Construction of the Virtual Fundamental Class and Applications

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Contents

1 Introduction

Virtual fundamental classes introduced by Li-Tian [8] and Behrend-Fantechi [3] are successful and crucial in the study of enumerative geometry, especially in Gromov-Witten theory, Donaldson-Thomas theory and Pandharipande-Thomas theory.

Roughly speaking, when the deformation-obstruction theory of the moduli space M is given by a two term complex $[E_0 \to E_1]$ of vector bundles, the virtual class $[M]^{\mathrm{vir}}$ can be constructed in the expected Chow group $A_{\mathrm{rk}(E_0)-\mathrm{rk}(E_1)}(M)$. The idea is to construct a cone of dimension $\mathrm{rk}(E_0)$ inside the vector bundle E_1 and take refined intersection with the zero section. We show that similar method can be applied to get the virtual fundamental class in algebraic cobordism (the universal oriented Borel-Moore homology theory constructed in [6] and [7]). Moreover, by the isomorphism between algebraic cobordism theory and twisted Chow theory (with $\mathbb{Q}$-coefficients), we prove that the cobordism virtual class is same as the Chow virtual class twisted by the universal inverse Todd class of virtual tangent bundle. As a consequence, Chern numbers can be computed in terms of the Chow virtual class and obstruction bundles.

In Section 2, we review some properties of cones and intersection theory. The definition of perfect obstruction theory is given in Section 3. In particular, we focus on the case when X is quasi-projective. Then we recall the construction of the virtual fundamental class in Chow theory and introduce a formula by Siebert [18] expressing the virtual fundamental class in terms of Fulton's canonical class in the fourth section.

The last three sections concern the virtual fundamental class in algebraic cobordism and its application to enumerative geometry. We briefly review some facts about algebraic cobordism in Section 5, and construct the cobordism virtual class in Section 6. Finally, as an application, we define the cobordism-valued Pandharipande-Thomas invariants which generalize the classical invariants of Calabi-Yau 3-folds. By applying the Grothendieck-Riemann-Roch formula, we show that our cobordism invariants can be represented by descendent invariants when the virtual dimension is positive. Hence the rationality of the cobordism partition function of a nonsingular toric 3-fold is obtained as a corollary.

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2 Preliminaries on cones

In this section, we review some basic properties of cones which play crucial roles in intersection theory, especially in the construction of virtual fundamental class. Fuller details can be found in [4].

2.1 Cones

We work on a scheme X. Let $S^\bullet = S^0 \oplus S^1 \oplus S^2 \oplus \cdots$ be a graded sheaf of $\mathcal{O}_X$-algebras, such that $\mathcal{O}_X$ is isomorphic to S^0 canonically, and $S^\bullet$ is locally generated by S^1 as an $\mathcal{O}_X$-algebra. Then the affine X-scheme $C = \mathrm{Spec}(S^\bullet)$ is called a cone over X. Naturally, there is a projection map $\pi : C \to X$. Also, we can define the projective cone of $S^\bullet$ as $P(C) = \mathrm{Proj}(S^\bullet)$ and there is a natural projection map $p : P(C) \to X$ which is proper.

Locally, when X is affine with coordinate ring A, then $S^\bullet$ is determined by a graded A-algebra. We denote it by the same notation $S^\bullet$. Assume that $x_0, x_1, \cdots, x_n$ are generators of S_1, then $S^\bullet$ can be represented by $A[x_0, \cdots, x_n]/I$ for a homogeneous ideal I. In this case C is the affine subscheme of $X \times \mathbb{A}^{n+1}$ and $P(C)$ is the subscheme of $X \times \mathbb{P}^n$. In general, the cones $C = \mathrm{Spec}(S^\bullet)$ and $P(C) = \mathrm{Proj}(S^\bullet)$ are constructed by gluing the local data.

Example The total space of a vector bundle E on X is the cone associated to the graded sheaf $\mathrm{Sym}(\mathcal{E}^\vee)$, where $\mathcal{E}$ is the sheaf of sections of E. The corresponding projective cone is just the total space of the projective bundle $P(E)$.

More generally, if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, we get an associate affine cone $C(\mathcal{F}) = \mathrm{Spec}(\mathrm{Sym}(\mathcal{F}))$. For any X-scheme Z we have $C(\mathcal{F})(Z) = \mathrm{Hom}(\mathcal{F}_Z, \mathcal{O}_Z)$, hence $C(\mathcal{F})$ is a group scheme over X. An affine cone of this form is called an **abelian cone**.

From now on, we mainly focus on affine cones. The only place we use projective cones is to interpret the virtual class in terms of Fulton's canonical class.

A morphism of cones over X is an X-morphism induced by a graded sheaves of $\mathcal{O}_X$-algebras. A closed subcone is the image of a closed immersion of cones. If $S^\bullet \to T^\bullet$ is a surjective graded homomorphism of graded sheaves of $\mathcal{O}_X$-algebras, then there is an imbedding $\mathrm{Spec}(T^\bullet) \hookrightarrow \mathrm{Spec}\ (S^\bullet)$.

Hence , we can define the zero section imbedding of X in $\mathrm{Spec}(S^\bullet)$ determined by the augmentation homomorphism from $S^\bullet$ to $\mathcal{O}_X$, which vanishes on S^i for $i > 0$ and is the canonical isomorphism between S^0 with $\mathcal{O}_X$.

If

$$\begin{array}{ccc} & & C_2 \\ & & \downarrow \\ C_1 & \longrightarrow & C_3 \end{array}$$

is a diagram of cones over X, the fibered product $C_1 \times_{C_3} C_2$ is a cone over X.

For any cone C defined by $S^\bullet$ over X, there is a canonical way to get an abelian cone $A(C)$ such that C is a closed sub cone of $A(C)$. Actually, define $A(C)$ to be the cone associated to $\mathrm{Sym}(S^1)$. Since there is a surjective map $\mathrm{Sym}(S^1) \to S^\bullet$, we get the closed imbedding $C \hookrightarrow A(C)$. Notice that if S^1 is locally free, then the abelian cone we obtain is a vector bundle.

2.2 Normal cones and normal sheaves

Now let $X \to Y$ be a closed imbedding of schemes with ideal sheaf I. Then

$$\bigoplus_{n\geq 0} I^n/I^{n+1}$$

is a sheaf of $\mathcal{O}_X$-algebras. The corresponding cone is denoted by $C_{X/Y}$, which is called the **normal cone** of X in Y. We call the corresponding abelian cone $\mathrm{Spec}(\mathrm{Sym}(I/I^2))$ the **normal sheaf** of X in Y, which is denoted by $N_{X/Y}$. The following lemma shows that the normal cone always has the same dimension as the ambient space.

Lemma 2.1 *With the notation as above, if Y has pure dimension k, then $C_{X/Y}$ also has pure dimension k.*

Proof Consider the imbedding $X \hookrightarrow Y \times \mathbb{A}^1$ which is the composition of the imbedding of X in Y and the imbedding of Y in $Y \times \mathbb{A}^1$ at $0 \in \mathbb{A}^1$. The space $\mathrm{Bl}_X(Y \times \mathbb{A}^1)$ is of pure dimension k. Since the exceptional divisor $P(C_{X/Y} \oplus 1)$ is a Cartier divisor on $\mathrm{Bl}_X(Y \times \mathbb{A}^1)$, it must have pure dimension k, and C is an open subscheme of $P(C_{X/Y} \oplus 1)$.

We recall the following definition from [4] Example 4.1.6. See also [3] Definition 1.2.

Definition 2.2 *A sequence of cone morphisms*

$$0 \longrightarrow E \longrightarrow C \longrightarrow D \longrightarrow 0$$

is exact if E is a vector bundle and locally over X there is a morphism of cones $C \to E$ splitting the first arrow and inducing an isomorphism $C \to E \times D$.

For a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow 0$$

of coherent sheaves on X with $\mathcal{E}$ locally free, then

$$0 \longrightarrow C(\mathcal{E}) \longrightarrow C(\mathcal{G}) \longrightarrow C(\mathcal{F}) \longrightarrow 0$$

is exact, and conversely.

Consider a commutative diagram of schemes

$$\begin{array}{ccc} X & \overset{i'}{\rightarrow} & Y' \\ & \searrow i & \downarrow \pi \\ & & Y \end{array} \tag{1}$$

where i and i' are immersions and π is smooth. Then there is an exact sequence of cones over X

$$0 \longrightarrow i'^{*} T_{Y'/Y} \longrightarrow C_{X/Y'} \longrightarrow C_{X/Y} \longrightarrow 0 \tag{2}$$

As a consequence, we have the following lemma.

Lemma 2.3 *([3], Lemma 3.2) Let*

$$f : X \longrightarrow Y$$

be a closed immersion of schemes where Y is smooth, then the pull-back of the tangent bundle $f^{*}T_Y$ acts on $N_{X/Y}$, and the normal cone $C_{X/Y} \hookrightarrow N_{X/Y}$ is invariant under the action of $f^{*}T_Y$.

2.3 Deformation to the normal cone

If X is a closed sub-scheme of Y, we have the normal cone $C_{X/Y}$ over X. Although it is impossible to construct an actual morphism from Y to $C_{X/Y}$, there is a morphism of Chow groups

$$\sigma : A_*(Y) \to A_*(C_{X/Y}) \tag{3}$$

The idea is to construct a family of imbeddings $X \hookrightarrow Y_t$ parametrized by $t \in \mathbb{P}^1$, such that for $t \neq 0$, the imbedding is the given imbedding of X in Y, and for $t = 0$ one has the zero section imbedding of X in $C_{X/Y}$.

Note that everything in this subsection works if Chow theory is replaced by algebraic K-theory or algebraic cobordism theory. The proofs are exactly the same. More details can be found in [4] Chapter 5 for Chow theory and [6] for algebraic cobordism theory. We briefly describe the construction below.

Consider the imbedding of X in $Y \times \mathbb{P}^1$ which is the composition of $X \hookrightarrow Y$ and the imbedding of Y in $Y \times \mathbb{P}^1$ at $0 \in \mathbb{P}^1$. Define

$$M := \operatorname{Bl}_{X\times\{0\}}(Y \times \mathbb{P}^1).$$

There is a projection

$$\pi : M \to \mathbb{P}^1$$

such that

1. $\pi^{-1}(t) = Y$ when $t \neq 0$;
2. $\pi^{-1}(0) = \operatorname{Bl}_X(Y) \cup P(C_{X/Y} \oplus 1)$.

Now we denote by M° the open subset of M obtained by removing the component $\operatorname{Bl}_X(Y)$ from the fiber over 0. Therefore the fiber over 0 in M° is $C_{X/Y}$.

Then we construct the morphism (3). Consider the diagram:

$$\begin{array}{ccccccc} A_{k+1}(C_{X/Y}) & \xrightarrow{j_*} & A_{k+1}(M^\circ) & \longrightarrow & A_{k+1}(Y \times \mathbb{A}^1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j^*} & & \uparrow{\scriptstyle \mathrm{pr}^*} & & \\ & & A_k(C_{X/Y}) & \longleftarrow & A_k(Y) & & \end{array} \tag{4}$$

The top row is the localization exact sequence. Since the divisor $C_{X/Y}$ moves in M°, the composition map $j^* j_*$ is zero. So we get the following map

$$\sigma : A_k(Y) \cong A_{k+1}(Y \times \mathbb{A}^1) \to A_k(C_{X/Y}).$$

In diagram (1), the flat surjection $p : C_{X/Y'} \to C_{X/Y}$ induces a morphism

$$p^* : A_*(C_{X/Y}) \to A_{*+d}(C_{X/Y'})$$

where $d = \mathrm{rk}(T_{Y'/Y})$. Together with the map (3), we obtain a diagram:

$$\begin{array}{ccc} A_*(Y) & \xrightarrow{\sigma_Y} & A_*(C_{X/Y}) \\ {\scriptstyle \pi^*}\Big\downarrow & & {\scriptstyle p^*}\Big\downarrow \\ A_{*+d}(Y') & \xrightarrow{\sigma_{Y'}} & A_{*+d}(C_{X/Y'}) \end{array} \tag{5}$$

Lemma 2.4 *The diagram (5) commutes.*

Proof Suppose M_Y° (resp. $M_{Y'}^\circ$) is the total space of the deformation to the normal cone constructed by the imbedding i (resp. i'). The projection $\pi : Y' \to Y$ extends to a smooth morphism $\pi_M : M_{Y'}^\circ \to M_Y^\circ$ inducing the natural map $C_{X/Y} \to C_{X/Y'}$. Take an element

$$\xi \in A_*(Y) \cong A_{*+1}(Y \times \mathbb{A}^1),$$

and let $\xi' \in A_{*+1}(M_Y^\circ)$ be a lifting of ξ (from (4)). By the Cartesian diagram

$$\begin{array}{ccc} Y' \times \mathbb{A}^1 & \longrightarrow & M_Y^\circ \\ {\scriptstyle \pi \times id}\Big\downarrow & & {\scriptstyle \pi_M}\Big\downarrow \\ Y \times \mathbb{A}^1 & \longrightarrow & M_Y^\circ \end{array}$$

the class $\pi_M^* \xi'$ is a lifting of $\pi^* \xi$. Hence

$$\begin{array}{rcl} p^* \sigma_Y(\xi) & = & p^*(\xi'|_{C_{X/Y}}) \\ & = & \pi_M^* \xi'|_{C_{X/Y'}} \\ & = & \sigma_{Y'}(\pi^* \xi) \end{array}$$

where $\xi'|_{C_{X/Y}}$ means the pull back of ξ' by the imbedding $C_{X/Y'} \hookrightarrow M_{Y'}^\circ$.

As an application of the deformation to the normal cone, we introduce refined Gysin homomorphisms.

Let $i : X \to Y$ be a regular imbedding of codimension d, and let $f : Y' \to Y$ be a morphism. From the Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle g}\Big\uparrow & & {\scriptstyle f}\Big\uparrow \\ X' & \xrightarrow{j} & Y' \end{array} \tag{6}$$

we can construct the following morphism

$$i^! : A_*(Y') \to A_{*-d}(X'). \tag{7}$$

Note that the codimension of X' in Y' may not be equal to d, however, we can also pull back a cycle from Y' to X' which has the expected dimension.

By the diagram (6), the normal cone $C_{X'/Y'}$ can be imbedded into a rank d vector bundle which is $g^*N_{X/Y}$. Hence (7) is constructed by:

$$A_*(Y') \to A_*(C_{X'/Y'}) \to A_*(g^*N_{X/Y}) \to A_{*-d}(X')$$

where the first map is just (3).

Remark 2.5 *The Gysin homomorphism constructed above can be extended to all l.c.i morphisms. See [4] Chapter 6.6.*

2.4 Segre class and Fulton's Canonical class

Let X be a scheme and $C = \mathrm{Spec}(S^\bullet)$ be a cone over X. The associated projective cone is $P(C)$ with the projection $p : P(C) \to X$ and $\xi = c_1(\mathcal{O}_{P(C)}(1))$. The Chow class

$$S(C) := \sum_{r \geq 0} p_*(\xi^r \cap [P(C)]),$$

is called the **Segre class** of C.

Proposition 2.6 *(1) If E is a vector bundle on X, then the segre class is the inverse of the total Chern class, i.e.*

$$s(E) = c(E)^{-1} \cap [X].$$

(2) For an exact sequence of cones

$$0 \longrightarrow E \longrightarrow C \longrightarrow D \longrightarrow 0$$

where E is a vector bundle, we have

$$s(C) = c(E) \cap s(D)$$

(3) If C is a cone on X and E is a vector bundle, then

$$s(C \times_X E) = c(E)^{-1} \cap s(C)$$

Now we turn to the definition of Fulton's canonical class. For a quasi-projective scheme X, which can be globally imbedded into a non-singular scheme Y. The following Chow class

$$c_F(X) := c(T_Y|_X) \cap s(C_{X/Y})$$

is called **Fulton's canonical class**.

Proposition 2.7 *The class $c_F(X)$ is independent of the choice of imbedding.*

Proof Two imbeddings are dominated by the diagonal, so it suffices to show the independence in the case of diagram (1). By the exact sequence (2), we have

$$c(T_{Y'/Y}|_X)^{-1} \cap s(C_{X/Y}) = s(C_{X/Y'}).$$

Hence $s(C_{X/Y'}) \cap c(T_Y|_X) = s(C_{X/Y'}) \cap c(T_{Y'}|_X)$.

3 Perfect obstruction theory

We work on a quasi-projective scheme X and assume it is globally imbedded into a nonsingular scheme Y with the corresponding ideal sheaf I. A perfect obstruction theory on X consists of the following data:

(1) A two term complex of locally free sheaves $E^\bullet = [E^{-1} \to E^0]$ on X.

(2) A morphism $\phi : E^\bullet \to L_X^\bullet$ in the derived category of coherent sheaves to the cotangent complex $L_X^\bullet$ satisfying that ϕ induces an isomorphism in cohomology in degree 0 and a surjection in cohomology in degree -1.

Although the cotangent complex seems mysterious and abstract, it is clear that only the information of the truncated complex $L_X^{\bullet \geq -1} (= [L^{-1}/\mathrm{Im}(L^{-2}) \to L^0 \to \cdots])$ is used in the data of a perfect obstruction theory. Hence we can use the explicit representative

$$L_X^{\bullet \geq -1} = [I/I^2 \to \Omega_Y|_X].$$

A morphism $\phi^\bullet : E^\bullet \to [I/I^2 \to \Omega_Y|_X]$ in derived category means there exists a two term complex $[G^{-1} \to G^0]$ of coherent sheaves such that $G^\bullet \xrightarrow{\varphi_1} E^\bullet$ and $G^\bullet \xrightarrow{\varphi_2} [I/I^2 \to \Omega_Y|_X]$ are both morphisms of complexes with φ_1 being a quasi-isomorphism and φ_2 inducing an isomorphism in h^0 and epimorphism in h^{-1}. Since X is quasi-projective, any coherent sheaf on X can be written as the quotient of a locally free sheaf. It implies that we may choose both G^0 and G^{-1} to be locally free. Hence we can use the following data as a perfect obstruction theory on a quasi-projective scheme X:

$$\begin{array}{ccc} E^{-1} & \longrightarrow & E^0 \\ \downarrow & & \downarrow \\ I/I^2 & \longrightarrow & \Omega_Y|_X \end{array} \tag{8}$$

where $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is an epimorphism.

Equivalently, such data can also be written in the language of abelian cones (or linear spaces in the sense of [18])

$$\Phi_\bullet : [T_Y|_X \to N_{X/Y}] \to [E_0 \to E_1]$$

where $E_i = (E^{-i})^\vee (i = 0, 1)$, and inducing an isomorphism on H^0 and a closed imbedding on H^1.

Lemma 3.1 *From (8) we get a short exact sequence of abelian cones*

$$0 \to TY|_X \to E_0 \times_X N_{X/Y} \to C(Q) \to 0 \tag{9}$$

where $C(Q)$ is a closed sub-abelian cone of E_1.

Proof Consider the cone of the map of complexes

$$\begin{array}{ccc} [E^{-1} & \longrightarrow & E^0] \\ {\scriptstyle \phi^{-1}}\downarrow & & {\scriptstyle \phi^0}\downarrow \\ [I/I^2 & \longrightarrow & \Omega_Y|_X] \end{array}$$

which can be represented by the complex

$$[E^{-1} \to E^{-1} \oplus I/I^2 \to \Omega_Y|_X].$$

The cohomology condition implies the right exactness

$$E^{-1} \to E^0 \oplus I/I^2 \xrightarrow{\gamma} \Omega_Y|_X \to 0.$$

Then Q is just the kernel of γ.

4 Construction of the virtual fundamental class in Chow theory

In this section, we review the construction of the virtual fundamental class in Chow theory. We work on a quasi-projective scheme X which admits a global imbedding into a nonsingular scheme Y with a perfect obstruction theory on X. The purpose is to construct a class in the expected Chow group $A_{\mathrm{rk}(E^0)-\mathrm{rk}(E^{-1})}(X)$ by the geometry of diagram (8). The main idea is to construct a cone of dimension $\mathrm{rk}(E_0)$ inside the vector bundle E_1. Finally, we introduce an interesting formula by Siebert [18] that the virtual fundamental class is obtained by cutting Fulton's canonical class by the Chern classes of the obstruction bundle. It should be mentioned that everything in this section works for Deligne-Mumford stacks.

4.1 Virtual fundamental class

By the exact sequence (9) and Lemma 2.3, the cone $C_{X/Y} \times_X E_0$ is a $T_Y|_X$-invariant subcone of the abelian cone $N_{X/Y} \times_X E_0$. Since (9) splits locally, $T_Y|_X$ acts on $C_{X/Y} \times_X E_0$ freely and fiberwise. Hence the quotient cone

$$D^{\mathrm{vir}} := \frac{C_{X/Y} \times_X E_0}{T_Y|_X}$$

is a closed subscheme of the vector bundle E_1 and by Lemma 2.1, we have $\dim(D^{\mathrm{vir}}) = \mathrm{rk}(E_0)$. Then we take the refined intersection of D^{vir} with the zero section of the vector bundle E_1

$$0_{E_1} : X \to E_1.$$

Definition 4.1 $[X]^{vir} := 0^!_{E_1}[D^{vir}] \in A_{rk(E_0)-rk(E_1)}(X)$ *is called the virtual fundamental class of* X *with respect to the perfect obstruction theory* $\phi : E^\bullet \to L^\bullet_X$.

Alternatively, we give another equivalent description of the virtual class. Consider the following Cartesian diagrams

$$\begin{array}{ccc} T_Y|_X & \longrightarrow & C_{X/Y} \times_X E_0 \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & D^{vir} \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & E_1. \end{array}$$

Then we have

$$[X]^{\mathrm{vir}} = 0^!_{T_Y} 0^!_{E_1} [C_{X/Y} \times E_0]. \tag{10}$$

Lemma 4.2 *The virtual class $[X]^{vir}$ is independent of the choice of imbedding and representative of obstruction bundles.*

Proof We use the language of cones. Assume there are two imbeddings $X \hookrightarrow Y$ and $X \hookrightarrow Z$, and two representatives of a perfect obstruction theory: $[T_Y|_X \to N_{X/Y}] \to [E_0 \to E_1]$ and $[T_Z|_X \to N_{X/Z}] \to [F_0 \to F_1]$. Then we can choose the diagonal imbedding $X \hookrightarrow Y' := Y \times Z$. The complex $[T_{Y'}|_X \to N_{X/Y'}]$ dominates both $[T_Y|_X \to N_{X/Y}]$ and $[T_Z|_X \to N_{X/Z}]$. Also, there is a two term vector bundles $[G_0 \to G_1]$ such that $E_\bullet \to G_\bullet$ and $F_\bullet \to G_\bullet$ are both quasi-isomorphisms. We only need to show that the virtual classes obtained by $[T_Y|_X \to N_{X/Y}] \to E_\bullet$ and $[T_{Y'}|_X \to N_{X/Y'}] \to G_\bullet$ are the same. The proof is divided into two steps:

First, we prove that $[T_Y|_X \to N_{X/Y}] \to E_\bullet$ and $[T_{Y'}|_X \to N_{X/Y'}] \to E_\bullet$ yield the same virtual class.

Under the case of diagram (1), we have the Cartesian diagrams:

$$\begin{array}{ccc} T_{Y'}|_X & \longrightarrow & C_{X/Y'} \times_X E_0 \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ T_Y|_X & \longrightarrow & C_{X/Y} \times_X E_0 \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & E_1 \end{array}$$

According to formula (10), it suffices to prove that

$$0^!_{E_1}[C_{X/Y'} \times_X E_0] = f^! 0^!_{E_1}[C_{X/Y} \times_X E_0].$$

This is a consequence of Theorem 6.2,(b) in [4] and the fact that

$$g^![C_{X/Y} \times_X E_0] = [C_{X/Y'} \times_X E_0]. \tag{11}$$

Second, we show that $[T_{Y'}|_X \to N_{X/Y'}] \to E_\bullet$ and $[T_{Y'}|_X \to N_{X/Y'}] \to G_\bullet$ yield the same virtual class.

Consider the following Cartesian diagram:

$$\begin{array}{ccccc} T_{Y'}|_X & \longrightarrow & C_{X/Y'} \times_X E_0 & \xrightarrow{j'} & C_{X/Y'} \times_X G_0 \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{0_{E_1}} & E_1 & \xrightarrow{j} & G_1. \end{array}$$

Both 0_{E_1} and j are l.c.i morphisms, therefore $(j \circ 0_{E_1})^! = 0^!_{E_1} j^!$. Hence it only suffices to prove that

$$j^![C_{X/Y} \times_X G_0] = [C_{X/Y'} \times_X E_0]. \tag{12}$$

Since j' is also l.c.i and $E_\bullet \to G_\bullet$ is quasi-isomorphism, we know that $j^! = j'^!$ ([4] Theorem 6.2(c) and Proposition 6.6(c)). Therefore

$$j^![C_{X/Y'} \times G_0] = j'^![C_{X/Y'} \times G_0] = [C_{X/Y'} \times E_0].$$

4.2 Fulton's canonical class and virtual class

During the construction of the virtual fundamental class, we obtained the following short exact sequence:

$$0 \to T_Y|_X \to C_{X/Y} \times_X E_0 \to D^{\mathrm{vir}} \to 0$$

By Proposition 2.6

$$c(E_0)^{-1} \cap s(C_{X/Y}) = c(T_Y|_X)^{-1} \cap s(D^{\mathrm{vir}}) \tag{13}$$

Since D^{vir} is a closed subcone of the vector bundle E_1 and $\dim(D^{\mathrm{vir}}) = \mathrm{rk}(E_0)$, the intersection of D^{vir} with the zero section of E_1 can be written in terms of the Chern classes of E_1 and Segre class of D^{vir} by the formula:

$$0^!_{E_1}[D^{\mathrm{vir}}] = \{c(E_1) \cap s(D^{\mathrm{vir}})\}_{\mathrm{rk}(E_0)-\mathrm{rk}(E_1)}$$

(See [4] Example 4.18.)

Therefore, by (13)

$$
\begin{aligned}
[X]^{\mathrm{vir}} &= 0^!_{E_1}[D^{\mathrm{vir}}] \\
&= \{c([E_1]-[E_0]) \cap (c(T_Y|_X) \cap s(C_{X/Y}))\}_{\mathrm{rk}(E_0)-\mathrm{rk}(E_1)} \\
&= \{c(E_\bullet)^{-1} \cap C_F(X)\}_{\mathrm{rk}(E_0)-\mathrm{rk}(E_1)}.
\end{aligned}
$$

We obtain the following theorem by Siebert:

Theorem 4.3 *([18] Theorem 4.6) Let X be a quasi-projective k-scheme with perfect obstruction theory $\phi_\bullet : E^\bullet \to L^\bullet_X$, then*

$$[X]^{vir} = \{c(E_\bullet)^{-1} \cap c_F(X)\}_{rk(E_0)-rk(E_1)}$$

Remark 4.4 *An immediate consequence of Theorem 4.3 is that the virtual fundamental class only depends on the K-theory class $[E_0]-[E_1]$, which also proves that the virtual class in Chow does not rely on the explicit choice of a representative of the obstruction theory or imbedding.*

5 Algebraic cobordism theory

Algebraic cobordism theory was first constructed in [6] as the universal Borel-Moore homology theory of schemes. Its geometric presentation in characteristic 0 is given in [7] via double point relations. In this section, we briefly review some properties of this theory. We work over a field k of characteristic 0. Let $\mathbf{Sch}_k$ be the category of separated schemes of finite type over k, and let $\mathbf{Sch'}_k$ be the subcategory of $\mathbf{Sch}_k$ with the same objects, but morphisms given by the projective morphisms in $\mathbf{Sch}_k$. Let $\mathbf{Ab}_*$ denote the category of graded abelian groups.

5.1 Oriented Borel-Moore homology and algebraic cobordism

First we recall the definition of an oriented Borel-Moore homology theory.

Definition 5.1 *An oriented Borel-Moore homology theory consists of the following data:*

(D1) An additive functor

$$H_* : \mathbf{Sch}'_k \to \mathbf{Ab}_*.$$

(D2) For each l.c.i morphism $f : Y \to X$ in $\mathbf{Sch}_k$ of relative dimension d, a homomorphism of graded groups

$$f^* : H_*(X) \to H_{*+d}(Y).$$

(D3) An element $\mathbf{1} \in H_{\mathbf{0}}(Spec(\mathbf{k}))$ and, for each pair (X, Y) of objects in $\mathbf{Sch}_k$, a bilinear graded pairing:

$$\begin{array}{rcl} H_*(X) \otimes H_*(Y) & \to & H_*(X \times Y) \\ u \times v & \mapsto & u \otimes v, \end{array}$$

called the external product, which is commutative, associative, and admits 1 as unit element.

These satisfying

(BM1) One has $Id_X^* = Id_{H_*(X)}$ for any $X \in \mathbf{Sch}_k$. Moreover, given composable l.c.i morphisms $f : Y \to X$ and $g : Z \to Y$ of pure relative dimension, one has $(f \circ g)^* = g^* \circ f^*$.

(BM2) *Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathbf{Sch}_k$. Suppose that f and g are transverse, i.e. $Tor_q^{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{O}_Y) = 0$ for all $q > 0$, that f is projective and g is an l.c.i morphism, giving the Cartesian diagram*

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \xrightarrow{g} & Z. \end{array}$$

Then $g^* f_* = f'_* g'^*$.

(BM3) *Let $f : X' \to X$ and $g : Y' \to Y$ be morphisms in $\mathbf{Sch}'_k$. Then for $u' \in H_*(X')$ and $v' \in H_*(Y')$, one has*

$$(f \times g)_*(u' \times v') = f_*(u') \times g_*(v').$$

If f and g are l.c.i morphisms, then for $u \in H_*(X)$ and $v \in H_*(Y)$ one has

$$(f \times g)^*(u \times v) = f^*(u) \times g^*(v).$$

(PB) *For $L \to Y$ a line bundle on $Y \in \mathbf{Sch}_k$ with zero section $0 : Y \to L$, define the operator*

$$c_1(L) : H_*(Y) \to H_{*-1}(Y)$$

by $c_1(L)(\eta) = s^*(s_*(\eta))$. Let $\mathcal{E}$ be a rank $n+1$ locally free coherent sheaf on $X \in \mathbf{Sch}_k$, with projective bundle $q : P(\mathcal{E}) \to X$. For $i = 0, 1, \dots, n$, let

$$\xi^{(i)} : H_{*+i-n}(X) \to H_*(P(\mathcal{E}))$$

be the composition of $q^* : H_{*+i-n}(X) \to H_{*+i}(P(\mathcal{E}))$ with $c_1(O(1))_{\mathcal{E}} : H_{*+i}(P(\mathcal{E})) \to H_*(P(E))$. Then the homomorphism

$$\sum_{i=0}^{n} \xi^{(i)} : \oplus_{i=0}^{n} H_{*+i-n}(X) \to H_*(P(\mathcal{E}))$$

is an isomorphism.

(EH) *Let $E \to X$ be a vector bundle of rank r over $X \in \mathbf{Sch}_k$, and let $p : V \to X$ be an E-torsor. Then $p^* : H_*(X) \to H_{*+r}(V)$ is an isomorphism.*

(CD) *For integers $r, N > 0$, let $W = \mathbb{P}^N \times \cdots \times \mathbb{P}^N$ (r factors), and let $p_i : W \to \mathbb{P}^N$ be the i-th projection. Let $X_0, \ldots, X_N$ be the standard homogeneous coordinates on $\mathbb{P}^N$, let $n_1, \ldots, n_r$ be non-negative integers, and let $i : E \to W$ be the subscheme defined by $\prod_{i=1}^r p_i^*(X_N)^{n_i} = 0$. Then $i_* : H_*(E) \to H_*(W)$ is injective.*

It is shown in [6] that there exists a universal oriented Borel-Moore homology theory which is called **algebraic cobordism**, denoted by the functor

$$\Omega_* : \mathbf{Sch}'_k \to \mathbf{Ab}_*.$$

For a k-scheme X, the cobordism group $\Omega_n(X)$ has a concrete description. It is generated by $[f : Y \to X]$ where Y is smooth and f is projective, quotient by double point relations. (See [7] for the precise definition.)

Remark 5.2 *1. When $k = \mathbb{C}$, singular homology is an example of Borel-Moore homology theory. Also, Chow theory (the Chow groups functor $X \mapsto A_*(X)$ with projective push-forwards, l.c.i pull backs and their properties) is a very important example of Borel-Moore homology theory.*

2. The functor Ω_* yields the universal oriented Borel-Moore cohomology theory for the subcategory of smooth quasi-projective schemes. In particular, for a smooth quasi-projective scheme X, the theory $\Omega_*(X)$ has the structure of a graded ring.

5.2 Todd classes and twisting an Oriented Borel-Moore theory

More details of this and next subsection can be found in 4.1 and 7.4 in [6].

Let H_* be an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ and $\tau = (\tau_i) \in \prod_{i=0}^{\infty} H_i(k)$, with $\tau_0 = 1$. Define the inverse Todd class operator of a line bundle $L \to X$ to be the operator on $H_*(X)$ given by

$$\mathrm{Td}_\tau^{-1}(L) = \sum_{i=0}^{\infty} c_1(L)^i \tau_i.$$

Note that by the definition of the first Chern class operator, there are only finitely many terms in this power series. We can extend such operator to all vector bundles, i.e. there is a homomorphism

$$\mathrm{Td}_\tau^{-1} : K^0(X) \to \mathrm{Aut}(H_*(X)).$$

Now we can twist an oriented Borel-Moore homology theory H_* by the inverse Todd class operator to get a new theory $H_*^{(\tau)}$. The construction is the following:

1. $H_*^{(\tau)} := H_*(X)$ for any $X \in \mathbf{Sch}_k$.

2. $f_*^{(\tau)} = f_*$.

3. For any l.c.i morphism $f : X \to Y$, choose a factorization of f as $f = qi$ with $i : Y \to P$ a regular imbedding and $q : P \to X$ a smooth morphism. The virtual normal bundle N_f is defined to be the K-theory class $[N_i] - [i^* T_q]$ which is independent the choice of the factorization. Then
$$f^*_{(\tau)} = \mathrm{Td}_\tau^{-1}([N_f]) \circ f^*.$$

The new theory $H_*^{(\tau)}$ is still a Borel-Moore homology theory.

Remark 5.3 *It should be mentioned that twisting a theory by some inverse Todd class operator actually changes the formal group law of that theory.*

5.3 Chow theory and algebraic cobordism theory

Let $\mathbb{Z}[\mathbf{t}] := \mathbb{Z}[t_1, t_2, \ldots, t_n, \ldots]$ be the graded ring of polynomials with integral coefficients on variables $t_i (i > 0)$ of degree i. Starting with Chow theory A_*, we consider the oriented Borel-Moore homology theory $X \mapsto A_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{t}]$ by extension of scalars. We take for the family $\tau = (\tau_i)$ given by $\tau_i = t_i$. Then by the construction above, we have an inverse Todd class operator $\mathrm{Td}_{\mathbf{t}}^{-1}$ such that for any line bundle L over X

$$\mathrm{Td}_{\mathbf{t}}^{-1}(L) = \sum_{i=0}^{\infty} c_1(L)^i t_i : A_*(X)[\mathbf{t}] \to A_*(X)[\mathbf{t}],$$

and an oriented Borel-Moore homology theory $A_*[\mathbf{t}]^{(\mathbf{t})}$.

Let E be any vector bundle over X, then $\mathrm{Td}_{\mathbf{t}}^{-1}(E)$ can be expanded as

$$\mathrm{Td}_{\mathbf{t}}^{-1}(E) = \sum_{I=(n_1,\cdots,n_d,\cdots)} c_I(E) t_1^{n_1} \ldots t_d^{n_d} \ldots.$$

The $c_I(E)$ are the **Conner-Floyed Chern class operators**. By the splitting principle, it can be checked easily that the free abelian group generated by $c_I(E)$ $(|I| = \Sigma k n_k = m)$ is same as the abelian group generated by $\prod_{I=(n_1,\ldots,n_d,\ldots)} c_i(E)^{n_i}$ $(|I| = m)$. The following theorem shows the relation between Chow theory and algebraic cobordism.

Theorem 5.4 *([6]) There is a canonical morphism*

$$\vartheta : \Omega_* \to A_*[\mathbf{t}]^{(\mathbf{t})}$$

which induces an isomorphism after $\otimes\mathbb{Q}$*.*

5.4 Cobordism classes of a point

The cobordism group $\Omega_*(\mathrm{pt}) := \Omega_*(\mathrm{Spec}(k))$ has the structure of a graded ring. Consider the morphism in Theorem 5.4

$$\vartheta : \Omega_*(\mathrm{pt}) \to \mathbb{Z}[\mathbf{t}].$$

It satisfies:

1. ϑ is injective. The ring $\Omega_*(\mathrm{pt})$ is isomorphic to the Lazard ring $\mathbb{L}_*$ which is a subring of $\mathbb{Z}[\mathbf{t}]$.
2. $\Omega_*(\mathrm{pt})_{\mathbb{Q}} \simeq \mathbb{Q}[\mathbf{t}] := \mathbb{Z}[\mathbf{t}] \otimes_{\mathbb{Z}} \mathbb{Q}$.

For convenience, we identify $\Omega_*(\mathrm{pt})$ with the subring of $\mathbb{Z}[\mathbf{t}]$. In particular, every class in $\Omega_*(\mathrm{pt})$ can be written in terms of homogeneous polynomials on variables $t_1, \ldots, t_d, \ldots$. Also, according to Theorem 5.4, we identify the theory $\Omega_* \otimes \mathbb{Q}$ with $A_*[\mathbf{t}]_{\mathbb{Q}}^{(\mathbf{t})}$.

Lemma 5.5 *Let X be a smooth quasi-projective k-scheme, then*

$$\mathbf{1}_X := [X \xrightarrow{id} X] = Td_{\mathbf{t}}^{-1}(-[T_X])[X].$$

where $[X]$ is the fundamental class of X in Chow. Here $\mathbf{1}_X$ is called the **fundamental class** *of X in cobordism.*

Proof Consider the projection $p_X : X \to \mathrm{pt}$. We have

$$\mathbf{1}_X = p_X^* \mathbf{1}_{\mathrm{pt}} = \mathrm{Td}_{\mathbf{t}}^{-1}(-[T_X])(p_X^*[\mathrm{pt}]) = \mathrm{Td}_{\mathbf{t}}^{-1}(-[T_X])[X].$$

Remark 5.6 *We see that if the projection $X \to pt$ is an l.c.i morphism, then X has the fundamental class which is just the pull back of the fundamental class of a point.*

Corollary 5.7 *Let X be a quasi-projective scheme. The following formula holds*

$$[X \to pt] = \sum_{|I|=dim(X)} \Big(\int_{[X]} c_I(-T_X)\Big)\mathbf{t}^I \tag{14}$$

where $I = (n_1, \dots, n_d, \dots)$ and $\mathbf{t}^I = t_1^{n_1} \dots t_d^{n_d} \dots$.

Proof Push forward the fundamental class of X to one point, and the formula is obtained immediately.

Define $c^I(E) := \prod_i c_i(E)^{n_i}$, $I = (n_1, \dots, n_d, \dots)$. Then the right hand side of (14) can be written as

$$\sum_{|I|=\mathrm{dim}(X)} \Big(\int_{[X]} c^I(T_X)\Big) P_I, \tag{15}$$

where $P_I \in \mathbb{Z}[\mathbf{t}]$ are homogeneous polynomials of degree $|I|$ forming a basis of degree $|I|$ part of $\mathbb{Z}[\mathbf{t}]$. We call $\int_{[X]} c^I(T_X)$ the **Chern numbers** of X. It is easy to see that $\int_{[X]} c^I(T_X) = 0$ if $|I| \neq \mathrm{dim}X$ for dimension reasons. Therefore the cobordism class $[X \to \mathrm{pt}]$ is just the computation of all the Chern numbers of X.

6 Constuction of the virtual fundamental class in algebraic cobordism theory

From now on, we work on schemes over the complex number $\mathbb{C}$. We show that similar methods as in Chow can be applied to construct the virtual fundamental class in algebraic cobordism theory.

6.1 Virtual fundamental class in $\Omega_*(X)$

The purpose of this section is to construct a cobordism class in $\Omega_{\mathrm{rk}(E_0)-\mathrm{rk}(E_1)}(X)$ from a perfect obstruction theory described before. The construction in cobordism is analogous to that in Chow theory. The main difficulty comes from that in general, the cones we used are singular, and they do not carry the fundamental classes in algebraic cobordism.

However. when X admits an imbedding into a smooth scheme Y, there is a canonical way to construct a cobordism class on the normal cone $C_{X/Y}$ serving as the "fundamental class " of $C_{X/Y}$. Such class is obtained by deformation to the normal cone.

Recall that we have the following map

$$\sigma : \Omega_*(Y) \to \Omega_*(C_{X/Y}).$$

We get the class $\alpha_{X/Y} := \sigma(\mathbf{1}_Y)$ where $\mathbf{1}_Y$ is the fundamental class of Y.

Since E_0 is a vector bundle over X, the cone $C_{X/Y} \times_X E_0$ is a vector bundle over $C_{X/Y}$. The projection $C_{X/Y} \times_X E_0 \to C_{X/Y}$ induces an isomorphism

$$\Omega_*(C_{X/Y}) \cong \Omega_{*+\mathrm{rk}(E_0)}(C_{X/Y} \times_X E_0).$$

Together with the diagram

$$\begin{array}{ccc} T_Y|_X & \longrightarrow & C_{X/Y} \times_X E_0 \\ \downarrow & & \downarrow \\ X & \longrightarrow & D^{\mathrm{vir}} \\ \downarrow & & \downarrow \\ X & \longrightarrow & E_1, \end{array} \tag{16}$$

we get the following chain of maps

$$\Omega_*(Y) \xrightarrow{\sigma} \Omega_*(C_{X/Y}) \xrightarrow{\simeq} \Omega_{*+\mathrm{rk}(E_0)}(C_{X/Y} \times_X E_0) \xrightarrow{0^!_{E_1}}$$

$$\Omega_{*+\mathrm{rk}(E_0)-\mathrm{rk}(E_1)}(T_Y|_X) \xrightarrow{0^*_{T_Y}} \Omega_{*+\mathrm{rk}(E_0)-\mathrm{rk}(E_1)-\dim(Y)}(X).$$

Theorem 6.1 *Let X be a quasi-projecitive $\mathbb{C}$-scheme, which is imbedded into a smooth $\mathbb{C}$-scheme Y. And $E^\bullet = [E^{-1} \to E^0] \to L_X^\bullet$ is a perfect obstruction theory for X, then there is a morphism of cobordism groups*

$$\sigma_{X/Y} : \Omega_*(Y) \to \Omega_{*+rk(E_0)-rk(E_1)-dim(Y)}(X). \tag{17}$$

Moreover, the image $\sigma_{X/Y}(\mathbf{1}_Y) \in \Omega_{rk(E_0)-rk(E_1)}(X)$ is independent of the choice of imbedding and representative of obstruction bundles.

Proof The proof of independence is almost the same as in Lemma 4.2.

For the **first step**, we only need to show

$$\pi^*\alpha_{X/Y} = \alpha_{X/Y'}$$

in stead of (11) because of the following Cartesian diagram

$$\begin{array}{ccc} C_{X/Y'} \times_X E_0 & \longrightarrow & C_{X/Y'} \\ \downarrow & & \downarrow \pi \\ C_{X/Y} \times_X E_0 & \longrightarrow & C_{X/Y} \end{array}$$

This is implied by (the cobordism version of) Lemma 2.4.

For the **second step**, by the commutative diagram

$$\begin{array}{ccc} C_{X/Y'} \times_X E_0 & \xrightarrow{j'} & C_{X/Y'} \times_X G_0 \\ & \searrow \pi_E & \downarrow \pi_G \\ & & C_{X/Y'} \end{array}$$

we have to prove $j^!\pi_G^*\alpha_{X/Y'} = \pi_E^*\alpha_{X/Y'}$ instead of (12). It is clear because

$$j^!\pi_G^*\alpha_{X/Y'} = j'^!\pi_G^*\alpha_{X/Y'} = \pi_E^*\alpha_{X/Y'}.$$

Definition 6.2 *With the same notation as above, we define*

$$[X]^{vir}_{\Omega_*} := \sigma_{X/Y}(1_Y) \in \Omega_{rk(E_0)-rk(E_1)}(X)$$

to be the virtual fundamental class of X in cobordism with respect to the perfect obstruction theory $E^\bullet \to L_X^\bullet$.

Recall that in Chow theory, the virtual fundamental class is defined to be $0^!_{E_1}[D^{\mathrm{vir}}]$. The analogous definition in cobordism is also possible. To do that, we have to define a class in $\Omega_{\mathrm{rk}(E_0)}(D^{\mathrm{vir}})$ serving as the fundamental class $[D^{\mathrm{vir}}]$ in cobordism. It can be done as follows:

First, we get a class $\beta \in \Omega_{\mathrm{rk}(E_0)+\dim(Y)}(C_{X/Y} \times_X E_0)$ by pulling back $\alpha_{X/Y} \in \Omega_{\dim(Y)}(C_{X/Y})$. Then since $T_Y|_X$ acts on $C_{X/Y} \times_X E_0$ freely and D^{vir} is the quotient cone, $C_{X/Y} \times_X E_0$ is a $p^*T_Y|_X$-principal homogeneous space over D^{vir} where $p : C_{X/Y} \to X$. By the homotopy invariance property ([6] Theorem 3.6.3), there is an isomorphism:

$$\Omega_*(D^{\mathrm{vir}}) \cong \Omega_{*+\dim(Y)}(C_{X/Y} \times_X E_0).$$

We get $\gamma \in \Omega_{\mathrm{rk}(E_0)}(D^{\mathrm{vir}})$ from $\beta \in \Omega_{\mathrm{rk}(E_0)+\dim(Y)}(C_{X/Y} \times_X E_0)$ and we can define the virtual class as $0^!_{E_1}(\gamma)$.

It can be seen easily from the diagram (16) that such definition is same as Definition 6.2.

6.2 Relation between the virtual fundamental classes in Chow theory and algebraic cobordism theory

Recall that for a smooth quasi-projective scheme X, by identifying $\Omega_*(X)_{\mathbb{Q}}$ with $A_*(X)[\mathbf{t}]^{(\mathbf{t})}_{\mathbb{Q}}$, we have the following formula for the fundamental class of X

$$\mathbf{1}_X = \mathrm{Td}_{\mathbf{t}}^{-1}(-[T_X])[X].$$

We show that a similar formula holds for the virtual fundamental classes by replacing the tangent bundle T_X by the K-theory class of virtual tangent bundle $[E_0] - [E_1]$.

Theorem 6.3 *Let X be a quasi-projective scheme, and $E^\bullet = [E^{-1} \to E^0] \to L^\bullet_X$ be a perfect obstruction theory for X. Then under the isomorphism between $\Omega_* \otimes \mathbb{Q}$ and $A_*[\mathbf{t}]^{(\mathbf{t})}_{\mathbb{Q}}$, the following formula holds:*

$$[X]^{vir}_{\Omega_*} = Td_{\mathbf{t}}^{-1}([E_1] - [E_0])[X]^{vir}.$$

Here $[X]^{vir}$ is the virtual class in $A_*(X)$ obtained by the same perfect obstruction theory.

The main purpose of this subsection is to prove Theorem 6.3. Before that, we state a few immediate corollaries. Let $p_X : X \to \mathrm{pt}$ be the projection.

Corollary 6.4 *With the same notation as above, the Chern numbers of* $p_{X*}[X]^{vir}_{\Omega_*}$ *are given by*

$$C^I(E_\bullet) := \int_{[X]^{vir}} c^I([E_0]-[E_1]).$$

In particular,

$$p_{X*}[X]^{vir}_{\Omega_*} = \sum_{|I|=rk(E_0)-rk(E_1)} C^I(E_\bullet)P_I,$$

where $I=(n_1,\dots,n_d,\dots)$ *and* P_I *are the same as in (15).*

Corollary 6.5 *The element* $\sum_{|I|=rk(E_0)-rk(E_1)} C^I(E_\bullet)P_I$ *lies in Lazard ring.*

Now we start to prove Theorem 6.3. According to the description at the end of last subsection, the virtual fundamental class can be obtained by the following chain of maps:

$$\Omega_*(Y) \xrightarrow{\ \sigma\ } \Omega_*(C_{X/Y}) \xrightarrow{\ \simeq\ } \Omega_{*+\mathrm{rk}(E_0)}(C_{X/Y}\times_X E_0) \simeq \Omega_{*+\mathrm{rk}(E_0)-\dim(Y)}(D^{\mathrm{vir}})$$
$$\longrightarrow \Omega_{*-\dim(Y)+\mathrm{rk}(E_0)}(E_1) \xrightarrow{\ 0^!_{E_1}\ } \Omega_{*+\mathrm{rk}(E_0)-\mathrm{rk}(E_1)-\dim(Y)}(X).$$

We chase the chain above.

Step 1. Let $\pi_1 : C_{X/Y}\to X$ be the projection. We claim that

$$\alpha_{X/Y} = \mathrm{Td}^{-1}_{\mathbf{t}}(-[\pi_1^*T_Y|_X])[C_{X/Y}] \in \Omega_*(C_{X/Y})_{\mathbb{Q}}(=A_*(C_{X/Y})[\mathbf{t}]^{(\mathbf{t})}_{\mathbb{Q}}).$$

Proof We use the same notation as section 2. Consider

$$\pi_M : M = \mathrm{Bl}_{X\times 0}(Y\times\mathbb{P}^1)\to Y.$$

We restrict the vector bundle $\pi_M^*T_Y$ on M to the open subset $M^\circ = M\setminus \mathrm{Bl}_X(Y)$. For convenience, this vector bundle is also denoted by $\pi_M^*T_Y$. Look at the following diagram

$$\begin{array}{ccccccc}
\Omega_{\dim(Y)+1}(C_{X/Y})_{\mathbb{Q}} & \xrightarrow{j_*} & \Omega_{\dim(Y)+1}(M^\circ)_{\mathbb{Q}} & \xrightarrow{u} & \Omega_{\dim(Y)+1}(Y\times\mathbb{A}^1)_{\mathbb{Q}} & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle j^*} & & \Big\uparrow{\scriptstyle \mathrm{pr}^*} & & \\
 & & \Omega_{\dim(Y)}(C_{X/Y})_{\mathbb{Q}} & \longleftarrow & \Omega_{\dim(Y)}(Y)_{\mathbb{Q}}. & &
\end{array}$$

As is shown before, $\mathbf{1}_Y = \mathrm{Td}_{\mathbf{t}}^{-1}(-[T_Y])[Y] \in \Omega_{\dim(Y)}(Y)_{\mathbb{Q}}$, then

$$\mathbf{1}_{Y\times\mathbb{A}^1} = \mathrm{pr}^*\mathbf{1_Y} = \mathrm{Td}_{\mathbf{t}}^{-1}(-[\mathrm{pr}^*T_Y])[Y\times\mathbb{A}^1].$$

Consider the class

$$\theta := \mathrm{Td}_{\mathbf{t}}^{-1}(-[\pi_M^*T_Y])[M^\circ] \in \Omega_{\dim(Y)+1}(M^\circ)_{\mathbb{Q}}.$$

It is clear that $u(\theta) = \mathbf{1}_{Y\times\mathbb{A}^1}$. Hence

$$\alpha_{X/Y} = \sigma(\mathbf{1}_Y) = j^*\theta = \mathrm{Td}_{\mathbf{t}}^{-1}(-[\pi_M^*T_Y|_{C_{X/Y}}])[C_{X/Y}],$$

and it is clear that $\pi_M^*T_Y|_{C_{X/Y}} = \pi_1^*T_Y|_X$.

Step 2. Let $\pi_2 : C_{X/Y}\times_X E_0 \to C_{X/Y}$ be the projection. We have

$$\beta := \pi_2^*\alpha_{X/Y} = \mathrm{Td}_{\mathbf{t}}^{-1}(-[T_Y]+[E_0])[C_{X/Y}\times_X E_0].$$

(Note that we just use T_Y and E_0 to represent the corresponding bundles pulled back from X for notational convenience).

Proof It follows from the definition of pull back in $A_*[\mathbf{t}]_{\mathbb{Q}}^{(\mathbf{t})}$, since $[N_{\pi_2}] = [E_0] \in K^0(C_{X/Y}\times E_0)$.

Step 3. Let $\pi_3 : C_{X/Y}\times_X E_0 \to D^{\mathrm{vir}}$ be the quotient map. Since

$$\pi_3^* : \Omega_*(D^{\mathrm{vir}}) \to \Omega_{*+\dim(Y)}(C_{X/Y}\times_X E_0)$$

induces an isomorphism, there exists $\gamma \in \Omega_{\mathrm{rk}(E_0)}(D^{\mathrm{vir}})$ such that $\pi_3^*\gamma = \beta$. We claim that

$$\gamma = \mathrm{Td}_{\mathbf{t}}^{-1}([E_0])[D^{\mathrm{vir}}] \in \Omega_{\mathrm{rk}(E_0)}(D^{\mathrm{vir}})_{\mathbb{Q}}.$$

Proof Since π_3^* induces an isomorphism, it suffices to prove

$$\pi_3^*(\mathrm{Td}_{\mathbf{t}}^{-1}([E_0])[D^{\mathrm{vir}}]) = \beta.$$

In fact, we have $[N_{\pi_3}] = [T_Y] \in K^0(C_{X/Y}\times_X E_0)$. Therefore

$$\pi_3^*(\mathrm{Td}_{\mathbf{t}}^{-1}([E_0])[D^{\mathrm{vir}}]) = \mathrm{Td}_{\mathbf{t}}^{-1}(-[T_Y])\mathrm{Td}_{\mathbf{t}}^{-1}([E_0])\pi_3^*[D^{\mathrm{vir}}] = \beta.$$

Step 4. Pushing forward by the imbedding $i : D^{\mathrm{vir}} \to E_1$, we get

$$\gamma' := i_* \gamma = \mathrm{Td}_{\mathbf{t}}^{-1}(-[E_0])[D^{\mathrm{vir}}] \in \Omega_{\mathrm{rk}(E_0)}(E_1).$$

Step 5. We prove Theorem 6.3

$$[X]^{\mathrm{vir}}_{\Omega_*} = \mathrm{Td}_{\mathbf{t}}^{-1}([E_1] - [E_0])[X]^{\mathrm{vir}} \in \Omega_{\mathrm{rk}(E_0)-\mathrm{rk}(E_1)}(X)_{\mathbb{Q}}.$$

Proof Let $\pi_4 : E_1 \to X$ be the projection. It suffices to prove that

$$\pi_4^*(\mathrm{Td}_{\mathbf{t}}^{-1}([E_1] - [E_0])[X]^{\mathrm{vir}}) = \gamma'.$$

Actually,

$$\begin{aligned} \pi_4^*(\mathrm{Td}_{\mathbf{t}}^{-1}([E_1] - [E_0])[X]^{\mathrm{vir}}) &= \mathrm{Td}_{\mathbf{t}}^{-1}(-[E_1])\mathrm{Td}_{\mathbf{t}}^{-1}([E_1] - [E_0])\pi_4^*[X]^{\mathrm{vir}} \\ &= \mathrm{Td}_{\mathbf{t}}^{-1}(-[E_0])[D^{\mathrm{vir}}]. \end{aligned}$$

7 Cobordism-valued invariants in Pandharipande-Thomas Theory

In this section, we work on schemes over $\mathbb{C}$. We define the cobordism invariants in Pandharipande-Thomas theory for arbitrary 3-folds which generalize the invariants for Calabi-Yau 3-folds. The rationality of these invariants is proven for nonsingular toric 3-folds.

7.1 Moduli space of stable pairs

We first review the moduli space of stable pairs constructed in [15].

Let X be a nonsingular 3-fold, and let

$$\beta \in H_2(X, \mathbb{Z})$$

be a non-zero class. A stable pair (F, s) consists of a coherent sheaf F with one-dimensional support in X and a section $s \in H^0(X, F)$ satisfying the following stability conditions:

1. F is pure.

2. the section s has zero dimensional cokenel.

Therefore we get a Cohen-Macaulay curve which is the support of F and a finite length subscheme Z of C which is the cokernel of the section. When the support C is nonsingular, the stable pair (F, s) is uniquely determined by $Z \subset C$.

Now we consider the moduli space of stable pairs

$$[O_X \xrightarrow{s} F] \in P_n(X, \beta)$$

which parameterizes stable pairs satisfying

$$\chi(F) = n, [F] = \beta.$$

(For notational convenience, sometimes we just write $P(X)$ to denote the moduli space $P_n(X, \beta)$.)

Theorem 7.1 *([15])*

1. *The moduli space $P(X)$ is projective.*

2. *There is a universal stable pair*

$$\mathcal{O}_{X\times P(X)} \to \mathbb{F}$$

on $X \times P(X)$ *which determines a universal complex*

$$\mathbb{I}^\bullet = [\mathcal{O}_{X\times P(X)} \to \mathbb{F}] \in D^b(X \times P(X))$$

with $\mathbb{F}$ *flat over* $P(X)$.

3. *Let* π *denote the projection* $\pi : X \times P(X) \to P(X)$. *Then the complex* $R\pi_* R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1]$ *determines an obstruction theory on* $P(X)$. *More precisely, there exist a 2-term complex of vector bundles* $[E_0 \to E_1]$ *which is quasi-isomorphic to* $R\pi_* R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1]$, *and* $[E_1^\vee \to E_0^\vee] \to L^\bullet_{P(X)}$ *is a perfect obstruction theory. Here* $R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0$ *stands for the traceless part of* $R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)$.

4. *The virtual dimension of* $P_n(X,\beta)$ *equals*

$$\int_\beta c_1(T_X).$$

Hence, we get the virtual fundamental class of $P_n(X,\beta)$ in both Chow theory and algebraic cobordism

$$[P_n(X,\beta)]^{\mathrm{vir}} \in A_{\mathrm{vdim}}(P_n(X,\beta)),$$

$$[P_n(X,\beta)]^{\mathrm{vir}}_{\Omega_*} \in \Omega_{\mathrm{vdim}}(P_n(X,\beta)).$$

Let $p : P_n(X,\beta) \to \mathrm{pt}$ be the projection. We define

$$[P_{n,\beta}] = p_*[P_n(X,\beta)]^{\mathrm{vir}}_{\Omega_*} \in \mathbb{L}_*.$$

When X is a Calabi-Yau 3-fold, the virtual dimension of $P_n(X,\beta)$ equals 0. Hence

$$[P_{n,\beta}] = \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}$$

is just the classical invariant.

7.2 Descendents, generalized descendents and Chern operators

If $\dim[P_n(X,\beta)]^{\text{vir}} = \int_\beta c_1(X) > 0$, in order to get numerical invariants, one should consider operators acting on the homology groups of moduli space. We introduce three kinds of operators in this subsection: descendants, generalized descendants and Chern operators, and prove that both generalized descendants and Chern operators can be expressed in terms of descendents. Let

$$\pi_X : X \times P(X) \to X$$

$$\pi_P : X \times P(X) \to P(X)$$

be the projections onto the first and second factors. Since X is nonsingular and $\mathbb{F}$ is π_P-flat (Theorem 7.1 2.), the universal sheaf $\mathbb{F}$ has a finite resolution by locally free sheaves. Hence the Chern character of $\mathbb{F}$ on $X \times P(X)$ is well defined.

The descendents $\tau_i(\gamma)$ where $\gamma \in H^*(X,\mathbb{Z})$ are defined to be the operators on the homology of $P(X)$

$$\pi_{P*}\big(\pi_X^*(\gamma) \cdot \text{ch}_{2+i}(\mathbb{F}) \cap \pi_P^*(\ \cdot\ )\big) : H_*(P(X)) \to H_*(P(X)).$$

Also, one can define the generalized descendents $\tau_{i,j}(\gamma)$ as the following operator:

$$\pi_{P*}\big(\pi_X^*(\gamma) \cdot \text{ch}_{2+i}(\mathbb{F}) \cdot \text{ch}_{2+j}(\mathbb{F}) \cap \pi_P^*(\ \cdot\ )\big) : H_*(P(X)) \to H_*(P(X)).$$

Since $\pi_P : P(X) \times X \to P(X)$ is an l.c.i and proper morphism, we have the Gysin homomorphism

$$\pi_{P*} : H^n(P(X) \times X) \to H^{n-3}(P(X))$$

satisfying

$$\pi_{P*}\beta \cap \alpha = \pi_{P*}(\beta \cap \pi_P^*\alpha)$$

where $\alpha \in H^*(P(X))$ and $\beta \in H_*(P(X) \times X)$. Hence descendents and generalized descendents lie in $H^*(P(X))$, i.e.

$$\tau_i(\gamma) = \pi_{P*}({\pi_X}^*\gamma \cdot \text{ch}_{2+i}(\mathbb{F})) \in H^*(P(X))$$

$$\tau_{i,j}(\gamma) = \pi_{P*}({\pi_X}^*\gamma \cdot \text{ch}_{2+i}(\mathbb{F}) \cdot \text{ch}_{2+j}(\mathbb{F})) \in H^*(P(X)).$$

We will show that generalized descendents can be expressed in terms of descendents.

Consider the diagonal imbedding $\delta : X \to X \times X$. For $\gamma \in H^*(X)$, the class $\delta_*\gamma$ has the Künneth decomposition $\sum_i u_i \otimes v_i$ where $u_i, v_i \in H^*(X)$. In other words, let $q_i : X \times X \to X$ be the projection to the i-th factor (i=1,2), then $\delta_*\gamma = \sum_i q_1^* u_i \cdot q_2^* v_i$.

We look at the space $P(X) \times X \times X$. From the two projections $p_i : P(X) \times X \times X \to P(X) \times X$ (i=1,2), we obtain two universal sheaves $\mathbb{F}_1$, $\mathbb{F}_2$ on $P(X) \times X \times X$ by pulling back the universal sheaf $\mathbb{F}$ via p_i. The diagonal imbedding δ yields the following commutative diagram:

$$\begin{array}{ccccc} P(X) \times X & \xrightarrow{\Delta} & P(X) \times X \times X & \xrightarrow{\pi} & P(X) \\ {\scriptstyle \pi_X}\downarrow & & {\scriptstyle \pi_{X\times X}}\downarrow & & \\ X & \xrightarrow{\delta} & X \times X & & \end{array} \qquad (18)$$

Lemma 7.2 *We have the following formula*

$$\tau_{i,j}(\gamma) = \pi_*\big(\pi_{X\times X}^*(\delta_*\gamma) \cdot ch_{2+i}(\mathbb{F}) \cdot ch_{2+j}(\mathbb{F})\big)$$

where $\gamma \in H^*(X)$.

Proof By definition

$$\begin{aligned} \tau_{i,j}(\gamma) &= \pi_*\Delta_*(\Delta^*(\mathrm{ch}_{2+i}(\mathbb{F}_1) \cdot \mathrm{ch}_{2+j}(\mathbb{F}_2)) \cdot \pi_X^*\gamma) \\ &= \pi_*(\mathrm{ch}_{2+i}(\mathbb{F}_1) \cdot \mathrm{ch}_{2+j}(\mathbb{F}_2) \cdot \Delta_*\pi_X^*\gamma) \\ &= \pi_*(\mathrm{ch}_{2+i}(\mathbb{F}_1) \cdot \mathrm{ch}_{2+j}(\mathbb{F}_2) \cdot \pi_{X\times X}^*(\delta_*\gamma)) \end{aligned}$$

Proposition 7.3 *With the notation as above, we have*

$$\tau_{k_1,k_2}(\gamma) = \sum_i \tau_{k_1}(u_i) \cdot \tau_{k_2}(v_i).$$

Proof Similarly, the strategy here is to use the projection formula several times.

$$\begin{array}{ccccc} P(X) \times X & \xrightarrow{\triangle} & P(X) \times X \times X & \xrightarrow{p_2} & P(X) \times X \\ & & {\scriptstyle p_1}\downarrow & & {\scriptstyle \pi_2}\downarrow \\ & & P(X) \times X & \xrightarrow{\pi_1} & X. \end{array}$$

By the diagram above and Lemma 7.2, we have

$$
\begin{aligned}
\tau_{k_1,k_2}(\gamma) &= \pi_{1*}p_{1*}(\pi^*_{X\times X}(\delta_*\gamma)\cdot p_1^*\mathrm{ch}_{2+k_1}(\mathbb{F})\cdot p_2^*\mathrm{ch}_{2+k_2}(\mathbb{F}))\\
&= \pi_{1*}p_{1*}((\sum_i p_1^*\pi_1^*u_i\cdot p_2^*\pi_2^*v_i)\cdot p_1^*\mathrm{ch}_{2+k_1}(\mathbb{F})\cdot p_2^*\mathrm{ch}_{2+k_2}(\mathbb{F}))\\
&= \sum_i \pi_{1*}p_{1*}(p_1^*(\pi_1^*u_i\cdot \mathrm{ch}_{2+k_1}(\mathbb{F}))\cdot p_2^*(\pi_2^*v_i\cdot \mathrm{ch}_{2+k_2}(\mathbb{F})))\\
&= \sum_i \pi_{1*}(\pi_1^*u_i\cdot \mathrm{ch}_{2+k_1}(\mathbb{F})\cdot p_{1*}p_2^*(\pi_2^*v_i\cdot \mathrm{ch}_{2+k_2}(\mathbb{F})))\\
&= \sum_i \pi_{1*}(\pi_1^*u_i\cdot \mathrm{ch}_{2+k_1}(\mathbb{F})\cdot \pi_1^*\pi_{2*}(\pi_2^*v_i\cdot \mathrm{ch}_{2+k_2}(\mathbb{F})))\\
&= \sum_i \pi_{1*}(\pi_1^*u_i\cdot \mathrm{ch}_{2+k_1}(\mathbb{F}))\cdot \pi_{2*}(\pi_2^*v_i\cdot \mathrm{ch}_{2+k_2}(\mathbb{F}))\\
&= \sum_i \tau_{k_1}(u_i)\cdot\tau_{k_2}(v_i).
\end{aligned}
$$

Finally, we consider the Chern operators. From the obstruction bundles $E_0 \to E_1$ on the moduli space $P(X)$, the **Chern operators** are defined to be

$$c_k(E_\bullet) := c_k([E_0]-[E_1]) : H_*(P(X)) \to H_*(P(X)).$$

Theorem 7.4 *The Chern operators $c_k(E_\bullet)$ can be written in terms of descendents.*

We prove that the Chern characters $\mathrm{ch}_k(-E_\bullet) = \mathrm{ch}_k([E_1]-[E_0])$ can be written in terms of generalized descendents. Then Theorem 7.4 is obtained as a consequence of Proposition 7.3.

Proof We apply the Grothendieck-Riemann-Roch formula ([1]) to the smooth map $\pi_P : P(X)\times X \to P(X)$. Since

$$
\begin{aligned}
[E_1]-[E_0] &= [R\pi_{P*}R\mathcal{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0]\\
&= [R\pi_{P*}R\mathcal{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)] - [R\pi_{P*}\mathcal{O}_{P(X)\times X}] \in K^0(P(X)\times X),
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathrm{ch}_k([E_1]-[E_0]) &= \mathrm{ch}_k([R\pi_{P*}R\mathcal{H}om(\mathbb{I}^\bullet,\mathbb{I}^\bullet)]) - \mathrm{ch}_k([R\pi_{P*}\mathcal{O}_{P(X)\times X}])\\
&= \mathrm{ch}_k([R\pi_{P*}(\mathbb{I}^\bullet\otimes(\mathbb{I}^\bullet)^\vee)]) - \mathrm{ch}_k([R\pi_{P*}\mathcal{O}_{P(X)\times X}]).
\end{aligned}
$$

By Grothendieck-Riemann-Roch

$$\begin{aligned}\mathrm{ch}([R\pi_{P*}(\mathbb{I}^\bullet \otimes (\mathbb{I}^\bullet)^\vee)]) &= \pi_{P*}(\mathrm{ch}(\mathbb{I}^\bullet \otimes (\mathbb{I}^\bullet)^\vee)) \cdot \mathrm{Td}(\pi_X^* T_X) \\ &= \pi_{P*}(\mathrm{ch}(\mathbb{I}^\bullet) \cdot \mathrm{ch}((\mathbb{I}^\bullet)^\vee) \cdot \pi_X^* \mathrm{Td}(T_X))\end{aligned}$$

Take the degree k part ($k > 0$), we see that $\mathrm{ch}_k([R\pi_{P*}(\mathbb{I}^\bullet \otimes (\mathbb{I}^\bullet)^\vee)])$ can be expressed as

$$\sum_{i+j+deg(\gamma_{ij})=k+3} a_{ij} \cdot \pi_{P*}(\mathrm{ch}_i(\mathbb{F}) \cdot \mathrm{ch}_j(\mathbb{F}) \cdot \pi_X^* \gamma_{ij}) \tag{19}$$

where $a_{ij} \in \mathbb{Z}$ and $\gamma_{ij} \in H^*(X)$. Similarly, we have

$$\mathrm{ch}(R\pi_{P*}\mathcal{O}_{P(X)\times X}) = \pi_{P*}(\pi_X^* \mathrm{Td}(T_X)).$$

Note that $\mathrm{ch}_k(R\pi_{P*}\mathcal{O}_{P(X)\times X}) = 0$ if $k > 0$ for dimension reasons. Hence $\mathrm{ch}_k([E_1] - [E_0])$ can be represented as in (19) when $k > 0$.

7.3 Cobordism invariants

Let X be a nonsingular 3-fold and $\beta \in H_2(X, \mathbb{Z})$, we have already defined the cobordism class $[P_{n,\beta}] \in \Omega_{\mathrm{vdim}}(\mathrm{pt})(\cong \mathbb{L}_*)$, where $\mathrm{vdim} = \int_\beta c_1(X)$. Define $P(X; q)_\beta$ to be the **cobordism partition function** of the Pandharipande-Thomas theory of X,

$$P(X; q)_\beta := \sum_{n\in\mathbb{Z}} [P_{n,\beta}] q^n \in \mathbb{L}_*[[q, q^{-1}]].$$

The function $P(X; q)_\beta$ is closely related to the descendants invariants. For non-zero $\beta \in H_2(X, \mathbb{Z})$ and arbitrary $\gamma_i \in H^*(X, \mathbb{Z})$, the PT descendent invariants are defined by

$$\begin{aligned}\left\langle \prod_{j=1}^m \tau_{i_j}(\gamma_j) \right\rangle_{n,\beta}^X &= \int_{[P_n(X,\beta)]^{\mathrm{vir}}} \prod_{j=1}^m \tau_{i_j}(\gamma_j) \\ &= \int_{[P_n(X,\beta)]} \prod_{j=1}^m \tau_{i_j}(\gamma_j)([P_n(X,\beta)]^{\mathrm{vir}})\end{aligned}$$

Then the partition function of descendents is

$$Z_\beta^X\left(\prod_{j=1}^m \tau_{i_j}(\gamma_j)\right) = \sum_n \left\langle \prod_{j=1}^m \tau_{i_j}(\gamma_j) \right\rangle_{n,\beta}^X q^n.$$

The following conjecture was made in [16] and proven for nonsingular toric case in [13].

Conjecture 7.5 *The partition function $Z^X_\beta\big(\prod_{j=1}^m \tau_{i_j}(\gamma_j)\big)$ is the Laurent expansion of a rational function in q.*

Theorem 7.6 *If Conjecture 7.5 holds for nonsingular 3-fold X and $\beta \in H_2(X,\mathbb{Z})$, then the cobordism partition function $P(X;q)_\beta$ is the Laurent expansion of a rational function in q. In particular, it is true when X is a nonsingular toric 3-fold.*

Proof By the result of Corollary 6.4, we have

$$[P_{n,\beta}] = \sum_{|I|=\text{vdim}} C^I(E^{(n)}_\bullet)P_I,$$

where $E^{(n)}_\bullet = [E^{(n)}_0] - [E^{(n)}_1] \in K^0(P_n(X,\beta))$ is the obstruction bundle. We can write

$$P(X;q)_\beta = \sum_{|I|=\text{vdim}} \Big(\sum_{n\in\mathbb{Z}} C^I(E^{(n)}_\bullet)q^n\Big)P_I.$$

Therefore the rationality of $P(X;q)_\beta$ holds if and only if all the partition functions of Chern numbers $C^I(E^{(n)}_\bullet)q^n \in \mathbb{Z}[[q,q^{-1}]]$ are rational in q. Since $C^I(E^{(n)}_\bullet)q^n$ can be expressed by the linear combination of the partition functions of descendents by Theorem 7.4, the theorem is proven.

Remark 7.7 *1. By the localization formula and computer calculation, we compute a few terms of $P(\mathbb{P}^3,q)_{[L]}$ and find out that $P(\mathbb{P}^3,q)_{[L]}$ behaves like a rational function satisfying the functional equation*

$$P(\mathbb{P}^3,q)_{[L]} = q^4 P(\mathbb{P}^3,1/q)_{[L]},$$

where L is a line in $\mathbb{P}^3$. It indicates that $P(X,q)_\beta$ may satisfy the functional equation for all nonsingular 3-fold X and $\beta \in H_2(X,\mathbb{Z})$:

$$P(X,q)_\beta = q^{vdim} P(X,1/q)_\beta,$$

where $vdim = \int_\beta c_1(X)$. As a special case, when X is a nonsingular Calabi-Yau 3-fold, it is just the symmetry of the partition function under the transformation $q \leftrightarrow 1/q$.

2. It is an interesting question whether the rationality and functional equation of the cobordism partition function can be reduced to toric case by degeneration method.

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