The Rectangular Symmetric Four-Body Problem

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Resonances in n-body systems

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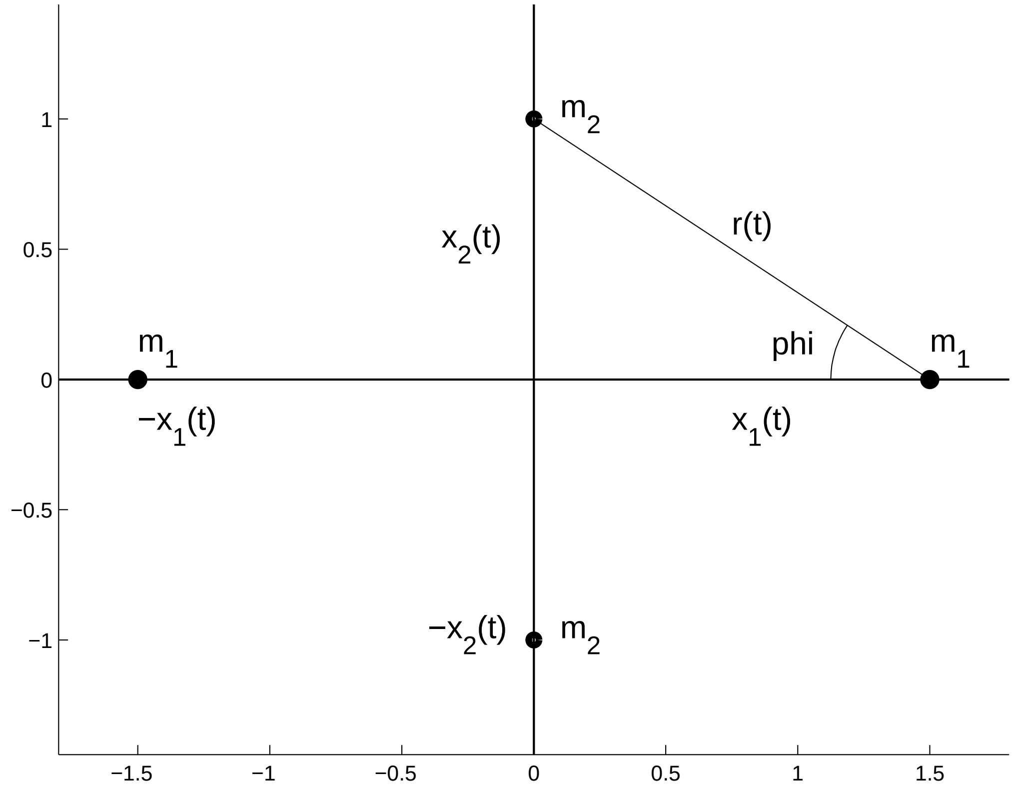
Abstract

We consider the symmetric planar four-body problem with two equal masses $m_1 > 0$ at positions $(\pm x_1(t), 0)$ and two equal masses $m_2 > 0$ at positions $(0, \pm x_2(t))$ at all times t, referred to as the rectangular symmetric 4-body problem. Owing to the simplicity of the equations of motion this problem is well suited to study regularization of the binary collisions, homothetic solutions and central configurations, as well as the four-body collision and escape manifolds. Furthermore, resonance phenomena between the two interacting rectilinear binaries play an important role.

Outline

1. Equations of Motion



Two equal masses $m_1 > 0$ at positions $(\pm x_1(t),\, 0)$, $\quad x_j(t) \geq 0$
Two equal masses $m_2 > 0$ at positions $(0,\, \pm x_2(t))$ at all times t.

Yields two binaries in coupled rectilinear motions on perpendicular lines.

Equations of motion

$$\ddot{x}_j + \frac{m_j}{4\,x_j^2} + \frac{2\,m_{3-j}\,x_j}{r^3} = 0\,, \quad j = 1,\,2\,, \quad r := \sqrt{x_1^2 + x_2^2}\,, \quad (\dot{\ }) = \frac{d}{dt}(\ ) \tag{1}$$

Energy integral: $\frac{1}{2}(T+U) =: H_0 = \text{const.}$ where

$$T = m_1\,\dot{x_1}^2 + m_2\,\dot{x_2}^2\,, \quad U = -\frac{m_1^2}{2\,x_1} - \frac{m_2^2}{2\,x_2} - \frac{4\,m_1\,m_2}{\sqrt{x_1^2+x_2^2}} \tag{2}$$

Hamiltonian

$$H = \frac{p_1^2}{2\,m_1} + \frac{p_2^2}{2\,m_2} - \frac{m_1^2}{4\,x_1} - \frac{m_2^2}{4\,x_2} - \frac{2\,m_1\,m_2}{\sqrt{x_1^2+x_2^2}} \quad \text{with} \quad p_j := m_j\,\dot{x}_j \tag{3}$$

Hamiltonian equations of motion

$$\dot{x}_j = \frac{\partial H}{\partial p_j}\,, \quad \dot{p}_j = -\frac{\partial H}{\partial x_j}\,, \quad j = 1,\,2\,, \quad H(t) = H_0 = \text{const.}$$

2. Levi-Civita Regularization

Step 1: time transformation

Fictitious time τ as new independent variable, new Hamiltonian K (Poincaré's device)

$$dt = x_1\, x_2\, d\tau\,, \quad K = x_1\, x_2\, (H - H_0)\,. \tag{4}$$

Step 2: Levi-Civita's canonical coordinate transformation

New coordinates ξ_j, new momenta $\pi_j\,,\ j = 1,\, 2$

$$x_j = \xi_j^2\,, \quad p_j = \frac{\pi_j}{2\,\xi_j}\,, \quad j = 1,\, 2 \tag{5}$$

This is obtained via the generating function $W(p, \xi) = p_1\, x_1 + p_2\, x_2$ as

$$\pi_j = \frac{\partial W}{\partial \xi_j}\,, \quad j = 1,\, 2\,.$$

The regularized Hamiltonian, $K(\tau) = 0$

$$K = \frac{1}{8}\left(\frac{\pi_1^2\,\xi_2^2}{m_1} + \frac{\pi_2^2\,\xi_1^2}{m_2}\right) - \frac{1}{4}\left(m_1^2\,\xi_2^2 + m_2^2\,\xi_1^2\right) - \frac{2\,m_1\,m_2\,\xi_1^2\,\xi_2^2}{\sqrt{\xi_1^4+\xi_2^4}} - H_0\,\xi_1^2\,\xi_2^2 \tag{6}$$

Regularized equations of motion

$$\xi_j{}' = \frac{\partial K}{\partial \pi_j}\,, \quad \pi_j{}' = -\frac{\partial K}{\partial \xi_j}\,, \quad j = 1,\,2\,, \quad (\ )' = \frac{d}{d\tau}(\ )$$

or, for $j = 1,\,2$ with $k := 3 - j$,

$$\begin{aligned} \xi_j{}' &= \frac{\pi_j\,\xi_k^2}{4\,m_j} \\ \pi_j{}' &= \xi_j\left(-\frac{\pi_k^2}{4\,m_k} + \frac{m_k^2}{2} + 4\,m_1\,m_2\left(\frac{\xi_k^4}{\xi_1^4+\xi_2^4}\right)^{3/2} + 2\,H_0\,\xi_k^2\right) \\ t' &= \xi_1^2\,\xi_2^2\,. \end{aligned} \tag{7}$$

Power series expansion in a binary collision

With no loss of generality consider collisions at $\tau = 0$ with $\xi_1(0) = 0$.

For $\xi_2(0) \neq 0\,,\;\; K = 0$ yields

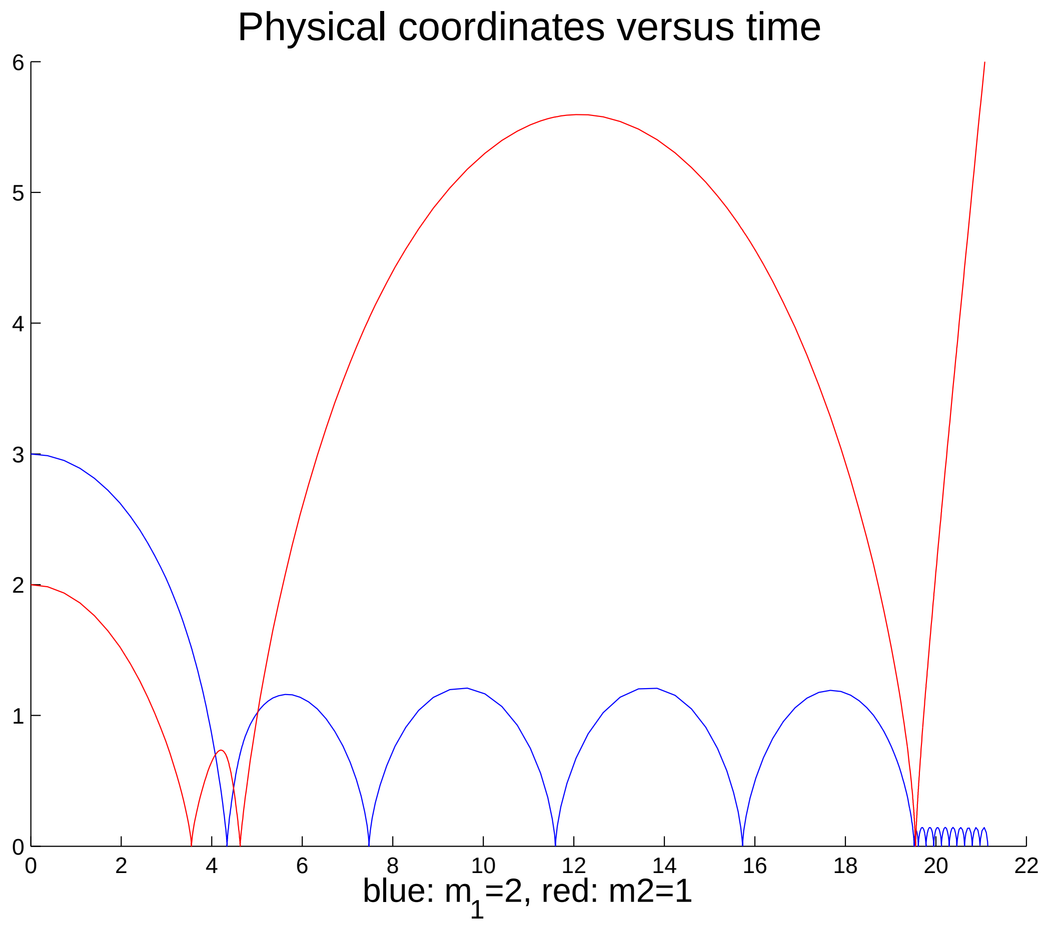
$$\pi_1(0) = \pm\sqrt{2\,m_1^3}\,.$$

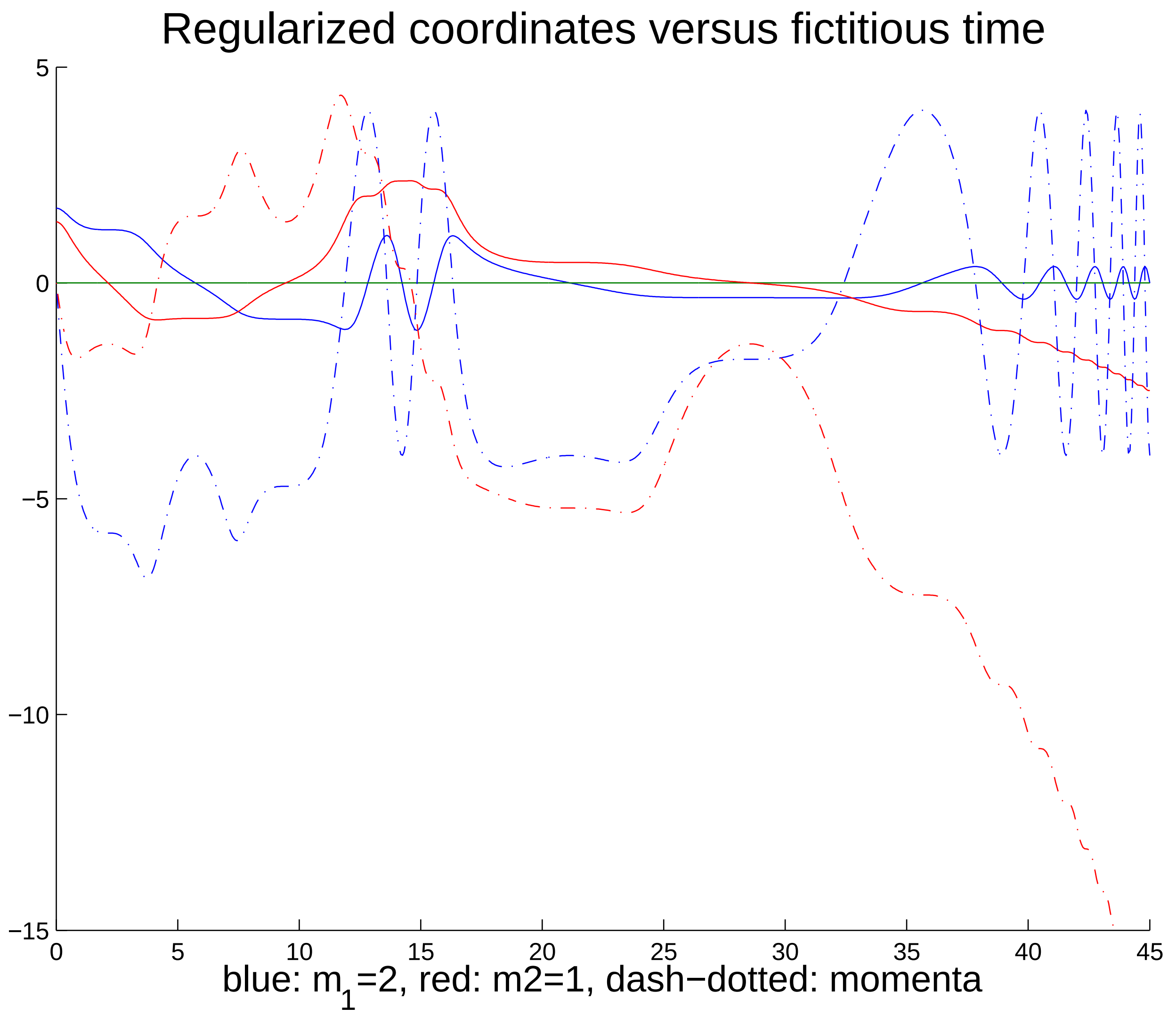
The three parameters of the motion are $A := \xi_2(0) \neq 0,\;\; B := \pi_2(0)$ and the total energy H_0:

$$\begin{array}{rcllcl}
\xi_1(\tau) & = & \pi_1(0)\,\dfrac{A^2}{4m_1}\,\tau & & + & O(\tau^3) \\
\xi_2(\tau) & = & A & & + & O(\tau^3) \\
\pi_1(\tau) & = & \pi_1(0) & & + & O(\tau^2) \\
\pi_2(\tau) & = & B & & + & O(\tau^3)
\end{array}$$

The series are uniquely determined (up to the signs) by $m_1, m_2,\; A,\; B,\; H_0$.

3. A Typical Example: Escape




Regularized coordinates versus fictitious time
5
0
−5
−10
−15
0
5
10
15
20
25
30
35
40
45
blue: m1=2, red: m2=1, dash−dotted: momenta

4. Periodic Solutions and Resonance

For finding periodic solutions we use initial conditions in a collision (see p. 8), e.g.

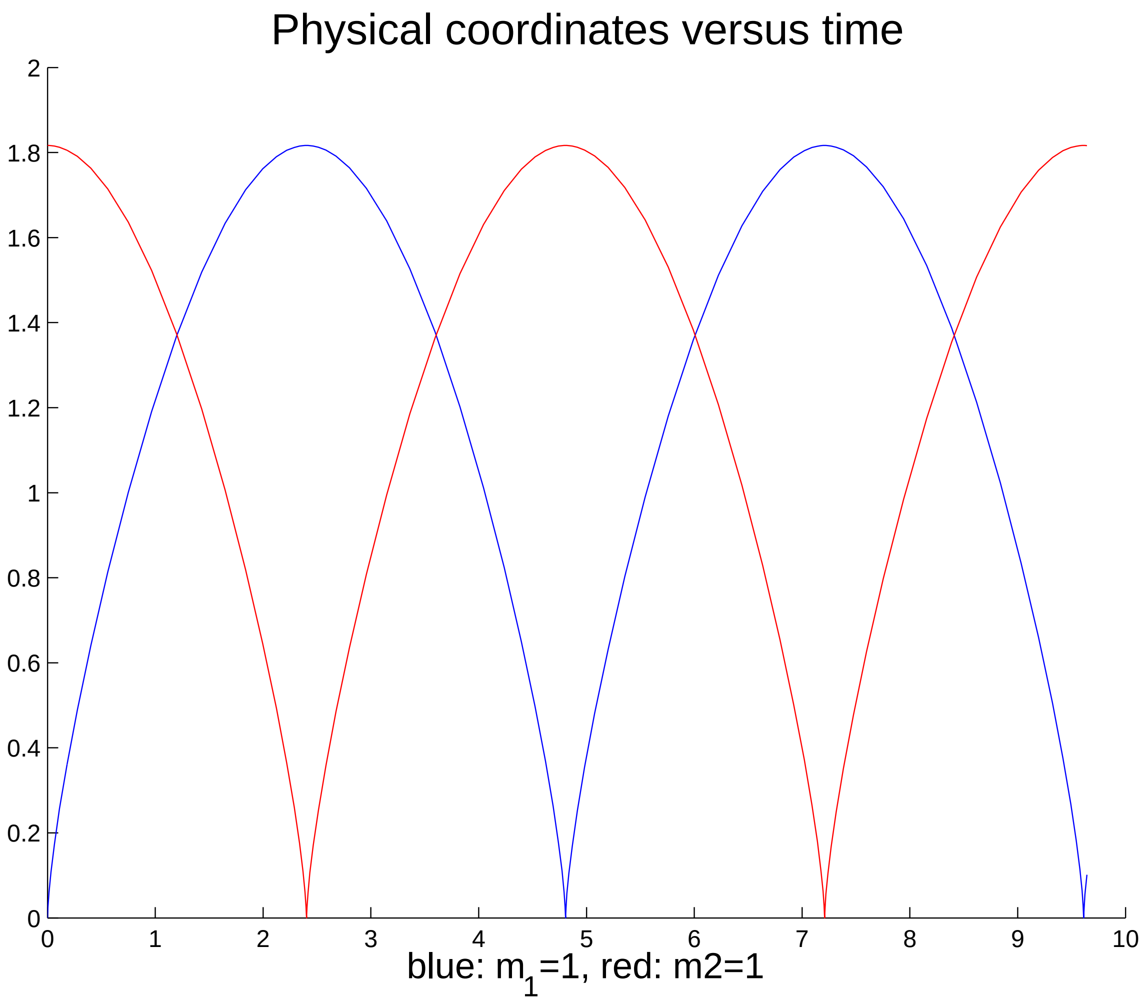
$$\xi_1(0) = 0\,, \quad \pi_1(0) = -\sqrt{2\,m_1^3}\,, \quad \pi_2(0) = 0$$

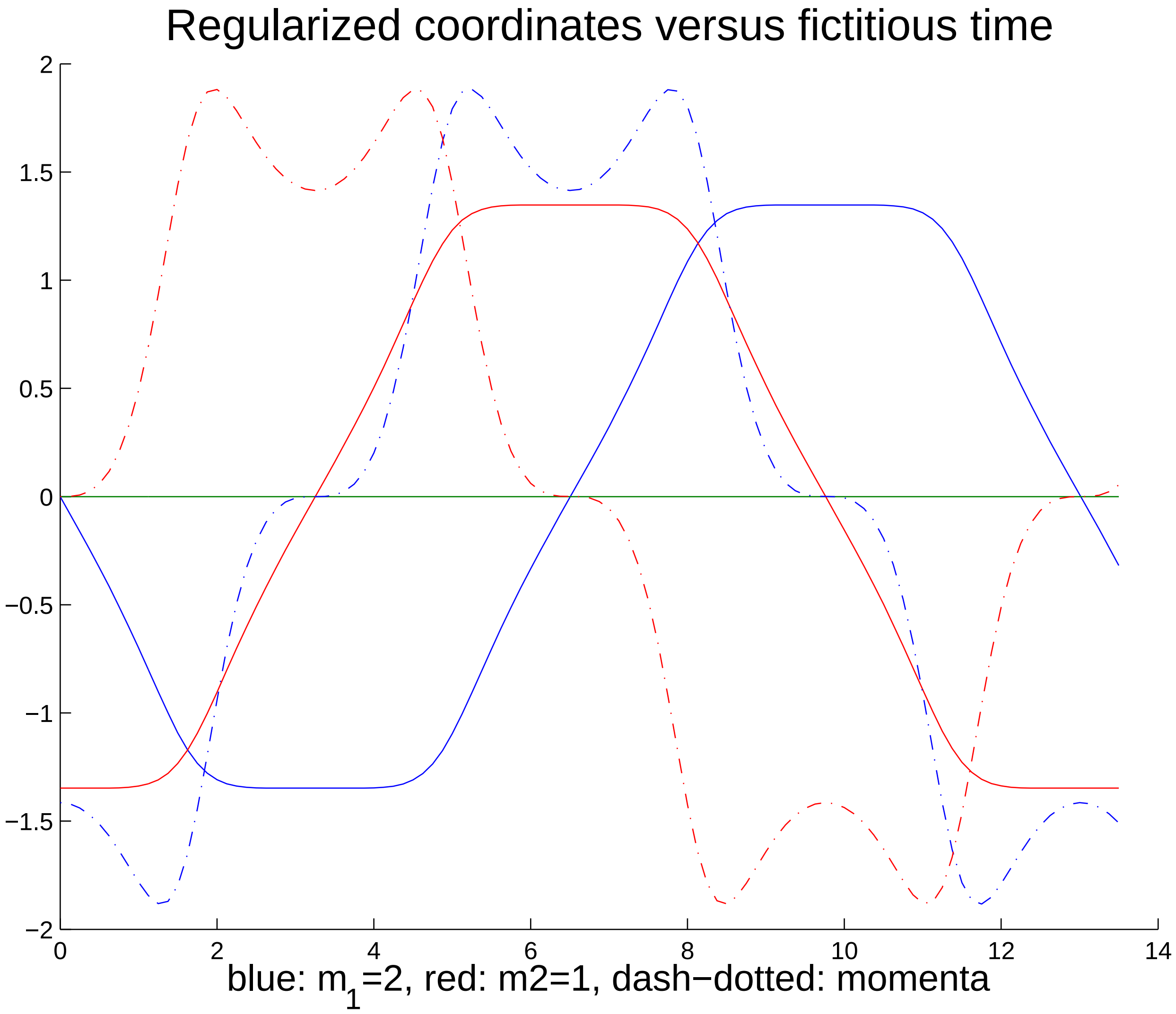
and fix the energy. For given masses and a given energy H_0, define a tentative quarter period q by $\xi_2(q) = 0$.

Periodicity condition: $\boxed{\pi_1(q) = 0}$

Example: $m_1 = m_2 = 1$, $H_0 = -0.9$. Numerical integration by an integrator with event capability and the secant method for solving nonlinear equations yields $\xi_2(0) = -1.34776\,71645\,4144$.

A periodic solution with equal masses and $H_0 = -0.9$




Regularized coordinates versus fictitious time
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
0
2
4
6
8
10
12
14
blue: m1=2, red: m2=1, dash−dotted: momenta

Resonance

This periodic orbit is remarkably robust against perturbations of the initial conditions (see Section 5).

Reason: The two binaries are in a 1:1 resonance. In this way they are locked away from a close quadruple encounter, which would eventually result in an escape (see p. 9).

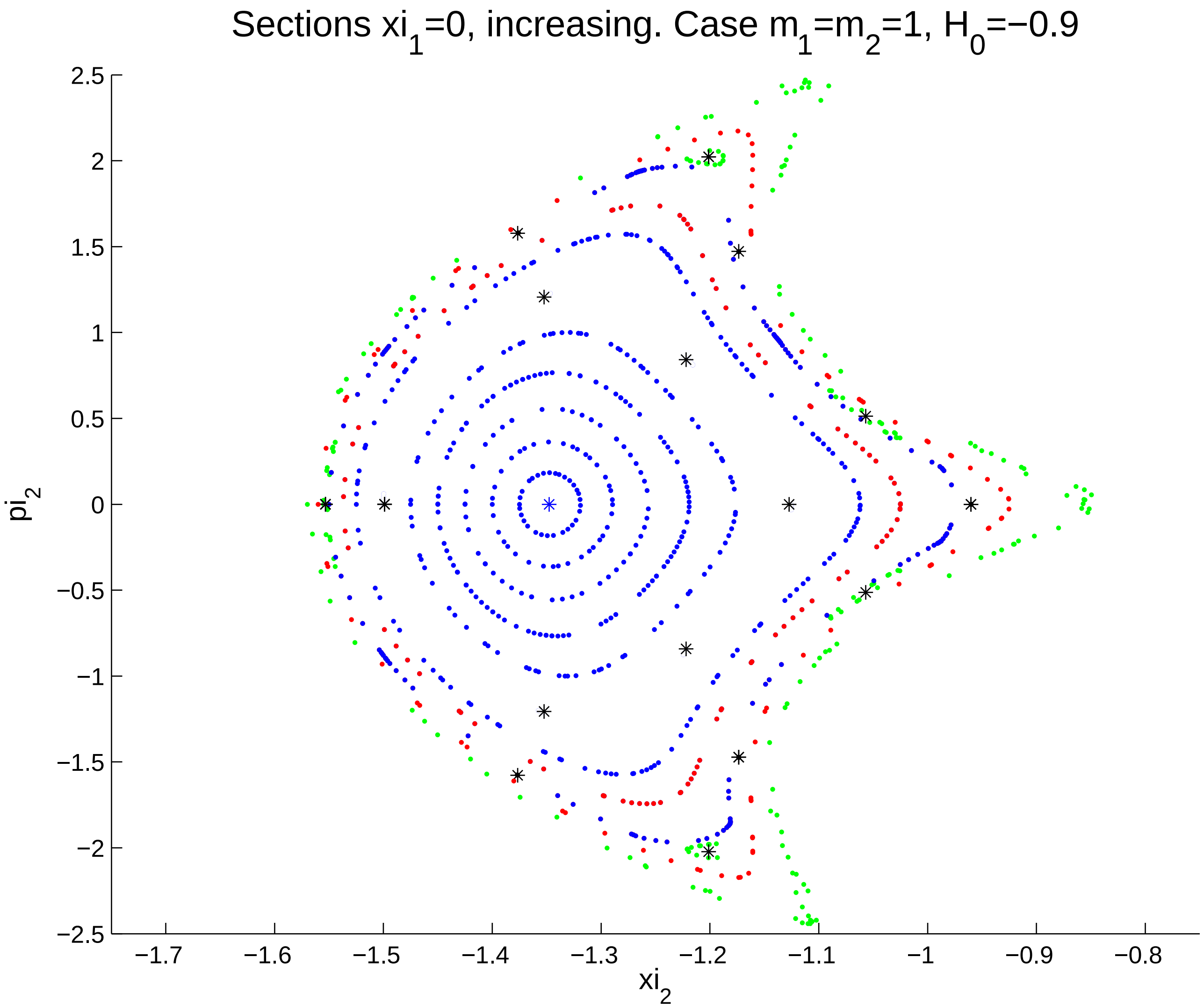
5. Poincaré Sections and Quasiperiodic Solutions

Instead of the entire orbit $\big(\xi_j(\tau),\, \pi_j(\tau)\big)$ we only consider its intersection points with the surface of section

$$\xi_1 = 0 \quad \text{with} \quad {\xi_1}' > 0\,, \quad \pi_1 = -\sqrt{2\, m_1^3}\,,$$

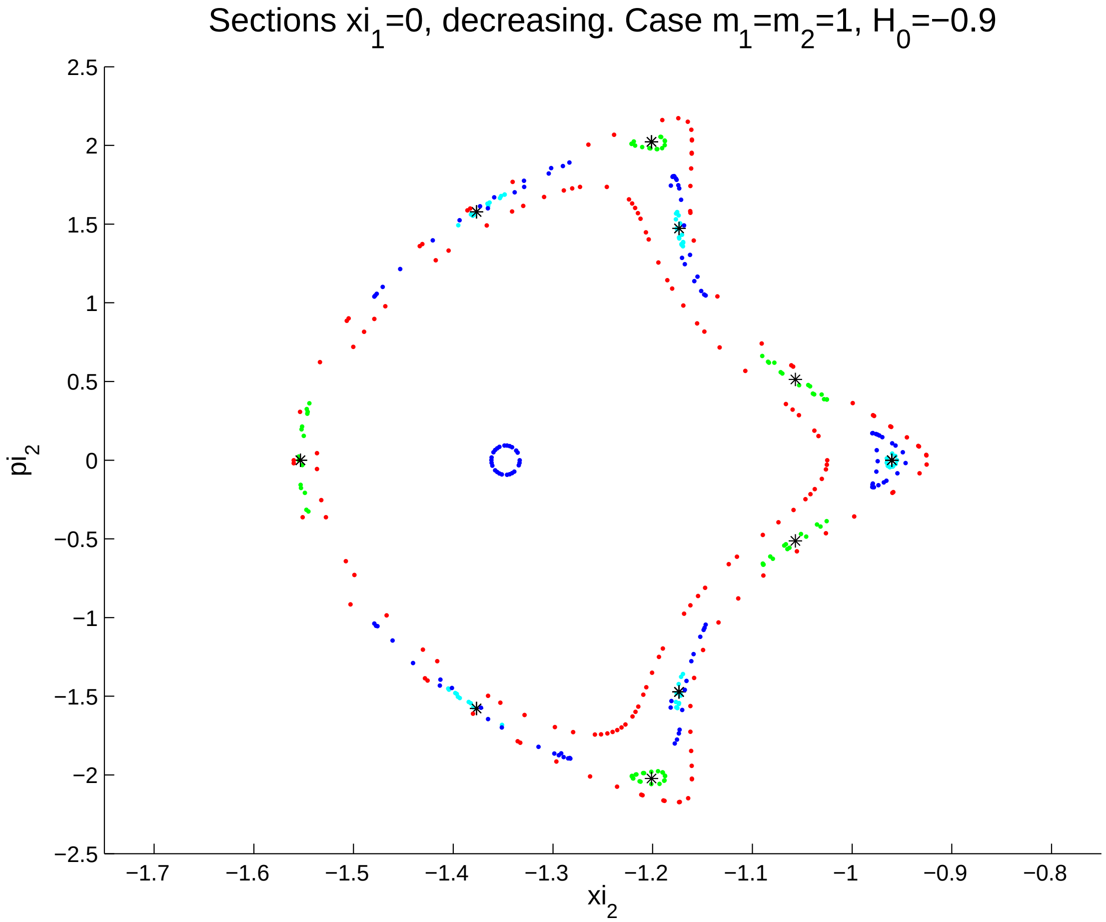
and we plot the sequence of points in the $(\xi_2,\, \pi_2)$-plane for fixed energy H_0 and various initial points.

In the plot on p. 16 the center corresponds to the periodic solution of p. 12. The ovals around it visualize quasiperiodic solutions (tori). The black asterisks mark periodic solutions of longer periods, e.g. near $\xi_2 = -1.49906$ (6-periodic) or near $\xi_2 = -1.55325$ (5-periodic, stable, with green islands) and near $\xi_2 = -0.96016$ (p. 17, 5-periodic, cyan islands). In the outermost green "curve" corresponding to $\xi_2 = -1.57$ the onset of chaos is visible.



Sections xi$_1$=0, increasing. Case m$_1$=m$_2$=1, H$_0$=−0.9
pi$_2$
xi$_2$
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−1.7
−1.6
−1.5
−1.4
−1.3
−1.2
−1.1
−1
−0.9
−0.8

Ten islands corresponding to two 5-periodic solutions



Hyperbolic points between the 5-periodic points generate a chaotic zone, marked by a few blue dots.

6. Homothetic Solutions and Central Configurations

Solve the equations of motion (Equ. (1), p. 5) by

$$x_j(t) = c_j\, f(t)\,, \quad j = 1,\, 2, \quad c_1 = c\, \cos(\varphi),\;\; c_2 = c\, \sin(\varphi)$$

with constants c, φ, and $f(t)$ describing a rectilinear Kepler motion,

$$\ddot{f}(t) + \frac{m}{f(t)^2} = 0\,.$$

This yields the two conditions

$$\frac{m_1}{\cos^3(\varphi)} + 8\, m_2 = \frac{m_2}{\sin^3(\varphi)} + 8\, m_1 = 4\, c^3\, m\,,$$

resulting in the following condition for symmetric diamond-shaped central configurations of four pairwise equal masses:

$$\frac{m_1}{\cos^3(\varphi)} - \frac{m_2}{\sin^3(\varphi)} = 8\,(m_1 - m_2)\,, \quad 0 < \varphi < \pi/2\,. \qquad (8)$$

Computation of the central configurations

Introduce the mass parameter $\mu := \dfrac{m_1 - m_2}{m_1 + m_2} \in (-1, 1)$.

For arbitrary (real) μ Equ. (8) has a unique real solution φ with

$$\left|\varphi - \frac{\pi}{4}\right| < 0.36474\,52742\,36650\,.$$

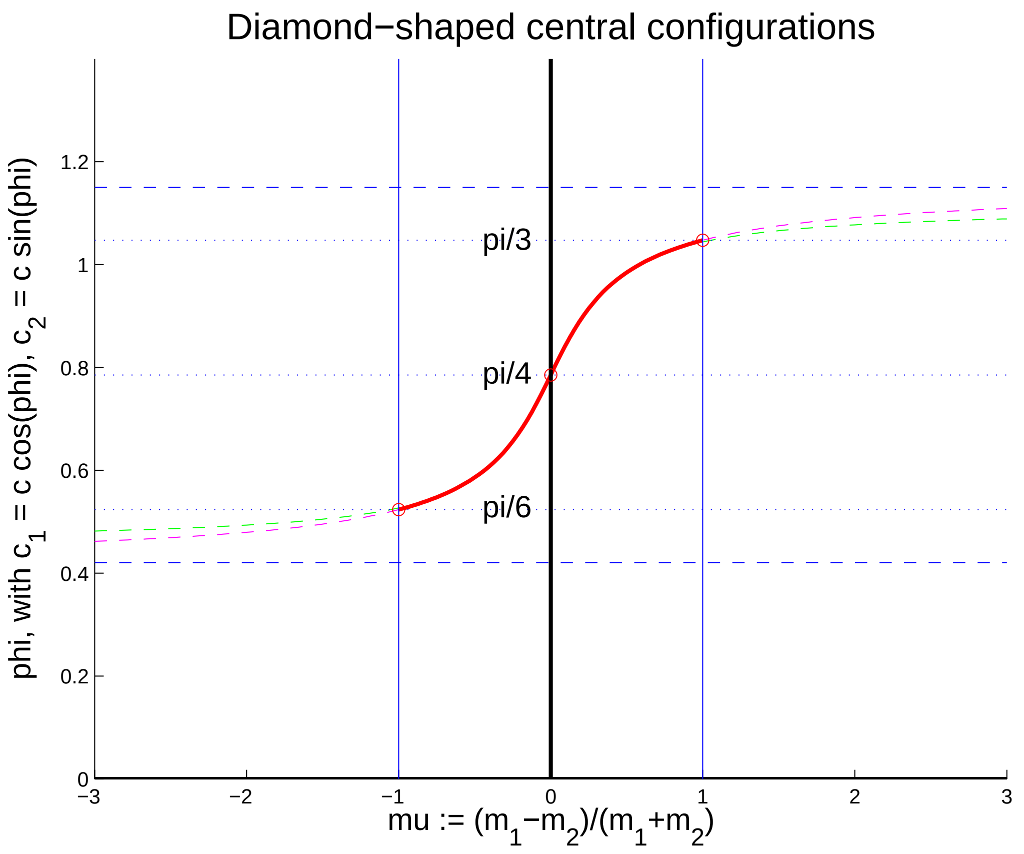
Equ. (8) reduces to a polynomial equation of degree 12 for $\tan(\frac{\varphi}{2})$.

Alternatively, Equ.(8) may be solved numerically by the Newton-Raphson iteration, e.g. by using the initial approximation (green curve on p. 20),

$$\varphi_0 = \frac{\pi}{4} + \frac{1}{f} \arctan\left(f\, b\, \mu\right) \quad \text{with} \quad f = 1.528545\,\pi,\ b = \frac{2\sqrt{2} - 1}{3}\,.$$

In the interval $-1 < \mu < 1$ the absolute error of φ_0 is less than 0.003255, and 3 iterations yield an accuracy of 15 digits.

Discussion of the central configurations. See figure on p.4



Particular values: $\varphi(-1) = \frac{\pi}{6}, \quad \varphi(0) = \frac{\pi}{4}, \quad \varphi(1) = \frac{\pi}{3},$

$$\varphi(\pm\infty) = \frac{\pi}{4} \pm 0.364745, \; \varphi'(0) = b := \frac{2\sqrt{2}-1}{3}\,.$$

7. The Quadruple-Collision Manifold

Idea: Introduce normalized coordinates, momenta, and fictitious time $\tilde{\xi}_j$, $\tilde{\pi}_j$, $\tilde{\tau}$ adapting to the current size and rate of change of a four-body system in a close quadruple encounter or in a quadruple collision.

A convenient length is the radius of inertia ρ, defined by means of the moment of inertia I (McGehee, JW):

$$\rho^2 = I = 2\left(m_1\, x_1^2 + m_2\, x_2^2\right). \tag{9}$$

We remark that Equs. (1) - (5) of p. 5, 6 easily imply

$$\dot{I} = 2\left(\pi_1\, \xi_1 + \pi_2\, \xi_2\right), \quad \ddot{I} = 4\,T + 2\,U = 8\,H_0 - 2\,U = 4\,H_0 + 2\,T\,. \tag{10}$$

Scaling transformations:

$$x_j = \rho\, \tilde{x}_j\,, \quad p_j = \rho^{-1/2}\, \tilde{p}_j\,, \quad \xi_j = \rho^{1/2}\, \tilde{\xi}_j\,, \quad \pi_j = \tilde{\pi}_j\,, \quad d\tau = \rho^{-1/2}\, d\tilde{\tau}\,. \tag{11}$$

Normalized equations of motion.

From Equ. (10_1) and the transformations (4_1), (9), (11) we obtain

$$\frac{d\rho}{d\tilde{\tau}} = \rho\, \tilde{\xi}_1^2\, \tilde{\xi}_2^2\, \left(\pi_1\, \tilde{\xi}_1 + \pi_2\, \tilde{\xi}_2\right), \tag{12}$$

a differential equation for ρ allowing $\rho(\tilde{\tau}) \equiv 0$ as a solution. The remaining four equations for equivalently describing the motion follow from Equs. (7) of p. 7:

$$\begin{aligned} \frac{d\tilde{\xi}_j}{d\tilde{\tau}} &= \tilde{\xi}_k^2 \left(\frac{\pi_j}{4\, m_j} - \frac{\tilde{\xi}_j^3}{2} (\pi_1\, \tilde{\xi}_1 + \pi_2\, \tilde{\xi}_2) \right), \quad k := 3 - j,\ j = 1,\, 2 \\ \frac{d\pi_j}{d\tilde{\tau}} &= \tilde{\xi}_j \left(-\frac{\pi_k^2}{4\, m_k} + \frac{m_k^2}{2} + 4\, m_1\, m_2 \left(\frac{\tilde{\xi}_k^4}{\tilde{\xi}_1^4 + \tilde{\xi}_2^4} \right)^{3/2} + 2\, \rho\, H_0\, \tilde{\xi}_k^2 \right) \\ \frac{dt}{d\tilde{\tau}} &= \rho^{3/2}\, \tilde{\xi}_1^2\, \tilde{\xi}_2^2\,. \end{aligned} \tag{13}$$

The collision manifold $\mathcal{M}$

is defined as the limiting solution of the system (12), (13) characterized by $\rho(\tilde{\tau}) \equiv 0$. Equ. (12) is satisfied, and (13_3) implies that time t does not advance. Therefore $\mathcal{M}$, i.e. the solution of $(13_1), (13_2)$ with $\rho = 0$, describes the very instant of collision as seen in an infinitely slowed down and blown-up slow-motion picture.

As a consequence of (9) and (6), the collision manifold has the two integrals of motion

$$m_1 \tilde{\xi_1}^4 + m_2 \tilde{\xi_2}^4 = \frac{1}{2}$$

$$\frac{1}{8}\left(\frac{\pi_1^2 \tilde{\xi_2}^2}{m_1} + \frac{\pi_2^2 \tilde{\xi_1}^2}{m_2}\right) - \frac{1}{4}\left(m_1^2 \tilde{\xi_2}^2 + m_2^2 \tilde{\xi_1}^2\right) - \frac{2\, m_1\, m_2 \tilde{\xi_1}^2 \tilde{\xi_2}^2}{\sqrt{\tilde{\xi_1}^4 + \tilde{\xi_2}^4}} = 0\,.$$

Furthermore, it can be shown that the flow on $\mathcal{M}$ is a gradient flow.

A few references

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4. B. A. Steves, A. E. Roy, 1998, *Some special restricted four-body problems I: modelling the Caledonian problem*. Planet Space Sci. **46**, 1465-1474.
5. J. Waldvogel, 1982, *Symmetric and regular coordinates on the plane triple collision manifold*. Celest. Mech. **28**, 69-82.

Conclusions

- A particular case of the "Caledonian" symmetric four-body problem is investigated: two pairs of equal masses are moving symmetrically in the plane on two fixed perpendicular axes.
- Motion is governed by a simple Hamiltonian with 2 degrees of freedom.
- The two types of binary collisions can be regularized by two one-dimensional Levi-Civita transformations.
- Periodic, quasiperiodic, and chaotic motion exists. As a consequence of a 1:1 resonance between the two binaries, orbits can be stable for very long time ("stickiness").
- The quadruple-collision manifold (McGehee) is governed by a rather simple 4th-order system with two integrals of motion.
- More results to come!