HOW CLOSE ARE THE OPTION PRICING FORMULAS OF BACHELIER AND BLACK-MERTON-SCHOLES?

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Abstract. We compare the option pricing formulas of Louis Bachelier and Black-Merton-Scholes and observe – theoretically as well as for Bachelier's original data – that the prices coincide very well. We illustrate Louis Bachelier's efforts to obtain applicable formulas for option pricing in pre-computer time. Furthermore we explain – by simple methods from chaos expansion – why Bachelier's model yields good short-time approximations of prices and volatilities.

1. Introduction

It is the pride of Mathematical Finance that L. Bachelier was the first to analyze Brownian motion mathematically, and that he did so in order to develop a theory of option pricing (see [2]). In the present note we shall review some of the results from his thesis as well as from his later textbook on probability theory (see [3]), and we shall work on the remarkable closeness of prices in the Bachelier and Black-Merton-Scholes model.

The "fundamental principle" underlying Bachelier's approach to option pricing is crystallized in his famous dictum (see [2], p.34)

"L'ésperance mathematique du spéculateur est nul",

i.e. "the mathematical expectation of a speculator is zero". His argument in favor of this principle is based on equilibrium considerations (see [2] and [11]), similar to what in today's terminology is called the "efficient market hypothesis" (see [10]), i.e. the use of martingales to describe stochastic time evolutions of price movements in ideal markets. L. Bachelier writes on this topic (see the original french version in [2], p. 31).

> "It seems that the market, the aggregate of speculators, can believe
> in neither a market rise nor a market fall, since, for each
> quoted price, there are as many buyers as sellers."

The reader familiar with today's approach to option pricing might wonder where the concepts of "risk free interest rate" and "risk neutral measure" have disappeared to, which seem crucial in the modern approach of pricing by no arbitrage arguments (recall that the *discounted* price process should be a martingale under the risk neutral measure). As regards the first issue L. Bachelier applied his "fundamental principle" in terms of "true" prices (this is terminology from 1900 which corresponds to the concept of forward prices in modern terminology), since all the payments involved (including the premium of the option) were done only at maturity of the contracts. See [2] for an explicit description of the trading rules at the bourse de Paris in 1900. It is well-known that the passage to forward prices makes the riskless interest rate disappear: in the context of the Black-Merton-Scholes formula, this is what amounts to the so-called Black's formula (see [4]). As regards the second issue L. Bachelier apparently believed in the martingale measure as the historical measure, i.e. for him the risk neutral measure conincides with the historical measure. For a discussion of this issue compare [10].

Summing up: Bachelier's "fundamental principle" yields *exactly* the same recipe for option pricing as we use today (for more details we refer to the first section of the St. Flour summer school lecture [11]): using forward prices ("true prices" in the terminology of 1900) one obtains the prices of options (or of more general derivatives of European style) by taking expectations. The expectation pertains to a probability measure under which the price process of the underlying security (given as forward prices) satisfies the fundamental principle, i.e. is a martingale in modern terminology.

It is important to emphasize that, although the *recipes* for obtaining option prices are the same for Bachelier's as for the modern approach, the *arguments* in favour of them are very different: an equilibrium argument in Bachelier's case as opposed to the no arbitrage arguments in the Black-Merton-Scholes approach. With all admiration for Bachelier's work, the development of a theory of hedging and replication by dynamic strategies, which is the crucial ingredient of the Black-Merton-Scholes-approach, was far out of his reach (compare [11] and section 2.1 below).

In order to obtain option prices one has to specify the underlying model. We fix a time horizon $T > 0$. As is well-known, Bachelier proposed to use (properly scaled) Brownian motion as a model for forward stock prices. In modern terminology this amounts to

$$S_t^B := S_0 + \sigma^B W_t, \tag{1.1}$$

for $0 \leq t \leq T$, where $(W_t)_{0 \leq t \leq T}$ denotes standard Brownian motion and the superscript B stands for Bachelier. The parameter $\sigma^B > 0$ denotes the volatility in the Bachelier model. Notice that in contrast to today's standard Bachelier measured volatility in absolute terms. In fact, Bachelier used the normalization $H = \frac{\sigma^B}{\sqrt{2\pi}}$ and called this quantity the "coefficient of instability" or of "nervousness" of the security S. The reason for the normalisation $H = \frac{\sigma^B}{\sqrt{2\pi}}$ is that $H\sqrt{T}$ then equals the price of an at the money option in Bachelier's model (see [2])

The Black-Merton-Scholes model (under the risk-neutral measure) for the price process is, of course, given by

$$S_t^{BS} = S_0 \exp(\sigma^{BS} W_t - \frac{(\sigma^{BS})^2}{2} t), \tag{1.2}$$

for $0 \leq t \leq T$. Here σ^{BS} denotes the usual volatility in the Black-Merton-Scholes model.

This model was proposed by P. Samuelson in 1965, after he had – led by an inquiry of J. Savage for the treatise [3] – personally rediscovered the virtually forgotten Bachelier thesis in the library of Harvard University. The difference between the two models is somewhat analogous to the difference between linear and compound interest, as becomes apparent when looking at the associated Itô stochastic differential equation,

$$dS_t^B = \sigma^B dW_t,$$
$$dS_t^{BS} = S_t^{BS}\sigma^{BS} dW_t.$$

This analogy makes us expect that, in the short run, both models should yield similar results while, in the long run, the difference should be spectacular. Fortunately, options usually have a relatively short time to maturity (the options considered by Bachelier had a time to maturity of less than 2 months).

2. Bachelier versus Black-Merton-Scholes

We now have assembled all the ingredients to recall the derivation of the price of an option in Bachelier's framework. Fix a strike price K, a horizon T and consider the European call C, whose pay-off at time T is modeled by the random variable

$$C_T^B = (S_T^B - K)_+.$$

Applying Bachelier‘s “fundamental principle” and using that S_T^B is normally distributed with mean S_0 and variance $(\sigma^B)^2 T$, we obtain for the price of the option at time $t = 0$

$$C_0^B = E[(S_T^B - K)_+]$$
$$= \int_{K-S_0}^{\infty} (S_0 + x - K)\frac{1}{\sigma^B\sqrt{2\pi T}} \exp(-\frac{x^2}{2(\sigma^B)^2 T})dx \tag{2.1}$$
$$= (S_0 - K)\Phi(\frac{S_0 - K}{\sigma^B\sqrt{T}}) + \sigma^B\sqrt{T}\phi(\frac{S_0 - K}{\sigma^B\sqrt{T}}), \tag{2.2}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$ denotes the density of the standard normal distribution. We applied the relation $\phi'(x) = -x\phi(x)$ to pass from (2.1) to (2.2). For details see, e.g. , [5].

For further use we shall need the very well-known Black-Merton-Scholes price, too,

$$C_0^{BS} = E[(S_T^{BS} - K)_+]$$
$$= \int_{-\infty}^{\infty} (S_0 \exp(-\frac{(\sigma^{BS})^2 T}{2} + \sigma^{BS}\sqrt{T}x) - K)_+ \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})dx \tag{2.3}$$
$$= \int_{\frac{\log\frac{K}{S_0} + \frac{(\sigma^{BS})^2 T}{2}}{\sigma^{BS}\sqrt{T}}}^{\infty} (S_0 \exp(-\frac{(\sigma^{BS})^2 T}{2} + \sigma^{BS}\sqrt{T}x) - K)\frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})dx \tag{2.4}$$
$$= S_0\Phi(\frac{\log\frac{S_0}{K} + \frac{1}{2}(\sigma^{BS})^2 T}{\sigma^{BS}\sqrt{T}}) - K\Phi(\frac{\log\frac{S_0}{K} - \frac{1}{2}(\sigma^{BS})^2 T}{\sigma^{BS}\sqrt{T}}). \tag{2.5}$$

Interestingly, Bachelier explicitly wrote down formula (2.1), but did not bother to spell out formula (2.2), see [2, p. 50]. The main reason seems to be that at his time option prices – at least in Paris – were quoted the other way around: while today the strike prices K is fixed and the option price fluctuates according to supply and demand, at Bachelier's times the option prices were fixed (at 10, 20 and 50 Centimes for a “rente”, i.e., a perpetual bond with par value of 100 Francs) and therefore the strike prices K fluctuated. What Bachelier really needed was the inverse version of the above relation between the option price C_0^B and the strike price K.

Apparently there is no simple "formula" to express this inverse relationship. This is somewhat analogous to the situation in the Black-Merton-Scholes model, where there is also no "formula" for the inverse problem of calculating the implied volatility as a function of the given option price.

We shall see below that L. Bachelier designed a clever series expansion for C_0^B as a function of the strike price K in order to derive (very) easy formulae which *approximate* this inverse relation and which were well suited to pre-computer technology.

2.1. **At the money options.** Bachelier first specializes to the case of at the money options (called "simple options" in the terminology of 1900), when $S_0 = K$. In this case (2.2) reduces to the simple and beautiful relation

$$C_0^B = \sigma^B \sqrt{\frac{T}{2\pi}}.$$

As explicitly noticed by Bachelier, this formula can also be used, for a given price $C = C_0^B$ of an at the money option with maturity T, to determine the "coefficient of nervousness of the security" $H = \frac{\sigma^B}{\sqrt{2\pi}}$, i.e., to determine the implied volatility in modern language. Indeed, it suffices to normalize the price C_0^B by $\sqrt{T}$ to obtain $H = \frac{C_0^B}{\sqrt{T}}$. We summarize this fact in the subsequent proposition. For convenience we phrase it rather in terms of σ^B than of H.

Proposition 1. *The volatility σ^B in the Bachelier model is determined by the price C_0^B of an at the money option with maturity T by the relation*

$$\sigma^B = C_0^B \sqrt{\frac{2\pi}{T}}. \tag{2.6}$$

In the subsequent Proposition, we compare the price of an at the money call option as obtained from the Black-Merton-Scholes and Bachelier's formula respectively. In order to relate optimally (see the last section) C_0^B and C_0^{BS} we choose $\sigma^B = S_0\sigma$ and $\sigma^{BS} = \sigma$ for some constant $\sigma > 0$. We also compare the implied volatilities, for given price C_0 of an at the money call with maturity T, in the Bachelier and Black-Merton-Scholes model. We denote the respective implied volatilities by σ^B and σ^{BS} and discover that the implied Bachelier volatility estimates the Black-Scholes implied volatility quite well at the money.

Proposition 2. *Fix $\sigma > 0$, $T > 0$ and $S_0 = K$ (at the money), let $\sigma^{BS} = \sigma$ and $\sigma^B = S_0\sigma$ and denote by C^B and C^{BS} the corresponding prices for a European call option in the Bachelier (1.1) and Black-Merton-Scholes model (1.2) respectively. Then*

$$0 \le C_0^B - C_0^{BS} \le \frac{S_0}{12\sqrt{2\pi}}\sigma^3 T^{\frac{3}{2}} = \mathcal{O}((\sigma\sqrt{T})^3). \tag{2.7}$$

The relative error can be estimated by

$$\frac{C_0^B - C_0^{BS}}{C_0^B} \le \frac{T}{12}\sigma^2. \tag{2.8}$$

Conversely, fix the price $0 < C_0 < S_0$ of an at the money option and denote by σ^B the implied Bachelier volatility and by σ^{BS} the implied Black-Merton-Scholes *volatility, then*

$$0 \leq \sigma^{BS} - \frac{\sigma^B}{S_0} \leq \frac{T}{12}(\sigma^{BS})^3. \tag{2.9}$$

Proof. (compare [5] and [11]). For $S_0 = K$, we obtain in the Bachelier and Black-Merton-Scholes model the following prices, respectively,

$$C_0^B = \frac{S_0\sigma}{\sqrt{2\pi}}\sqrt{T}$$
$$C_0^{BS} = S_0(\Phi(\frac{1}{2}\sigma\sqrt{T}) - \Phi(-\frac{1}{2}\sigma\sqrt{T})).$$

Hence

$$\begin{aligned}
C_0^B - C_0^{BS} &= (\frac{S_0}{\sqrt{2\pi}}x - S_0(\Phi(\frac{x}{2}) - \Phi(-\frac{x}{2})))|_{x=\sigma\sqrt{T}} \\
&= \frac{S_0}{\sqrt{2\pi}}\left(\int_{-\frac{x}{2}}^{\frac{x}{2}}(1-\exp(-\frac{y^2}{2}))dy\right)\Bigg|_{x=\sigma\sqrt{T}} \\
&\leq \frac{S_0}{\sqrt{2\pi}}\int_{-\frac{x}{2}}^{\frac{x}{2}}\frac{y^2}{2}dy|_{x=\sigma\sqrt{T}} \\
&= \frac{S_0}{\sqrt{2\pi}}\frac{x^3}{12}|_{x=\sigma\sqrt{T}} = \frac{S_0}{24\sqrt{2\pi}}\sigma^3T^{\frac{3}{2}} = \mathcal{O}((\sigma\sqrt{T})^3),
\end{aligned}$$

since $e^y \geq 1 + y$ for all y, so that $\frac{y^2}{2} \geq 1 - e^{-\frac{y^2}{2}}$ for all y. Clearly we obtain $C_0^B - C_0^{BS} \geq 0$ again from the first line.

For the second assertion note that solving equation

$$C_0 = \frac{\sigma^B}{\sqrt{2\pi}}\sqrt{T} = S_0(\Phi(\frac{1}{2}\sigma^{BS}\sqrt{T}) - \Phi(-\frac{1}{2}\sigma^{BS}\sqrt{T}))$$

for given $\sigma^B > 0$ yields the Black-Merton-Scholes implied volatility σ^{BS}. We obtain similarly as above

$$\begin{aligned}
0 \leq \sigma^{BS} - \frac{\sigma^B}{S_0} &= \sigma^{BS} - \frac{\sqrt{2\pi}}{\sqrt{T}}(\Phi(\frac{1}{2}\sigma^{BS}\sqrt{T}) + \Phi(-\frac{1}{2}\sigma^{BS}\sqrt{T})) \\
&= \frac{\sqrt{2\pi}}{\sqrt{T}}(\frac{1}{\sqrt{2\pi}}x - (\Phi(\frac{x}{2}) - \Phi(-\frac{x}{2})))|_{x=\sigma^{BS}\sqrt{T}} \\
&\leq \frac{\sqrt{2\pi}}{\sqrt{T}}\frac{1}{12\sqrt{2\pi}}(\sigma^{BS})^3T^{\frac{3}{2}} = \frac{(\sigma^{BS})^3T}{12}.
\end{aligned}$$

□

Proposition 1 and 2 yield in particular the well-known asymptotic behaviour of an at the money call price in the Black-Merton-Scholes model for $T \to \infty$ as described in [1].

Proposition 2 tells us that for the case when $(\sigma\sqrt{T}) \ll 1$ (which typically holds true in applications), formula (2.6) gives a satisfactory approximation of the implied Black-Merton-Scholes volatility, and is very easy to calculate. We note that for the data reported by Bachelier (see [2] and [11]), the yearly relative volatility was of the order of 2.4% p.a. and T in the order of one month, i.e $T = \frac{1}{12}$ years so that

$\sqrt{T} \approx 0.3$. Consequently we get $(\sigma\sqrt{T})^3 \approx (0.008)^3 \approx 5 \times 10^{-7}$. The estimate in Proposition 2 yields a right hand side of $\frac{S_0}{12\sqrt{2\pi}} 5 \times 10^{-7} \approx 1.6 \times 10^{-8} S_0$, i.e. the difference of the Bachelier and Black-Merton-Scholes price (when using the same volatility $\sigma = 2.4\%$ p.a.) is of the order 10^{-8} of the price S_0 of the underlying security.

The above discussion pertains to the limiting behaviour, for $\sigma\sqrt{T} \to 0$, of the Bachelier and Black-Merton-Scholes prices of at the money options, i.e. , where $S_0 = K$. One might ask the same question for the case $S_0 \neq K$. Unfortunately, this question turns out not to be meaningful from a financial point of view. Indeed, if we fix $S_0 \neq K$ and let $\sigma\sqrt{T}$ tend to zero, then the Bachelier price as well as the Black-Scholes price tend to the pay-off function $(S_0 - K)_+$ of order higher than $(\sigma\sqrt{T})^n$, for every $n \geq 1$. Hence, in particular, their difference tends to zero faster than any power of $(\sigma\sqrt{T})$. This does not seem to us an interesting result and corresponds – in financial terms – to the fact that, for fixed $S_0 \neq K$ and letting $\sigma\sqrt{T}$ tend to zero, the hinked pay-off function $(S_0 - K)_+$ essentially amounts to the same as the linear pay-off functions $S_0 - K$, in the case $S_0 > K$, and 0, in the case $S_0 - K < 0$. In particular the functional dependence of the prices on $\sigma\sqrt{T}$ is not analytical. This behaviour is essentially different from the behaviour at the money, $S_0 = K$, since there the prices depend analytically on $\sigma\sqrt{T}$ and the difference tends to zero of order 3.

3. Further results of L. Bachelier

We now proceed to a more detailed analysis of the option pricing formula (2.2) for general strike prices K. Let $C = C_0^B$ denote the option price from (2.2). We shall introduce some notation used by L. Bachelier for the following two reasons: firstly, it should make the task easier for the interested reader to look up the original texts by Bachelier; secondly, and more importantly, we shall see that his notation has its own merits and allows for intuitive and economically meaningful interpretations (as we have already seen for the normalization $H = \frac{\sigma^B}{\sqrt{2\pi}}$ of the volatility, which equals the time-standardized price of an at the money option).

L. Bachelier found it convenient to use a parallel shift of the coordinate system moving S_0 to 0, so that the Gaussian distribution will be centered at 0. We write

$$a = \frac{\sigma^B\sqrt{T}}{\sqrt{2\pi}}, \quad m := K - S_0, \quad P := m + C. \tag{3.1}$$

The parameter a equals, up to the normalizing factor $\frac{S_0}{\sqrt{2\pi}}$, the *time standardized absolute volatility* $\sigma^B\sqrt{T}$ *at maturity* T. Readers familiar, e.g. with the Hull-White model of stochastic volatility, will realize that this is a very natural parametrization for an option with maturity T.

In any case, the quantity a was a natural parametrization for L. Bachelier, as it is the *price of the at the money option* with the maturity T (see formula 2.6), so that it can be directly observed from market data.

The quantity m is the difference between the strike price K and S_0 and needs no further explanation. P has a natural interpretation (in Bachelier's times it was called "écart", i.e. the "spread" of an option): it is the price P of a european put with maturity T and strike price K, as was explicitly noted by Bachelier (using, of course, different terminology). In today's terminology this amounts to the put-call parity. Bachelier interpreted P as the premium of an insurance against prices falling below K.

This is nicely explained in [2]: a speculator "á la hausse", i.e. hoping for a rise of S_T may buy a forward contract with maturity T. Using the "fundamental principle", which in this case boils down to elementary no arbitrage arguments, one concludes that the forward price must equal S_0, so that the total gain or loss of this operation is given by the random variable $S_T - S_0$ at time T.

On the other hand , a more prudent speculator might want to limit the maximal loss by a quantity $K > 0$. She thus would buy a call option with price C, which would correspond to a strike price K (here we see very nicely the above mentioned fact that in Bachelier's times the strike price was considered as a function of the option price C – la "prime" in french – and not vice versa as today). Her total gain or loss would then be given by the random variable

$$(S_T - K)_+ - C.$$

If at time T it indeed turns out that $S_T \geq K$, then the buyer of the forward contract is, of course, better off than the option buyer. The difference equals

$$(S_T - S_0) - [(S_T - K) - C] = K - S_0 + C = P,$$

which therefore may be interpreted as a "cost of insurance". If $S_T \leq K$, we obtain

$$(S_T - S_0) - [0 - C] = (S_T - K) + K - S_0 + C = (S_T - K) + P.$$

By the Bachelier's "fundamental principle" we obtain

$$P = E[(S_T - K)_-].$$

Hence Bachelier was led by no-arbitrage considerations to the put-call parity. For further considerations we denote the put price in the Bachelier model at time $t = 0$ by P_0^B or $P(m)$ respectively. Clearly, the higher the potential loss C is, which the option buyer is ready to accept, the lower the costs of insurance P should be and vice versa, so that we expect a monotone dependence of these two quantities.

In fact, Bachelier observed that the following pretty result holds true in his model (see [3], p.295):

Proposition 3 (Theorem of reciprocity)**.** *For fixed $\sigma^B > 0$ and $T > 0$ the quantities C and P are reciprocal in Bachelier's model, i.e. there is a monotone, strictly decreasing and self-inverse (that is $I = I^{-1}$) function $I : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $P = I(C)$.*

Proof. Denote by ψ the density of $S_T - S_0$, then

$$C(m) = \int_m^\infty (x - m)\psi(x)dx,$$
$$P(m) = \int_{-\infty}^m (m - x)\psi(x)dx.$$

Hence we obtain that $C(-m) = P(m)$. We note in passing that this is only due to the symmetry of the density ψ with respect to reflection at 0. Since $C'(m) < 0$ (see the proof of Proposition 1) we obtain $P = P(m(C)) := I(C)$, where $C \mapsto m(C)$ inverts the function $m \mapsto C(m)$. C maps $\mathbb{R}$ in a strictly decreasing way to $\mathbb{R}_{>0}$ and

P maps $\mathbb{R}$ in a strictly increasing way to $\mathbb{R}_{>0}$. The resulting map I is therefore strictly decreasing, and – due to symmetry – we obtain

$$I(P) = P(m(P)) = P(-m(C)) = C,$$

so I is self-inverse. □

Using the above notations (3.1), equation (2.2) for the option price C_0^B (which we now write as $C(m)$ to stress the dependence on the strike price) obtained from the fundamental principle becomes

$$C(m) = \int_m^\infty (x - m)\mu(dx), \tag{3.2}$$

where μ denotes the distribution of $S_T - S_0$, which has the Gaussian density $\mu(dx) = \psi(x)dx$,

$$\psi(x) = \frac{1}{\sigma^B\sqrt{2\pi T}} \exp(-\frac{x^2}{2(\sigma^B)^2 T}) = \frac{1}{2\pi a} \exp(-\frac{x^2}{4\pi a^2}).$$

As mentioned above, Bachelier does not simply calculate the integral (3.2). He rather does something more interesting (see [3, p. 294]): "Si l'on développe l'integrale en série, on obtient ", i.e. "if one develops the integral into a series one obtains..."

$$C(m) = a - \frac{m}{2} + \frac{m^2}{4\pi a} - \frac{m^4}{96\pi^2 a^3} + \frac{m^6}{1920\pi^3 a^5} + \ldots'' . \tag{3.3}$$

In the subsequent theorem we justify this step. It is worth noting that the method for developing this series expansion is not restricted to Bachelier's model, but holds true in general (provided that μ, the probability distribution of $S_T - S_0$, admits a density function ψ, which is analytic in a neighborhood of 0).

Theorem 1. *Suppose that the law μ of the random variable S_T admits a density*

$$\mu(dx) = \psi(x)dx,$$

such that ψ is analytic in a ball of radius $r > 0$ around 0, and that

$$\int_{-\infty}^\infty x\psi(x)dx < \infty.$$

Then the function

$$C(m) = \int_m^\infty (x - m)\mu(dx)$$

is analytic for $|m| < r$ and admits a power series expansion

$$C(m) = \sum_{k=0}^\infty c_k m^k,$$

where $c_0 = \int_0^\infty x\psi(x)dx$, $c_1 = -\int_0^\infty \psi(x)dx$ and $c_k = \frac{1}{k!}\psi^{(k-2)}(0)$ for $k \geq 2$.

Proof. Due to our assumptions C is seen to be analytic as sum of two analytic functions,

$$C(m) = \int_m^\infty x\psi(x)dx - m\int_m^\infty \psi(x)dx.$$

Indeed, if ψ is analytic around 0, then the functions $x \mapsto x\psi(x)$ and $m \mapsto \int_m^\infty x\psi(x)dx$ are analytic with the same radius of convergence r. The same holds true for the function $m \mapsto m\int_m^\infty \psi(x)dx$. The derivatives can be calculated by the Leibniz rule,

$$\begin{aligned} C'(m) &= -m\psi(m) - \int_m^\infty \psi(x)dx + m\psi(m) \\ &= -\int_m^\infty \psi(x)dx, \\ C''(m) &= \psi(m), \end{aligned}$$

whence we obtain for the k-th derivative,

$$C^{(k)}(m) = \psi^{(k-2)}(m),$$

for $k \geq 2$. □

Remark 1. *If we assume that $m \mapsto C(m)$ is locally analytic around $m = 0$ (without any assumption on the density ψ), then the density $x \mapsto \psi(x)$ is analytic around $x = 0$, too, by inversion of the above argument.*

Remark 2. *In the case when ψ equals the Gaussian distribution, the calculation of the Taylor coefficients yields*

$$\begin{aligned} \frac{d}{dy}\left(\frac{1}{2\pi a}\exp(-\frac{y^2}{4\pi a^2})\right) &= -\frac{1}{4}\frac{y}{\pi^2 a^3}e^{-\frac{1}{4}\frac{y^2}{\pi a^2}}, \\ \frac{d^2}{dy^2}\left(\frac{1}{2\pi a}\exp(-\frac{y^2}{4\pi a^2})\right) &= -\frac{1}{8}\frac{2\pi a^2 - y^2}{\pi^3 a^5}e^{-\frac{1}{4}\frac{y^2}{\pi a^2}}, \\ \frac{d^3}{dy^3}\left(\frac{1}{2\pi a}\exp(-\frac{y^2}{4\pi a^2})\right) &= \frac{1}{16}\frac{6\pi a^2 y - y^3}{\pi^4 a^7}e^{-\frac{1}{4}\frac{y^2}{\pi a^2}}, \\ \frac{d^4}{dy^4}\left(\frac{1}{2\pi a}\exp(-\frac{y^2}{4\pi a^2})\right) &= \frac{1}{32}\frac{12\pi^2 a^4 - 12y^2\pi a^2 + y^4}{\pi^5 a^9}e^{-\frac{1}{4}\frac{y^2}{\pi a^2}}, \end{aligned}$$

Consequently $\psi(0) = \frac{1}{2\pi a}$, $\psi'(0) = 0$, $\psi''(0) = -\frac{1}{4\pi^2 a^3}$, $\psi'''(0) = 0$ and $\psi''''(0) = \frac{3}{8}\frac{1}{\pi^3 a^5}$, hence with $C(0) = a$ and $C'(0) = -\frac{1}{2}$,

$$C(m) = a - \frac{m}{2} + \frac{m^2}{4\pi a} - \frac{m^4}{96\pi^2 a^3} + \frac{m^6}{1920\pi^3 a^5} + \mathcal{O}(m^8) \tag{3.4}$$

and the series converges for all m, as the Gaussian distribution is an entire function. This is the expansion indicated by Bachelier in [3]. *Since $P(-m) = C(m)$, we also obtain the expansion for the put*

$$P(m) = a + \frac{m}{2} + \frac{m^2}{4\pi a} - \frac{m^4}{96\pi^2 a^3} + \frac{m^6}{1920\pi^3 a^5} + \mathcal{O}(m^8). \tag{3.5}$$

Remark 3. *Looking once more at Bachelier's series one notes that it is rather a Taylor expansion in $\frac{m}{a}$ than in m. Note furthermore that $\frac{m}{a}$ is a dimensionless quantity. The series then becomes*

$$\begin{aligned} C(m) &= a\ F(\frac{m}{a}) \\ F(x) &= 1 - \frac{x}{2} + \frac{x^2}{4\pi} - \frac{x^4}{96\pi^2} + \frac{x^6}{1920\pi^3} + \mathcal{O}(x^8). \end{aligned}$$

We note as a curiosity that already in the second order term the number π appears. Whence – if we believe in Bachelier's formula for option pricing – we are able to *determine π at least approximately (see (3.6) below) – from financial market data (compare Georges Louis Leclerc Comte de Buffon's method to determine π by using statistical experiments).*

Let us turn back to Bachelier's original calculations. He first truncated the Taylor series (3.4) after the quadratic term, i.e.

$$C(m) \approx a - \frac{m}{2} + \frac{m^2}{4\pi a}. \tag{3.6}$$

This (approximate) formula can easily be inverted by solving a quadratic equation, thus yielding an explicit formula for m as a function of C. Bachelier observes that the approximation works well for small values of $\frac{m}{a}$ (the cases relevant for his practical applications) and gives some numerical estimates. We summarize the situation.

Proposition 4 (Rule of Thumb 1)**.** *For given maturity T, strike K and $\sigma^B > 0$, let $m = K - S_0$ and denote by $a = C(0)$ the Bachelier price of the at the money option and by $C(m)$ the Bachelier price of the call option with strike $K = S_0 + m$. Define*

$$\widehat{C}(m) = a - \frac{m}{2} + \frac{m^2}{4\pi a} \tag{3.7}$$

$$= C(0) - \frac{m}{2} + \frac{m^2}{4\pi C(0)}, \tag{3.8}$$

then we get an approximation of the Bachelier price $C(m)$ of order 4, i.e. $C(m) - \widehat{C}(m) = \mathcal{O}(m^4)$.

Remark 4. *Note that the value of $\widehat{C}(m)$ only depends on the price $a = C(0)$ of an at the money option (which is observable at the market) and the given quantity $m = K - S_0$.*

Remark 5. *Given any stock price model under a risk neutral measure, the above approach of quadratic approximation can be applied if the density ψ of $S_T - S_0$ is locally analytic and admits first moments. The approximation then reads*

$$\widehat{C}(m) = C(0) - B(0)m + \frac{\psi(0)}{2}m^2 \tag{3.9}$$

up to a term of order $\mathcal{O}(m^3)$. Here $C(0)$ denotes the price of the at the money european call option (pay-off $(S_T - S_0)_+$), $B(0)$ the price of the at the money binary option (pay-off $1_{\{S_T \geq S_0\}}$) and $\psi(0)$ the value of an at the money "Dirac" option (pay-off δ_{S_0}, with an appropriate interpretation as a limit). Notice that $B(0)$ can also be interpreted, in Bachelier's model, as the hedging ration Delta.

Example 1. *Take for instance the Black-Merton-Scholes model, then the terms of the quadratic approximation (3.9) can be calculated easily,*

$$\begin{aligned} C(0) &= S_0(\Phi(\frac{1}{2}\sigma^{BS}\sqrt{T}) - \Phi(-\frac{1}{2}\sigma^{BS}\sqrt{T})) \\ B(0) &= \Phi(-\frac{1}{2}\sigma^{BS}\sqrt{T}), \\ \psi(0) &= \frac{1}{\sigma^{BS}\sqrt{2\pi T}}\frac{1}{S_0}\exp(-\frac{1}{8}(\sigma^{BS})T). \end{aligned}$$

Notice that the density ψ of $S_T - S_0$ in the Black-Merton-Scholes model is not an entire function, hence the Taylor expansion only converges with a finite radius of convergence.

Although Bachelier had achieved with formula (3.7) a practically satisfactory solution, which allowed to calculate (approximately) m as a function of C by only using pre-computer technology, he was not entirely satisfied. Following the reflexes of a true mathematician he tried to obtain better approximations (yielding still easily computable quantities) than simply truncating the Taylor series after the quadratic term. He observed that, using the series expansion for the put option (see formula 3.5)

$$P(m) = a + \frac{m}{2} + \frac{m^2}{4\pi a} - \frac{m^4}{96\pi^2 a^3} + ...$$

and computing the product function $C(m)P(m)$ or, somewhat more sophisticatedly, the triple product function $C(m)P(m)\frac{C(m)+P(m)}{2}$, one obtains interesting cancellations in the corresponding Taylor series,

$$C(m)P(m) = a^2 - \frac{m^2}{4} + \frac{m^2}{2\pi} + \mathcal{O}(m^4),$$

$$C(m)P(m)\frac{C(m)+P(m)}{2} = a^3 - \frac{m^2 a}{4} + \frac{3m^2 a}{4\pi} + \mathcal{O}(m^4).$$

Observe that $(C(m)P(m))^{\frac{1}{2}}$ is the geometric mean of the corresponding call and put price, while $(C(m)P(m)\frac{C(m)+P(m)}{2})^{\frac{1}{3}}$ is the geometric mean of the call, the put and the arithmetic mean of the call and put price.

The latter equation yields the approximate identity

$$(C(m) + P(m))C(m)P(m) \approx 2a^3$$

which Bachelier rephrases as a cooking book recipe (see [3], p.201):

> On additionne l'importance de la prime et son écart.
>
> On multiplie l'importance de la prime par son écart.
>
> On fait le produit des deux résultats.
>
> Ce produit doit être le même pour toutes les primes
>
> qui ont même échéance,

i.e. "One adds up the call and put price, one multiplies the call and the put price, one multiplies the two results. The product has to be the same for all premia, which correspond to options of the same maturity a." This recipe allows to approximately calculate for $m \neq m'$ in a quadruple

$$(C(m), P(m), C(m'), P(m'))$$

any one of these four quantities as an easy (from the point of view of pre-computer technology) function of the other three. Note that, in the case $m = 0$, we have $C(0) = P(0) = a$, which makes the resulting calculation even easier.

We now interpret these equations in a more contemporary language (but, of course, only rephrasing Bachelier's insight in this way).

Proposition 5 (Rule of Thumb 2). *For given $T > 0$, $\sigma^B > 0$ and $m = K - S_0$ denote by $C(m)$ and $P(m)$ the prices of the corresponding call and put options in* *the Bachelier model. Denote by*

$$a(m) := C(m)P(m)$$
$$b(m) = C(m)P(m)(\frac{C(m)+P(m)}{2})$$

the products considered by Bachelier, then we have $a(0) = a^2$ *and* $b(0) = a^3$, *and*

$$\frac{a(m)}{a(0)} = 1 - \frac{(\pi-2)(\frac{m}{a})^2}{4\pi} + \mathcal{O}(m^4),$$
$$\frac{b(m)}{b(0)} = 1 - \frac{(\pi-3)(\frac{m}{a})^2}{4\pi} + \mathcal{O}(m^4).$$

Remark 6. *We rediscover an approximation of the reciprocity relation of Proposition 3.1 in the first of the two rules of the thumb. Notice also that this rule of thumb only holds up to order* $(\frac{m}{a})^2$. *Finally note that* $\pi - 3 \approx 0.1416$ *while* $\pi - 2 \approx 1.1416$, *so that the coefficient of the quadratic term in the above expressions is smaller for* $\frac{b(m)}{b(0)}$ *by a factor of* 8 *as compared to* $\frac{a(m)}{a(0)}$. *This is why Bachelier recommended this slightly more sophisticated product.*

4. Bachelier versus other models

Bachelier's model yields very good approximation results with respect to the Black-Scholes-model for at the money options. This corresponds to notions of weak (absolute or relative) errors of approximation in short time asymptotics, i.e. estimates of the quantitiy

$$|E(f(S_T^B)) - E(f(S_T^{BS}))|$$

for $T \to 0$. Compare [6] for the notion of weak and strond errors of approximation in the realm of numerical methods for SDE. This can be very delicate as seen in Proposition 2 and the remarks thereafter. However, in this section we concentrate on the easier notion of L^2-strong errors, which means estimates on the L^2-distance between models at given points in time T.

We ask two questions: first, how to generalize Bachelier's model in order to obtain better approximations of the Black-Scholes-model and second, how to extend this approach beyond the Black-Scholes model. Both questions can be answered by the methods from chaos expansion. There are possible extensions of the Bachelier model in several directions, but extensions, which improve – in an optimal way – the L^2-distance and the short-time asymptotics (with respect to a given model), are favorable from the point of view of applications. This observation in mind we aim for best (in the sense of L^2-distance) approximations of a given general process $(S_t)_{0 \leq t \leq T}$ by iterated Wiener-Ito integrals up to a certain order. We demand that the approximating processes are martingales to maintain no arbitrage properties. The methodology results into the one of chaos expansion or Stroock-Taylor Theorems (see [8]). Methods from chaos expansion for the (explicit) construction of price processes have already proved to be very useful, see for [7]. In the sequel we shall work on a probability space $(\Omega, \mathcal{F}, P)$ carrying a one-dimensional Brownian motion $(W_t)_{0 \leq t \leq T}$ with its natural filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. For all necessary details on Gaussian spaces, chaos expansion, n-th Wiener chaos, etc, see [7]. Certainly, the following considerations easily generalize to multi-dimensional Brownian motions.

We shall call any Gaussian martingale in this setting a *(general) Bachelier model.*

Definition 1. *Fix $N \geq 1$. Denote by $(M_t^{(n)})_{0 \leq t \leq T}$ martingales with continuous trajectories for $0 \leq n \leq N$, such that $M_t^{(n)} \in \mathcal{H}_n$ for $0 \leq t \leq T$, where $\mathcal{H}_n$ denotes the n-th Wiener chaos in $L^2(\Omega)$. Then we call the process*

$$S_t^{(N)} := \sum_{n=0}^{N} M_t^{(n)}$$

an extension of degree N of the Bachelier model. *Note that $M_t^{(0)} = M^{(0)}$ is constant, and that $S_t^{(1)} = M^{(0)} + M_t^{(1)}$ is a (general) Bachelier model.*

Given an L^2-martingale $(S_t)_{0 \leq t \leq T}$ with $S_0 > 0$ and $N \geq 1$. There exists a unique extension of degree N of the Bachelier model (in the sense that the martingales $(M_t^{(n)})_{0 \leq t \leq T}$ are uniquely defined for $0 \leq n \leq N$) minimizing the L^2-norm $E[(S_t - S_t^{(N)})^2]$ for all $0 \leq t \leq T$. Furthermore $S_t^{(N)} \to S_t$ in the L^2-norm as $N \to \infty$, uniformly for $0 \leq t \leq T$.

Indeed, since the orthogonal projections $p_n : L^2(\Omega, \mathcal{F}_T, P) \to L^2(\Omega, \mathcal{F}_T, P)$ onto the n-th Wiener chaos commute with conditional expectations $E(.|\mathcal{F}_t)$ (see for instance [8]), we obtain that

$$M_t^{(n)} := p_n(S_t),$$

for $0 \leq t \leq T$, is a martingale with continuous trajectories, because

$$E(p_n(S_T)|\mathcal{F}_t) = p_n(E(S_T|\mathcal{F}_t) = p_n(S_t)$$

for $0 \leq t \leq T$. Consequently

$$S_t^{(N)} := \sum_{n=0}^{N} M_t^{(n)}$$

is a process minimizing the distance to S_t for any $t \in [0, T]$. Clearly we have that

$$S_t := \sum_{n=0}^{\infty} M_t^{(n)}$$

in the L^2-topology.

Example 2. *For the Black-Merton-Scholes model with $\sigma^{BS} = \sigma$ we obtain that*

$$M_t^{(n)} = S_0 \sigma^n t^{\frac{n}{2}} H_n(\frac{W_t}{\sqrt{t}}),$$

where H_n denotes the n-th Hermite polynomial, i.e. $(n+1)H_{n+1}(x) = xH_n(x) - H_{n-1}(x)$ and $H_0(x) = 1$, $H_1(x) = x$ for $n \geq 1$ (for details see [9]*). Hence*

$$\begin{aligned} M_t^{(0)} &= S_0, \\ M_t^{(1)} &= S_0 \sigma W_t, \\ M_t^{(2)} &= S_0 \frac{\sigma^2}{2}(W_t^2 - t). \end{aligned}$$

We recover Bachelier's model as extension of degree 1 minimizing the distance to the Black-Merton-Scholes model. Note that we have

$$||S_t - S_t^{(N)}||_2 \leq C_N S_0 \sigma^{N+1} t^{\frac{N+1}{2}},$$

for $0 \leq t \leq T$. Furthermore we can calculate a sharp constant C_N, namely

$$C_N^2 = \sup_{0\leq t\leq T} \sum_{m\geq N+1} (\sigma\sqrt{t})^{2(m-N-1)} E[(H_m(\frac{W_t}{\sqrt{t}}))^2]$$
$$= \sum_{m\geq N+1} (\sigma\sqrt{t})^{2(m-N-1)} \frac{1}{m!}.$$

since $E[(H_m(\frac{W_t}{\sqrt{t}}))^2] = \frac{1}{m!}$ for $0 < t \leq T$ and $m \geq 0$.

Due to the particular structure of the chaos decomposition we can prove the desired short-time asymptotics:

Theorem 2. *Given an L^2-martingale $(S_t)_{0\leq t\leq T}$, assume that $S_T = \sum_{i=0}^{\infty} W_i(f_i)$ with symmetric L^2-functions $f_i : [0,T]^i \to \mathbb{R}$ and iterated Wiener-Ito integrals*

$$W_t^i(f_i) := \int_{0\leq t_1\leq\cdots\leq t_i\leq t} f_i(t_1,\ldots,t_i)dW_{t_1}\cdots dW_{t_i}.$$

If there is an index i_0 such that for $i \geq i_0$ the functions f_i are bounded and $K^2 := \sum_{i\geq i_0} \frac{(T)^{i-i_0}}{i!}||f_i||_\infty^2 < \infty$, then we obtain $||S_t - S_t^{(N)}||_2 \leq K t^{\frac{n+1}{2}}$ for each $N \geq i_0$ and $0 \leq t \leq T$.

Proof. We apply that $E[(E(W_T^i(f_i)|\mathcal{F}_t))^2] = \frac{1}{i!}||1_{[0,t]}^{\otimes i} f_i||^2_{L^2([0,T]^i)} \leq \frac{t^i}{i!}||f_i||_\infty^2$ for $i \geq i_0$. Hence the result by applying

$$||S_T - S_T^{(N)}||_2 \leq \sum_{i\geq i_0} E[(E(W_T^i(f_i)|\mathcal{F}_T))^2] \leq KT^{\frac{N+1}{2}}.$$

□

Remark 7. *Notice that the Stroock-Taylor Theorem (see [8], p.161) tells that for $S_T \in \mathcal{D}^{2,\infty}$ the series*

$$\sum_{i=0}^{\infty} W_T^i((t_1,\ldots,t_i) \mapsto E(D_{t_1,\ldots,t_i} S_T)) = S_T$$

converges in $\mathcal{D}^{2,\infty}$. Hence the above condition is a statement about boundedness of higher Malliavin derivatives. The well-known case of the Black-Merton-Scholes model yields

$$D_{t_1,\ldots,t_i} S_T^{BS} = 1_{[0,T]}^{\otimes i}$$

for $i \geq 0$, so the condition of the previous theorem is satisfied.

References

[1] Ananthanarayanan, A.L. and Boyle, Ph., *The Impact of Variance Estimation on Option Valuation Models*, Journal of Financial Economics **6**, 4, 375-387, 1977.

[2] Bachelier, L., *Theorie de la Speculation*, Paris, 1900, see also: http://www.numdam.org/en/

[3] Bachelier, L., *Calcul de Probabilités*, Paris, 1912, Les Grands Classiques Gauthier-Villars, Editions Jacques Gabay 1992.

[4] Black, F., *The Pricing of Commodity Contracts*, Journal of Financial Economics **3**, 167–179 (1976).

[5] Delbaen, F. and Schachermayer, W., *The mathematics of Arbitrage*, Springer Finance (2005).

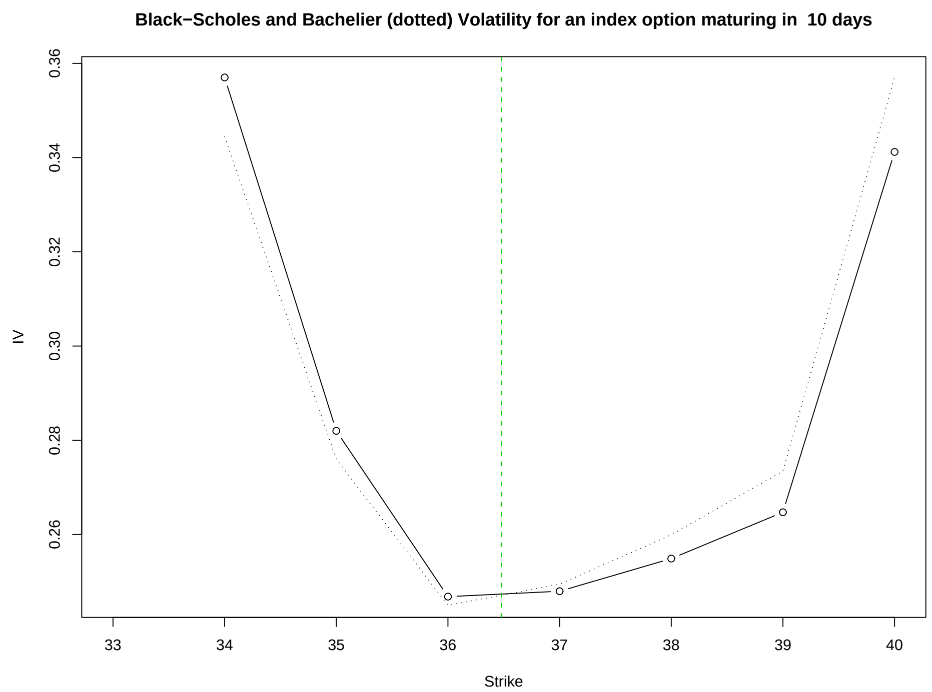
[6] P. E. Kloeden and E. Platen, *Numerical solution of stochastic differential equations*, Applications of Mathematics, **23**, Springer-Verlag, Berlin (1992).

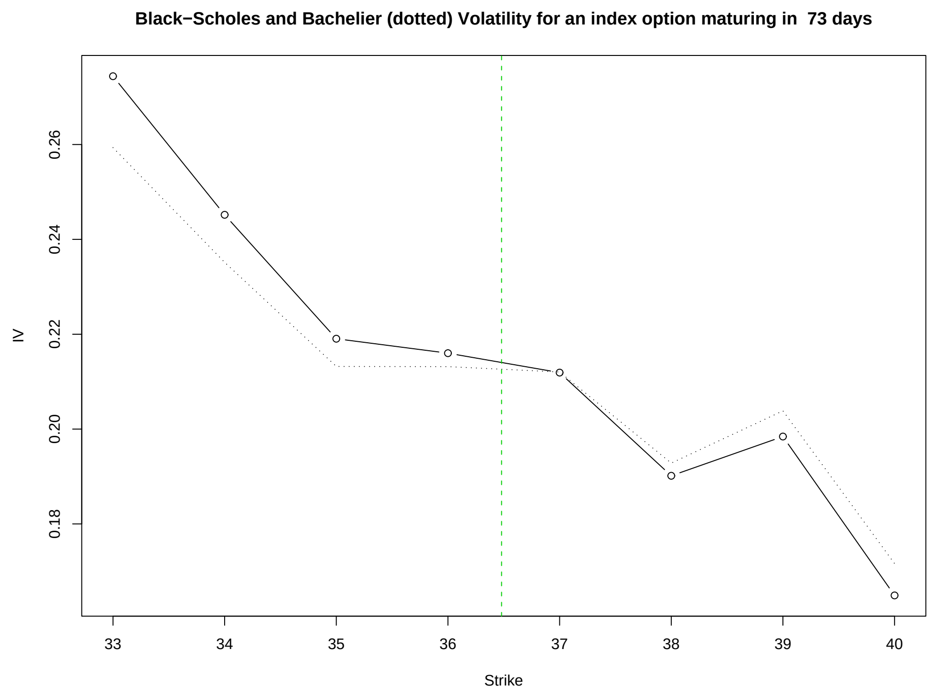
[7] Malliavin, P. and Thalmaier, A., *Stochastic Calculus of Variations in Mathematical Finance*, Springer Finance, Springer Verlag (2005).

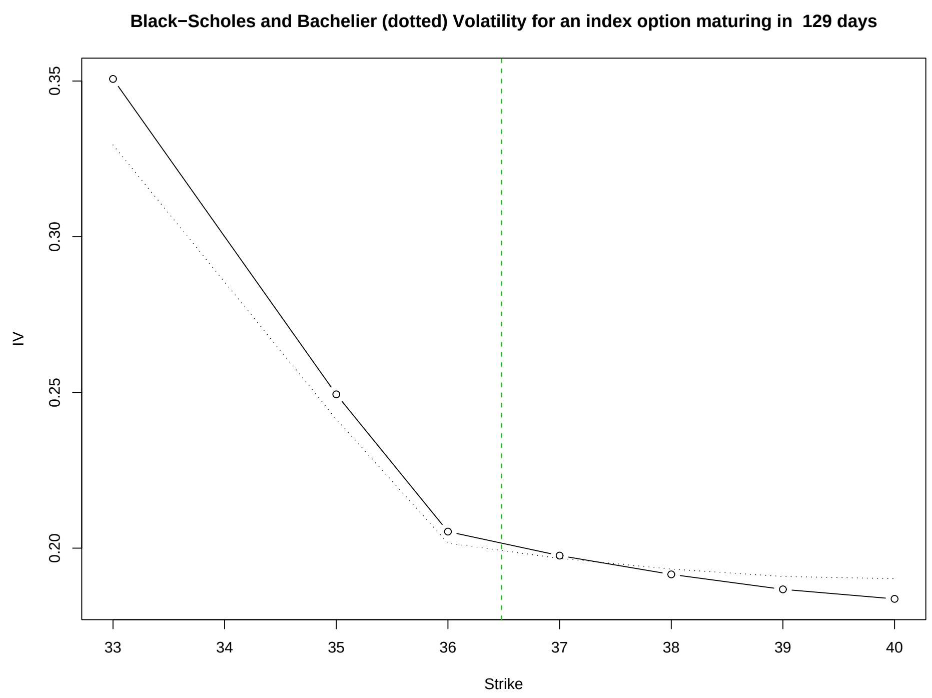
[8] Malliavin, P., *Stochastic Analysis*, Grundlehren der mathematischen Wissenschaften **313**, Springer Verlag (1995).
[9] Nualart, D., *The Malliavin Calculus and Related Topics*, Springer-Verlag (1995).
[10] Samuelson, P. A., *Proof that Properly Anticipated Prices Fluctuate Randomly*, Industrial Management Review **6**, 41–49 (1965).
[11] Schachermayer, W., *Introduction to Mathematics of financial markets*, Lecture Notes in Mathematics 1816 - Lectures on Probability Theory and Statistics, Saint-Flour summer school 2000 (Pierre Bernard, editor), Springer Verlag, Heidelberg, pp. 111-177.

5. Appendix

We provide tables with Bachelier and Black-Scholes implied volatilities for different times to maturity, where we have used Nasdaq-index-option data in discounted prices in order to compare the results. Circles represent the Black-Scholes implied volatility of data points above a certain level of trade volume. The dotted line represents the implied Bachelier (relative) volatility $\sigma^B = S_0\sigma_{rel}$.




Black−Scholes and Bachelier (dotted) Volatility for an index option maturing in 73 days
IV
0.18
0.20
0.22
0.24
0.26
33
34
35
36
37
38
39
40
Strike


Black−Scholes and Bachelier (dotted) Volatility for an index option maturing in 129 days
IV
0.20
0.25
0.30
0.35
33
34
35
36
37
38
39
40
Strike

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