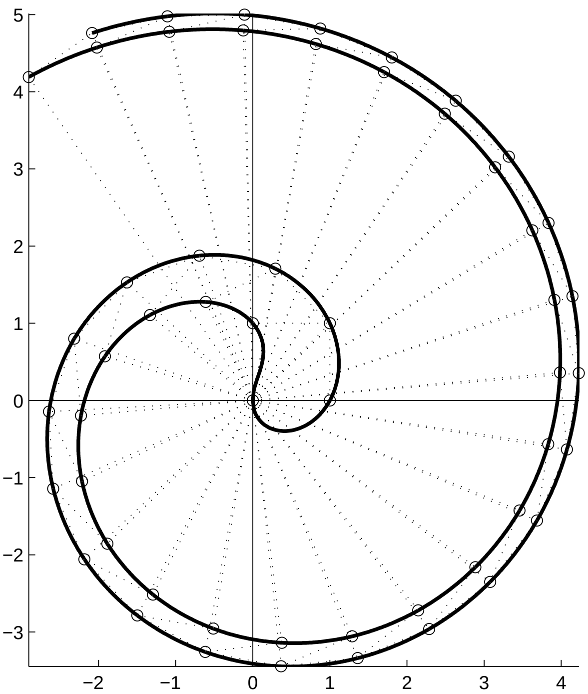


Figure 2: Davis' interpolant of the outer Theodorus spiral and its analytic continuation to the inner spiral. A single analytic curve, convex at infinity.

3.2 The Polar Angle

The polar angle $\varphi(r)$ as a function of $r \in \mathbb{R}$ may be found from (12) as $\varphi(r) = \text{Im } \log\big(f(r)\big)$. By using the analytic continuation (7) of $\arctan(r)$ through $r = 0$ we obtain

$$\varphi(r) = -\frac{\pi}{4} + \text{Arctan}(r) + \sum_{k=2}^{\infty} \left\{ \arctan\left(\frac{1}{\sqrt{k}}\right) - \arctan\left(\frac{1}{\sqrt{k-1+r^2}}\right) \right\}\ , \quad (15)$$

where $r \in \mathbb{R}$. The above sum converges absolutely since its terms are of the order $O(k^{-3/2})$,

$$\Big\{ \dots \Big\} = \arctan\left(\frac{r^2 - 1}{(k+r^2)\sqrt{k} + (k+1)\sqrt{k+r^2-1}} \right). \tag{16}$$

The inner spiral is characterized by negative values of the parameter r. Equ. (15) directly yields

$$\varphi(-r) = \varphi(r) - 2\arctan(r)\,, \quad r \in \mathbb{R}\,, \tag{17}$$

and we obtain the following list of special values, in perfect agreement with Figure 1:

r	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$	$\sqrt{3}$
$\varphi(r)$	$\frac{\pi}{4} - 2\,\mathrm{Arctan}(\sqrt{2})$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	0	$\frac{\pi}{4}$	$\frac{\pi}{4} + \arctan(1/\sqrt{2})$

As follows from Equ. (11) and the analytic continuation (7), the function $\varphi(r)$ satisfies the functional equation

$$\varphi(\sqrt{r^2+1}) = \varphi(r) + \frac{\pi}{2} - \mathrm{Arctan}(r)\,, \quad r \in \mathbb{R}\,. \tag{18}$$

The data for generating Figures 2, 3 were obtained by algorithms described in the next two sections.

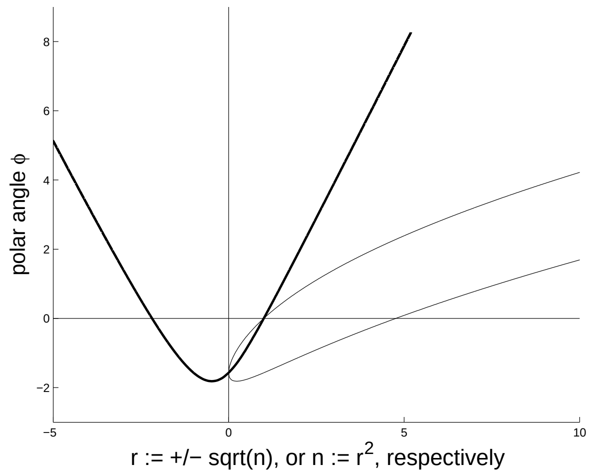


Figure 3: The polar angle $\varphi(r)$ (bold) and two branches of $\Phi(n)$.

4 Fast Evaluation

For generating the plots of Figures 2 and 3 a fast evaluation algorithm for slowly convergent series, such as (15) for $\varphi(r)$, is needed. It suffices to evaluate (15), e.g., in the interval $0 \le r \le 1$; these values can then be propagated to $\mathbb{R}_+$ by the functional equation (18). Values for $r < 0$ may be obtained by the reflection formula (17), and the complex positions are given by $f(r) = r\, e^{i\,\varphi(r)}$.

There is a wide literature on techniques for accelerating the convergence of series like (15), see, e.g. [3], Appendix A. As an alternative, we propose the technique of converting the sum into a contour integral, which typically yields exponential convergence for sums of terms analytic in the index, see, e.g. [3], Chapter 3. Slowly convergent products such as (12) may be handled by the same technique via the logarithm.

4.1 Summation formulas of exponential convergence

Consider the infinite series with terms $s(k)$,

$$S := \sum_{k=1}^{\infty} s(k)\,, \tag{19}$$

where $s(k)$ is real-analytic in the summation index k. A sufficient condition for the absolute convergence of the series (19) is that $s(k)$ is of the order $s(k) = O(k^{-\alpha}),\ \alpha > 1$ as $k \to \infty,\ k \in \mathbb{N}$. We restrict ourselves to the class of real-analytic terms analytic in $|\arg(z)| < \pi/2$ and satisfying

$$s(z) = O(z^{-\alpha})\,, \quad \alpha > 1 \qquad \text{as} \qquad |z| \to \infty\,, \quad |\arg(z)\,| < \pi/2\,.$$

Using the residue theorem of complex analysis in the reverse direction, as well as the fact that the analytic function $z \mapsto \pi \operatorname{cotan}(\pi\, z)$ has first-order poles with unit residues in all integer points $z = k \in \mathbb{Z}$ as its only singularities, the sum (19) may be written as the contour integral

$$S := \frac{\pi}{2\,\pi\, i} \int_C s(z)\ \operatorname{cotan}(\pi\, z)\, dz\,.$$

Here the contour C passes from infinity in the first quadrant to infinity in the fourth quadrant, and intersects the real axis in the interval $(0, 1)$. The path of integration C may be deformed into the line $z = \frac{1}{2} + i\, y,\ \infty >$

$y > -\infty$ since no singularities are crossed and the semi-circle of radius R does not contribute to the integral in the limit $R \to \infty$.

With the identity $\operatorname{cotan}\big(\pi\,(\frac{1}{2} + i\,y)\big) = -i\,\tanh(\pi\,y)$ we obtain

$$S = -\frac{1}{2}\int_{-\infty}^{\infty} \operatorname{Im} s(\frac{1}{2} + i\,y)\;\tanh(\pi\,y)\;dy\,. \tag{20}$$

In summary, we have

Theorem 2. Let $s : z \mapsto s(z)$ be analytic in $D := \{z \;\big|\; |\arg(z)\,| < \pi/2\}$ with $s(\bar{z}) = \overline{s(z)}$ and $s(z) = O(z^{-\alpha})\,,\ \alpha > 1$ as $|z| \to \infty\,,\ z \in D$. Then

$$S := \sum_{k=1}^{\infty} s(k) = -\frac{1}{2}\int_{-\infty}^{\infty} \operatorname{Im} s(\frac{1}{2} + i\,y)\;\tanh(\pi\,y)\;dy\,. \qquad \square$$

On first sight, the evaluation of the integral (20) seems to be no simpler than the evaluation of the sum (19) if $s(z)$ decays slowly as $z \to \infty$. However, compared to the sum, the integral has the advantage of freedom of choice of the evaluation points. By appropriate transformations of the integration variable a quickly decaying integrand can easily be obtained. Then the trapezoidal rule provides an efficient tool for aproximating the transformed integral. Typically, the discretization error is exponentially small with respect to the reciprocal step size, see, e.g. [12].

4.2 The Theodorus Constant

As an introductory example consider the *Theodorus constant* T_1, mentioned in [4], [5], [11], defined as

$$T_1 := \varphi'(1)\,, \quad (\,\cdot\,)' = \frac{d}{dr} \tag{21}$$

with $\varphi(r)$ from (15). T_1 is the slope of the Davis interpolant at the point $F_1 = 1$. Differentiation of (15) yields

$$\varphi'(r) = \frac{1}{1+r^2} + \sum_{k=2}^{\infty} \frac{r}{(k+r^2)\sqrt{k-1+r^2}} \tag{22}$$

which implies

$$T_1 = \sum_{k=1}^{\infty} \frac{k^{-1/2}}{k+1}\,. \tag{23}$$

For applying Theorem 2 we therefore define

$$s(z) := \frac{z^{-1/2}}{z+1}\,. \tag{24}$$

Transforming the integral (20) by means of $y = \sinh(v)$ yields an integrand of exponential decay in v; the change of variables $y = \sinh(\sinh(v))$ produces an integrand of doubly exponential decay. In both cases the discretization error of the trapezoidal rule with step size $h > 0$ is found to be roughly of the order $O(e^{-\gamma\,\omega})$ with $\omega := 2\,\pi/h,\ \gamma > 0$ as $h \to 0$ [12].

The results of a few experiments using the language PARI [2] on a 1.6 GHz processor are collected in the following table:

Transformation	$y = \sinh(t)$		$y = \sinh(\sinh(t))$			
Working precision	38	77	38	77	105	202
Accuracy, digits	35	73	38	75	104	200
Execution time, sec	0.40	3.12	0.02	0.12	0.29	2.07
Step size h	1/25	1/52	1/25	1/66	1/98	1/220

The algorithms using an integrand of doubly exponential decay turn out to be more efficient in speed and accuracy. This is mainly due to shorter and smaller tails of the truncated infinite trapezoidal sums. However, enforcing a fast decay of the integrand by iterated sinh transformations bears the danger of "breeding" complex singularities, which may slow down the convergence with respect to step refinement [12]. Indeed, in the doubly exponential case the discretization error seems to be of the order $O(\exp\big(-\gamma\,\omega/\log(\omega)\big))$ rather than $O(e^{-\gamma\,\omega})$.

The Theodorus constant, to a precision of 50 dgits, is found to be

$$T_1 = 1.86002\,50792\,21190\,30718\,06959\,15717\,14332\,46665\,24121\,52345\ldots.$$

By using Laplace transforms, W. Gautschi [5] found an integral representation of T_1 involving Dawson's integral in its integrand. We will briefly digress to convert Gautschi's integral to an integral involving elementary functions only.

According to [1], Equ. 29.3.44, the term (24) may be written as a Laplace integral:

$$s(k) = \frac{1}{\sqrt{k}\,(k+1)} = \int_0^\infty e^{-k\,t}\,\left(\frac{1}{i}\,e^{-t}\,\mathrm{erf}(i\,\sqrt{t})\right)\,dt\,, \tag{25}$$

where erf(.) is the error function, see [1], Equ. 7.1.1. By carrying out the summation of (23) explicitly under the integral (25) and by using

$$\frac{1}{i}\,\mathrm{erf}(i\,\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{\tau^2}\,d\tau\,,$$

we obtain T_1 as the double integral

$$T_1 = \frac{2}{\sqrt{\pi}}\int_0^{\infty} dt\,\frac{e^{-t}}{e^t-1}\int_0^{\sqrt{t}} d\tau\; e^{\tau^2}\,.$$

Exchanging the order of the integrations and carrying out the integration with respect to t explicitly yields the elegant result

$$\boxed{T_1 = \frac{2}{\sqrt{\pi}}\int_0^{\infty}\Big(-\,e^{\tau^2}\;\log\big(1-e^{-\tau^2}\big)-1\Big)\;d\tau\,.} \tag{26}$$

The numerical evaluation of this integral to high precision requires a few precautions to avoid cancellation. Denoting the integrand in (26) by $L(u) := -u\,\log(1-u^{-1})-1$ with $u := \exp(\tau^2)$, we suggest to use the equivalent expression

$$L(u) = u\;\mathrm{asinh}\bigg(\frac{1}{4\,u}\Big(3+\frac{1}{\tanh(\tau^2/2)}\Big)\bigg)\;-\;1$$

for $u \le 16$ and forward evaluation of the continued fraction

$$\begin{aligned} L(u) &= \frac{u^{-1}}{2}+\frac{u^{-2}}{3}+\frac{u^{-3}}{4}+\dots \\ &= \frac{q^{-1}\,|}{|\,2\,q}-\frac{2^2\,|}{|\,3\,q}-\frac{1^2\,|}{|\,4\,q}-\frac{3^2\,|}{|\,5\,q}-\frac{2^2\,|}{|\,6\,q}-\frac{4^2\,|}{|\,7\,q}-\dots\;,\quad q=\sqrt{u} \end{aligned}$$

for $u > 16$. The change of variables $\tau = \exp(v - e^{-v})$, $v \in \mathbb{R}$ in (26) yields an integrand of doubly exponential decay as well as exponential convegence of the trapezoidal rule with respect to step refinement. At most two digits are lost due to round-off errors, and execution times roughly agree with the times required when applying Theorem 2 with the change of variables $y = \sinh\big(\sinh(v)\big)$ to the sum (23).

4.3 The Taylor series at $r = 0$

A natural generalization of the discussion in the preceding section is the construction of the Taylor series of $\varphi(r)$ and $f(r)$ centered at the origin

$r = 0$. We will first expand $\varphi'(r)$, defined in Equ. (22), in a Taylor series centered at $r = 1$. Substituting $r = 1 + \rho$ with $|\,\rho\,| < 1$ in (22) and expanding with respect to ρ yields

$$\varphi'(1+\rho) = \sum_{k=1}^{\infty} \frac{k^{-1/2}}{k+1} \,(1+\rho)\, \Big(1 - \frac{2\,\rho + \rho^2}{k+1} + \ldots\Big)\, \Big(1 - \frac{2\,\rho + \rho^2}{2\,k} + \ldots\Big)\,.$$

The leading term of the series is the Theodorus constant T_1 of Equ. (23); including first-order terms we obtain $\varphi'(1+\rho) = T_1 + \rho\,\varphi''(1) + O(\rho^2)$, where

$$\varphi''(1) = 2\,T_1 - 2\,T_2 - \zeta\big(\tfrac{3}{2}\big)\,, \quad T_2 = \sum_{k=1}^{\infty} \frac{k^{-1/2}}{(k+1)^2}\,, \tag{27}$$

and $\zeta(\,.\,)$ is the Riemann zeta function. The technique of Section 4.1 yields $T_2 = 0.43916\,45765\,56176\,63266$, and the software package [2] finds $\zeta(\frac{3}{2}) = 2.61237\,53486\,85488\,34335$. Integrating and using the initial condition $\varphi(1) = 0$ then yields

$$\varphi(1+\rho) = T_1\,\rho + \varphi''(1)\,\frac{\rho^2}{2} + O(\rho^3)\,. \tag{28}$$

Now, Equ. (18), valid for $r \in \mathbb{R}$, allows to carry over the Taylor series (28) in $r = 1$ to a series centered at $r = 0$ by putting

$$\rho = \sqrt{1 + r^2} - 1 = \frac{1}{2}\,r^2 - \frac{1}{8}\,r^4 + O(\rho^6)\,.$$

The result is

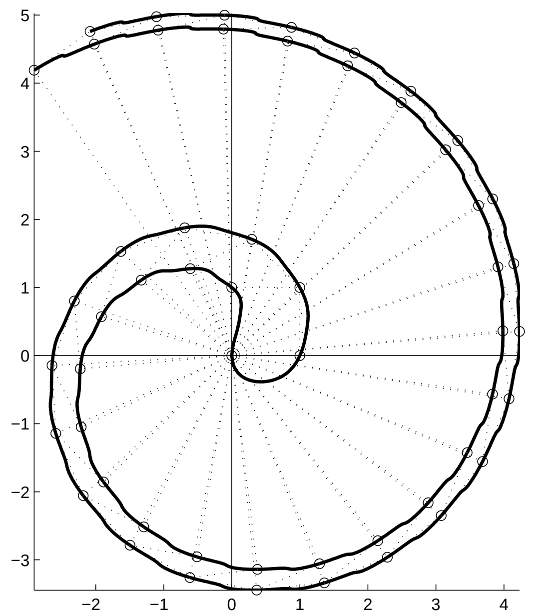
$$\varphi(r) = -\frac{\pi}{2} + \mathrm{Arctan}(r) + T_1\,\frac{r^2}{2} + \big(\varphi''(1) - T_1\big)\,\frac{r^4}{8} + O(r^6)\,. \tag{29}$$

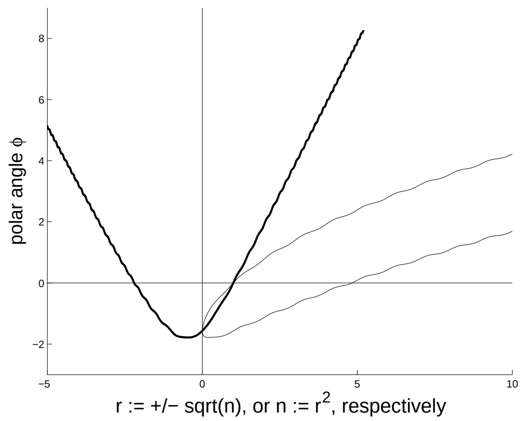
Finally, the complex position $f(r) = r\,e^{i\,\varphi(r)}$ is obtained as

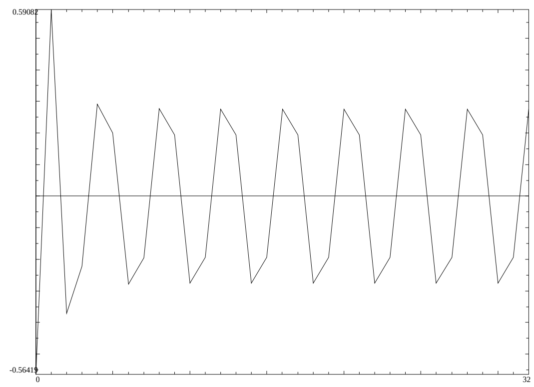
$$f(r) = (-i\,r{+}r^2)\,\Big[1 + (-1 + i\,T_1)\frac{r^2}{2} + \Big(3 - {T_1}^2 + i\big(\varphi''(1) - 3\,T_1\big)\Big)\frac{r^4}{8} + O(r^6)\Big]\,.$$

5 Asymptotics

- Functional equations. A non-monotonic solution of (11), Fig. 4, 5
- Gronau
- Asymptotic series for Φ_n as $n \to +\infty$. Conjectured growth rate of the coefficients c_n. Figure 6: Plot of $n^{5/2}\,(2\,\pi)^n/n!\;c_n$
- Euler-Maclaurin expansion, integral for the Euler constant of the square root spiral, $\gamma = -2.157782996659446$




polar angle ϕ
r := +/− sqrt(n), or n := r^2, respectively



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