Regularization of the Symmetric Four-Body Problem by Elliptic Functions

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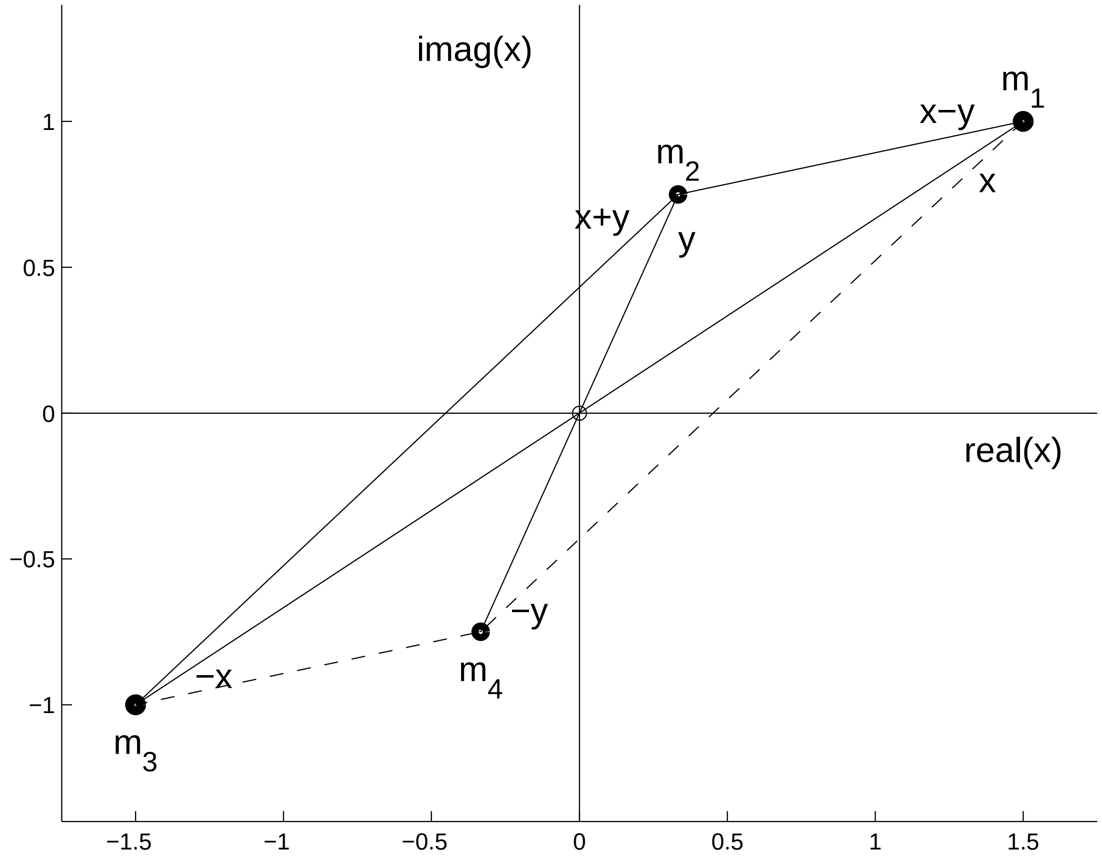
Abstract

Consider 4 point masses $m_k > 0$ at positions $\mathbf{x}_k(t) \in \mathbb{R}^2,\ k = 1, 2, 3, 4,$ moving under Newtonian forces and satisfying the symmetry relations $m_1 = m_3,\ m_2 = m_4,\ \mathbf{x}_1(t) + \mathbf{x}_3(t) = 0,\ \mathbf{x}_2(t) + \mathbf{x}_4(t) = 0$ at all times t. This system, referred to as the Caledonian Four-Body Problem, has been extensively studied B.A. Steves, A.E. Roy, and many others. Binary collisions can occur as single collisions between m_1 and m_3 or between m_2 and m_4. Also, simultaneous collisions (m_1, m_2) and (m_3, m_4) or (m_1, m_4) and (m_2, m_3) can occur. Regularization according to Levi-Civita is possible in every case (for the simultaneous collisions as a consequence of the symmetry). A single coordinate transformation involving elliptic functions is able to regularize every binary collision.

Outline

1. Cartesian Equations of Motion



Complex notation: $\mathbf{x} = \binom{x_1}{x_2} \in \mathbb{R}^2 \quad \longrightarrow \quad x = x_1 + i\, x_2 \in \mathbb{C}$

Two equal masses $m_1 = m_3$ at positions $x(t) \in \mathbb{C}$ and $-x(t) \in \mathbb{C}$

Two equal masses $m_2 = m_4$ at positions $y(t) \in \mathbb{C}$ and $-y(t) \in \mathbb{C}$
at all times t.

Kinetic energy: $T = m_1\,|\dot{x}|^2 + m_2\,|\dot{y}|^2\,, \qquad \dot{(\;)} = \frac{d}{dt}(\;)$

Potential energy: $U = -\dfrac{m_1^2}{2\,|x|} - \dfrac{m_2^2}{2\,|y|} - \dfrac{2\,m_1\,m_2}{|x+y|} - \dfrac{2\,m_1\,m_2}{|x-y|}$

Energy integral: $T + U = \text{const.} =: \mathrm{H}_0$

Hamiltonian, with complex momenta $p = m_1\,\dot{x},\; q = m_2\,\dot{y},\; |x| = \sqrt{x\,\bar{x}}$

$$H(x,\bar{x},y,\bar{y},p,\bar{p},q,\bar{q}) = \frac{p\,\bar{p}}{m_1} + \frac{q\,\bar{q}}{m_2} - \frac{m_1^2}{2\,|x|} - \frac{m_2^2}{2\,|y|} - \frac{2\,m_1\,m_2}{|x+y|} - \frac{2\,m_1\,m_2}{|x-y|}$$

Hamiltonian equations of motion, complex notation

$$\dot{x} = \frac{\partial H}{\partial \bar{p}},\; \dot{y} = \frac{\partial H}{\partial \bar{q}},\; \dot{p} = -\frac{\partial H}{\partial \bar{x}},\; \dot{q} = -\frac{\partial H}{\partial \bar{y}}, \qquad H(t) = H_0 = \text{const.}$$

2. Jacobian Elliptic Functions

Definition:

Elliptic integral of first kind with “modulus” $k \in (0,1)$:

$$s = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)\,(1-k^2\,\zeta^2)}}\,;$$

then

$$\begin{array}{lclclcl} s & := & \mathrm{sn}^{[-1]}(z,k), & & z & = & \mathrm{sn}(s,k) = \mathrm{sn}(s) \\ \mathrm{cn}(s) & := & \sqrt{1-\mathrm{sn}^2(s)}, & & \mathrm{dn}(s) & := & \sqrt{1-k^2\,\mathrm{sn}^2(s)}. \end{array}$$

Identities:

$$\mathrm{sn}^2(s) + \mathrm{cn}^2(s) = 1, \quad \mathrm{dn}^2(s) + k^2\,\mathrm{sn}^2(s) = 1$$

This implies

$$\mathrm{dn}^2(s) - k'^2\,\mathrm{sn}^2(s) = \mathrm{cn}^2(s) \quad \text{with} \quad k^2 + k'^2 = 1$$

Derivatives

$$
\begin{aligned}
\frac{d}{ds}\,\mathrm{sn}(s,k) &= \mathrm{cn}(s,k)\;\mathrm{dn}(s,k)\\
\frac{d}{ds}\,\mathrm{cn}(s,k) &= -\,\mathrm{sn}(s,k)\;\mathrm{dn}(s,k)\\
\frac{d}{ds}\,\mathrm{dn}(s,k) &= -\,k^2\;\mathrm{sn}(s,k)\;\mathrm{cn}(s,k)
\end{aligned}
$$

Remark: Many software packages have implemented the Jacobian elliptic functions; e.g. the MATLAB call

```
[sn, cn, dn] = ellipj(s,k^2)
```

generates all three function values by the AGM algorithm (see p. 8).

Efficient Evaluation of sn(s,k), cn(s,k), dn(s,k) by the AGM

First step (depending on k only): Arithmetic-geometric mean algorithm until $c_n <$ tol:

$$
\begin{array}{lll}
a_0 = 1, & b_0 = k' = \sqrt{1-k^2}, & c_0 = k \\
a_1 = \frac{1}{2}(a_0 + b_0), & b_1 = \sqrt{a_0\, b_0}, & c_1 = \frac{1}{2}(a_0 - b_0) \\
\ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \\
a_n = \frac{1}{2}(a_{n-1} + b_{n-1}), & b_n = \sqrt{a_{n-1}\, b_{n-1}}, & c_n = \frac{1}{2}(a_{n-1} - b_{n-1}) < \mathrm{tol}
\end{array}
$$

Second step

$$
\begin{array}{lcl}
\varphi_n & = & 2^n\, s\, a_n \\
\varphi_{n-1} & = & \frac{1}{2}\Big(\varphi_n + \arcsin(\frac{c_n}{a_n}\, \sin(\varphi_n))\Big) \\
\ldots & & \ldots \quad \ldots \quad \ldots \\
\varphi_0 & = & \frac{1}{2}\Big(\varphi_1 + \arcsin(\frac{c_1}{a_1}\, \sin(\varphi_1))\Big)
\end{array}
$$

Then: sn$(s,k) =$ sin(φ_0), cn$(s,k) =$ cos(φ_0), dn$(s,k) = \dfrac{\cos(\varphi_0)}{\cos(\varphi_1 - \varphi_0)}$

Complex Arguments

Let $s = s_1 + i\,s_2, \quad k' = \sqrt{1-k^2}$ and

$$S_1 = \mathrm{sn}(s_1,k), \quad C_1 = \mathrm{cn}(s_1,k), \quad D_1 = \mathrm{dn}(s_1,k),$$
$$S_2 = \mathrm{sn}(s_2,k'), \quad C_2 = \mathrm{cn}(s_2,k'), \quad D_2 = \mathrm{dn}(s_2,k').$$

Then

$$\begin{aligned}
\mathrm{sn}(s_1 + i\,s_2, k) &= \frac{S_1\,D_2 + i\,S_2\,D_1\;C_1\,C_2}{1 - S_2^2\,D_1^2} \\
\mathrm{cn}(s_1 + i\,s_2, k) &= \frac{C_1\,C_2 - i\,S_1\,S_2\;D_1\,D_2}{1 - S_2^2\,D_1^2} \\
\mathrm{dn}(s_1 + i\,s_2, k) &= \frac{C_2\;D_1\,D_2 - i\,k^2\,C_1\;S_1\,S_2}{1 - S_2^2\,D_1^2}\,.
\end{aligned}$$

3. Levi-Civita Type Coordinate Transformation

For regularizing the Kepler problem with complex Hamiltonian

$$H(x, \bar{x}, p, \bar{p}) = \frac{p\,\bar{p}}{m} - \frac{2\,m^2}{|x|}, \quad x, p \in \mathbb{C}$$

Levi-Civita used a new complex coordinate ξ with $x = \xi^2$. Therefore, for analoguously regularizing all binary collisions of the Hamiltonian $H(x, \bar{x}, y, \bar{y}, p, \bar{p}, q, \bar{q})$ of p. 5 we need two new complex variables such that all four denominator variables of H appear as complete squares:

$$\boxed{x = \xi^2, \quad y = \eta^2, \quad x + y = r^2, \quad x - y = d^2}\ .$$

We will use the last two identities of p. 6, involving Jacobian elliptic functions,

$$\mathrm{dn}^2(s,k) + k^2\,\mathrm{sn}^2(s,k) = 1, \quad \mathrm{dn}^2(s,k) - k'^2\,\mathrm{sn}^2(s,k) = \mathrm{cn}^2(s,k)\,.$$

Levi-Civita type transformation, continued

The equations for $x+y$ and $x-y$ can easily be satisfied with the choice

$$k = k' = \frac{1}{\sqrt{2}}\,.$$

For regularizing all four types of binaries we now reparametrize ξ, η in terms of two new complex variables, $r, s \in \mathbb{C}$, as

$$\boxed{\xi = r \cdot \mathrm{dn}(s, \frac{1}{\sqrt{2}})\,, \quad \eta = r \cdot \ \frac{1}{\sqrt{2}}\,\mathrm{sn}(s, \frac{1}{\sqrt{2}})}\,.$$

The first identity is equivalent with $x+y=r^2$, and the second one implies $d = \sqrt{x-y} = r \cdot \mathrm{cn}(s,k)$.

In the following, the elliptic functions will only be used with the second parameter $k = 1/\sqrt{2}$; it will therefore be suppressed: $\mathrm{sn}(s), \mathrm{cn}(s), \mathrm{dn}(s)$.

Quarter period: $\mathsf{K} = \mathsf{K}' = \frac{\pi}{\mathrm{agm}(2,\sqrt{2})} = 1.85407\,46773\,01371\,91843\ldots$

4. Canonically Conjugated Momenta

Theory of canonical transformations, e.g. Siegel-Moser (Springer 1971), or, for the complex notation, e.g., JW 1999, p. 258-262 (Ref. on p. 18).

List of complex variables	Hamiltonian p.5	Regularized
Coordinates	$x = x_1,\ y = x_2$	$r = r_1,\ s = r_2$
Momenta	$p = p_1,\ q = p_2$	$u = u_1,\ v = u_2$

Given coordinate transformation:

$$x_1 = f_1(r_1, r_2), \quad x_2 = f_2(r_1, r_2), \quad f_1, f_2 \ \text{analytic in} \ r_1, r_2$$

Generating function:

$$W(p_1, \overline{p_1}, p_2, \overline{p_2}, r_1, \overline{r_1}, r_2, \overline{r_2}) =$$
$$p_1\, f_1(\overline{r_1}, \overline{r_2}) + p_2\, f_2(\overline{r_1}, \overline{r_2}) + \overline{p_1}\, f_1(r_1, r_2) + \overline{p_2}\, f_2(r_1, r_2)$$

Conjugated momenta, continued

From the generating function W the new momenta are obtained as

$$u_1 = \frac{\partial W}{\partial \,\overline{r_1}}\,, \quad u_2 = \frac{\partial W}{\partial \,\overline{r_2}}\,.$$

With $f_1(r,s) = r^2\,\mathrm{dn}^2(s),\; f_2(r,s) = \dfrac{r^2}{2}\,\mathrm{sn}^2(s)$ the generating function becomes

$$W = r^2\left(\bar p\;\mathrm{dn}^2(s) + \frac{\bar q}{2}\,\mathrm{sn}^2(s)\right) + \bar r^2\left(p\;\mathrm{dn}^2(\bar s) + \frac{q}{2}\,\mathrm{sn}^2(\bar s)\right).$$

Finally, the partial derivatives above, together with the differentiation rules of p. 7, yield the momenta u, v conjugated to r, s:

$$u = \bar r\left(2\,p\;\mathrm{dn}^2(\bar s) + q\;\mathrm{sn}^2(\bar s)\right), \quad v = \bar r^2\;\mathrm{sn}(\bar s)\;\mathrm{cn}(\bar s)\;\mathrm{dn}(\bar s)\,(q-p)\,.$$

5. Transformation of the Hamiltonian

Solving for p, q yields the following (written without denominators):

$$p\,\bar{r}^2\,\mathrm{sn}(\bar{s})\,\mathrm{cn}(\bar{s})\,\mathrm{dn}(\bar{s}) = \tfrac{u}{2}\,\bar{r}\,\mathrm{sn}(\bar{s})\,\mathrm{cn}(\bar{s})\,\mathrm{dn}(\bar{s}) - \tfrac{v}{2}\,\mathrm{sn}^2(\bar{s})$$
$$q\,\bar{r}^2\,\mathrm{sn}(\bar{s})\,\mathrm{cn}(\bar{s})\,\mathrm{dn}(\bar{s}) = \tfrac{u}{2}\,\bar{r}\,\mathrm{sn}(\bar{s})\,\mathrm{cn}(\bar{s})\,\mathrm{dn}(\bar{s}) + v\,\mathrm{dn}^2(\bar{s})\,.$$

By using this and the framed equations on p. 10, 11 the Hamiltonian H of p.5 becomes a function of $r, \bar{r}, s, \bar{s}, u, \bar{u}, v, \bar{v}$:

$$H = \frac{p\,\bar{p}}{m_1} + \frac{q\,\bar{q}}{m_2} - \frac{1}{r\,\bar{r}}\left(\frac{\frac{1}{2}\,m_1^2}{\mathrm{dn}(s)\,\mathrm{dn}(\bar{s})} + \frac{m_2^2}{\mathrm{sn}(s)\,\mathrm{sn}(\bar{s})} + \frac{2\,m_1\,m_2}{\mathrm{cn}(s)\,\mathrm{cn}(\bar{s})} + 2\,m_1\,m_2\right)\,.$$

It is natural to choose the common denominator, f, of H as the dilatation factor in the regularizing time transformation, see p.15.

6. The Time Transformation

We introduce a new independent variable, the fictitious time τ, by Sundman's technique, using the factor f in order to produce a regularized Hamiltonian K:

$$dt = f \cdot d\tau \qquad \Longrightarrow \qquad K\big(r, \bar{r}, s, \bar{s}, u, \bar{u}, v, \bar{v}\big) = f \cdot \big(H - H_0\big)\,,$$

where H_0 is the fixed value of H on the orbit. From p. 14 the common denominator is found to be

$$f = r^2 \,\mathrm{sn}(s)\,\mathrm{cn}(s)\,\mathrm{dn}(s) \cdot \bar{r}^2 \,\mathrm{sn}(\bar{s})\,\mathrm{cn}(\bar{s})\,\mathrm{dn}(\bar{s}) = \frac{2\,|x|\,|y|\,|x-y|}{|x+y|}\,,$$

where the last expression (the physical meaning of f) follows from p. 10 and 11.

7. Global Regularization

The regularized Hamiltonian:

$$K\big(r,\bar r,s,\bar s,u,\bar u,v,\bar v\big) =$$

$$r\,\bar r\,C_0(s)\,u\,\bar u + r\,C_1(s)\,\bar u\,v + \bar r\,C_1(\bar s)\,u\,\bar v + C_2(s)\,v\,\bar v + r\,\bar r\,C_3(s) + \big(r\,\bar r\big)^2\,C_4(s)\,,$$

where, by omitting the argument s for simplicity, $\mathrm{sn} = \mathrm{sn}(s)$, $\overline{\mathrm{sn}} = \mathrm{sn}(\bar s)$,

$$\begin{aligned}
C_0(s) &= \Big(\frac{1}{4\,m_1} + \frac{1}{4\,m_2}\Big)\,\mathrm{sn}\,\overline{\mathrm{sn}}\,\mathrm{cn}\,\overline{\mathrm{cn}}\,\mathrm{dn}\,\overline{\mathrm{dn}} \\
C_1(s) &= \overline{\mathrm{sn}}\,\overline{\mathrm{cn}}\,\overline{\mathrm{dn}}\,\Big(\frac{\mathrm{dn}^2}{2\,m_2} - \frac{\mathrm{sn}^2}{4\,m_1}\Big) \\
C_2(s) &= \frac{\mathrm{sn}^2\,\overline{sn}^2}{4\,m_1} + \frac{\mathrm{dn}^2\,\overline{dn}^2}{m_2} \\
C_3(s) &= -\frac{m_1^2}{2}\,\mathrm{sn}\,\overline{\mathrm{sn}}\,\mathrm{cn}\,\overline{\mathrm{cn}} - m_2^2\,\mathrm{cn}\,\overline{\mathrm{cn}}\,\mathrm{dn}\,\overline{\mathrm{dn}} - 2\,m_1\,m_2\,\mathrm{sn}\,\overline{\mathrm{sn}}\,\mathrm{dn}\,\overline{\mathrm{dn}}\,\big(1 + \mathrm{cn}\,\overline{\mathrm{cn}}\big) \\
C_4(s) &= -H_0\,\mathrm{sn}\,\overline{\mathrm{sn}}\,\mathrm{cn}\,\overline{\mathrm{cn}}\,\mathrm{dn}\,\overline{\mathrm{dn}}
\end{aligned}$$

8. Regularized Equations of Motion

in terms of complex variables:

$$\frac{dr}{d\tau} = \frac{\partial K}{\partial \bar{u}}\,, \quad \frac{ds}{d\tau} = \frac{\partial K}{\partial \bar{v}}\,, \quad \frac{du}{d\tau} = -\frac{\partial K}{\partial \bar{r}}\,, \quad \frac{dv}{d\tau} = -\frac{\partial K}{\partial \bar{s}}\,.$$

Outlook to numerical experiments.

- The regularized Hamiltonian K is a polynomial in r, u, v.
- For the partial derivative of K with respect to $\bar{s}$ see the differentiation rules on p. 7.
- The Jacobian elliptic functions (doubly periodic in the complex plane) have first-order poles at points in a quadratic grid with mesh 2 K containing the point $s = i\,$K (see p.11). These non-physical singularities may need special attention in practical implementations.

A few references

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5. J. Waldvogel, 1982, *Symmetric and regular coordinates on the plane triple collision manifold*. Celest. Mech. **28**, 69-82.

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Conclusions

- The "Caledonian" symmetric four-body problem in two dimensions, introduced by B.A. Steves and A.E. Roy in 1998, allows for four types of binary collisions (besides the simultaneous collision of all four bodies).

- A technique doubling the number of degrees of freedom (suggested by D. Heggie in 1974) was used by Sivasankaran, Steves, and Sweatman in 2010 for regularizing all binary collisions in the Caledonian problem.

- In this study the Jacobian elliptic functions $\mathrm{sn}(s,k),\ \mathrm{cn}(s,k),\ \mathrm{dn}(s,k)$ with a complex argument s and $k = 1/\sqrt{2}$ were used for achieving Levi-Civita-type regularization of all binary collisions.

- The regularized equations of motion, although somewhat long, have an elegant and transparent stucture.

- Owing to the existence of an efficient algorithm for evaluating sn, cn, and dn (arithmetic-geometric mean), the variables suggested here are a promising tool for numerical applications.

- Experiments with numerical integration upcoming!