Semi-Static Completeness and Model-independent Pricing by Informed Investors

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Outline

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2. Setup
3. Semi-static completeness and the Jacod-Yor theorem
4. Semi-static completeness and filtration structure
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6. Conclusions
Model-independent framework:

- $X$: path-space, $S$: canonical process on $X$
- $\Psi$: set of claims $\psi$ available for buy-and-hold trading
- $\mathcal{M}$: martingale measures consistent w/ the market price of $\psi$’s
- $\Phi$: a given derivative, robust pricing: $\sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_\mathbb{Q} [\Phi]$
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A central problem in model-independent finance is to prove:

$$\sup_{Q \in \mathcal{M}} \mathbb{E}_Q [\Phi] = \inf \left\{ c \in \mathbb{R} : \Phi \text{ can be hedged pathwise starting with initial capital } c \right\}$$

Beiglböck, H.-Labordère, Penkner ‘13; Galichon, H.-Labordère, Touzi ‘14; Acciaio, Beiglböck, Penkner, Schachermayer ‘13; Bouchard, Nutz ‘13; Dolinsky, Soner ‘14a,‘14b; Beiglböck, Cox, Huesmann ‘14; Biagini, Bouchard, Kardaras, Nutz ‘14; Beiglböck, Nutz, Touzi ‘15; Guo, Tan, Touzi ‘15; Hou, Obłój ‘15; Beiglböck, Cox, Huesmann, Perkowski, Prömel ‘15, Beiglböck, Nutz, Touzi ’15,...
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- Note: $\mathcal{M}$ clearly depends on the underlying filtration, as does the set of available trading strategies.

- Question: What can be said about the relation between the super-hedging price and the choice of filtration? In particular, when passing from $\mathcal{F}$ to $\mathcal{G} \supseteq \mathcal{F}$?
Insider information

- Uninformed agent $F \subseteq G$ Informed agent
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- How do things change?

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- Informed agent has more trading strategies
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- **Question**: Which measures in $\mathcal{M}(\mathcal{F})$ are still relevant for pricing for the informed agent?
Setup

- \((\Omega, \mathbb{F}, \mathcal{F})\): Filtered measurable space with \(\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) right-continuous.
  - Later we will consider other filtrations.

- \(S = (S_t)_{0 \leq t \leq T}\): càdlàg \(\mathbb{F}\)-adapted discounted price process of an asset available for dynamic trading. We assume \(S_0 = 0\). (Everything works the same for multiple assets.)

- A risk-free asset with price \(\equiv 1\) available for dynamic trading.

- \(\Psi = \{\psi_1, \ldots, \psi_n\}\) a set of \(\mathcal{F}_T\)-measurable payoffs available for buy-and-hold trading. Today’s price of \(\psi_i\) is zero for each \(i\).
Martingale measures

Calibrated martingale measures:

\[ \mathcal{M}(\mathcal{F}) = \left\{ Q \in \mathcal{P} : \begin{array}{l}
\text{S is an } \mathcal{F}\text{-martingale, } \mathbb{E}_Q[S_T^2] < \infty, \\
\mathbb{E}_Q[\psi | \mathcal{F}_0] = 0, \mathbb{E}_Q[\psi^2] < \infty \text{ for all } \psi \in \Psi
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- We want to study \( \mathcal{M}(\mathcal{F}) \) w.r.to \( \mathcal{F} \)
- \( \mathcal{M}(\mathcal{F}) \) is “huge”
  
  \( \hookrightarrow \) Can we reduce to the study of a special subset?
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- \( \mathcal{M}(\mathcal{F}) \) is “huge”
  - Can we reduce to the study of a special subset?
  - For example, if \( \mathcal{P} \) is endowed with a topology s.t. \( \mathcal{M}(\mathcal{F}) \) is compact, then
    \[ \mathcal{M}(\mathcal{F}) = \text{conv}(\text{ext } \mathcal{M}(\mathcal{F})), \]
    where \( \text{ext } \mathcal{M}(\mathcal{F}) \) is the set of all extreme points in \( \mathcal{M}(\mathcal{F}) \).
Extreme points

**Extreme points:** $Q \in \mathcal{M}(\mathbb{F})$ is called an extreme point if

$$Q = \lambda Q^1 + (1 - \lambda) Q^2$$

for $Q^i \in \mathcal{M}(\mathbb{F}), \lambda \in (0, 1)$

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- Consider an $\mathcal{F}_T$-measurable payoff $\Phi$ and endow $\mathcal{P}$ with a topology such that
  - $\mathcal{M}(\mathcal{F})$ is compact and $Q \mapsto \mathbb{E}_Q[\Phi]$ is continuous.

  Then $\sup_{Q \in \mathcal{M}(\mathcal{F})} \mathbb{E}_Q[\Phi] = \sup_{Q \in \text{ext} \mathcal{M}(\mathcal{F})} \mathbb{E}_Q[\Phi]$. 
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  Then
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  \sup_{Q \in \mathcal{M}(\mathbb{F})} \mathbb{E}_Q[\Phi] = \sup_{Q \in \text{ext } \mathcal{M}(\mathbb{F})} \mathbb{E}_Q[\Phi].
  \]

- **Note:** The notion of extreme point is purely algebraic, independent of any topology we may put on the space of probability measures.
### Example (Discrete time and bounded prices)

- $\Omega = [a, b]^T$, $S$ is the coordinate process,
- each $\omega \mapsto \psi_i(\omega)$ is continuous,
- $\mathcal{F}$ is generated by $S$

Then $\mathcal{M}(\mathcal{F})$ is weakly compact.
Examples

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Example (Continuous time and bounded volatility)

- $\Omega = C_0[0, T]$, $S$ is the coordinate process,
- $\omega \mapsto \psi_i(\omega)$ bounded and continuous, $\mathcal{F}$ generated by $S$
- $\mathcal{P} = \{ Q : \mathbb{E}_Q\left[ X \sup_{s \leq u \leq t} |S_u - S_s|^p \right] \leq C_p \sigma^p (t - s)^{p/2} \mathbb{E}_Q [X] \}$, for all $0 \leq s < t \leq T$, $X \geq 0$ $\mathcal{F}_s$-measurable, $p \geq 1$.

Then $\mathcal{M}(\mathcal{F})$ is weakly compact.
Example (Jakubowski topology)

- $\Omega = D_0([0, T], [-1, 1])$ with Jakubowski’s S-topology,
- $S$ is the coordinate process, $\psi_i$ suitable continuity conditions,
- $\mathcal{F}$ is generated by $S$

Semi-static completeness and the Jacod-Yor theorem
The classical Jacod-Yor theorem

Suppose $\Psi = \emptyset$ (no static claims).

For $Q \in \mathcal{M}(\mathbb{F})$, by the classical Jacod-Yor (1977) theorem:

$$Q \in \text{ext } \mathcal{M}(\mathbb{F}) \iff L^2(\mathcal{F}_T) = \{x + (H \cdot S)_T : H \in L^2(S)\}$$

This result can be generalized to the semi-static case.
Generalization of the Jacod-Yor theorem

Definition

For $Q \in \mathcal{M}(\mathbb{F})$, we say that **semi-static completeness** holds if any $X \in L^2(\mathcal{F}_T)$ can be represented as

$$X = x + a_1 \psi_1 + \cdots + a_n \psi_n + (H \cdot S)_T$$

for some $x, a_1, \ldots, a_n \in \mathbb{R}$ and $H \in L^2(S)$.

Notation:

$$\text{SSC}(\mathbb{F}) = \{Q \in \mathcal{M}(\mathbb{F}) : \text{semi-static completeness holds}\}$$
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**Theorem (semi-static Jacod-Yor theorem)**

*The extreme martingale measures are exactly the semi-statically complete models, i.e.*

$$\text{ext } \mathcal{M}(\mathcal{F}) = \text{SSC}(\mathcal{F}).$$
Generalization of the Jacod-Yor theorem

About the proof.

- The proof is very close to the classical case …
- … but uses duality for random variables \((L^1 - L^\infty)\) instead of processes \((H^1 - BMO)\):
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About the proof.

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- . . . but uses duality for random variables ($L^1 - L^\infty$) instead of processes ($H^1 - BMO$):

1. Fix $Q \in \text{ext } \mathcal{M}(\mathbb{F})$ and show that this set is dense in $L^1(\mathcal{F}_T)$

\[ \{ x + \sum_i a_i \psi_i + (H \cdot S)_T : x, a_i \in \mathbb{R}, H \in L^2(S) \} . \]
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\]

2. Prove it is dense and closed in \(L^2(\mathcal{F}_T)\) using Hahn-Banach and a result by Yor (see also Delbaen/Schachermayer, 1999):

**Theorem (Yor (1978))**

Let \(H^n \in L(S)\) be such that \(H^n \cdot S\) is a martingale for each \(n\), and suppose \(\lim_n (H^n \cdot S)_T = X\) in \(L^1\) for some r.v. \(X\). Then there is \(H \in L(S)\) such that \(H \cdot S\) is a martingale with \((H \cdot S)_T = X\).
Remarks.

- Infinitely many $\psi_i$’s would allow to treat the case of a fixed (by the market) marginal law $S_T \sim \mu$

- But the arguments we use in the above proof break down in this case – for the moment we are only able to deal with finitely many $\psi_i$’s
Generalization of the Jacod-Yor theorem

Can we say more?

- Already in the classical case ($\Psi = \emptyset$), completeness is a strong property, but yet we do not have “control” on the complete models. For instance, completeness holds if $\mathbb{F} = \mathbb{F}^S$, and $S$ is a strong solution to an SDE of the form

$$
    dS_t = \sigma(t; S_u : u \leq t) dW_t, \quad (W_t)_{t \geq 0} \text{ BM, } \sigma > 0.
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Generalization of the Jacod-Yor theorem

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- Should we expect some additional structure in the semi-static case? – We shall see an interesting consequence of SSC
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**Notation:** For any martingale $N$, denote

\[ S(N) = \left\{ H \cdot N : H \in L^2(N) \right\}. \]

This is a closed subspace of $H^2$ (stable subspace generated by $N$).
A curious consequence of semi-static completeness

- For simplicity let $\Psi = \{\psi\}$, and fix $Q \in \text{SSC}(\mathcal{F})$
- Let $K \cdot S$ be the orthogonal projection of $\mathbb{E}_Q[\psi | \mathcal{F}_t]$ onto $S(S)$, and define
  \[ M_t = \mathbb{E}_Q[\psi | \mathcal{F}_t] - (K \cdot S)_t \]

**Note:** $M_T$ is the part of $\psi$ which is not replicable by trading on $S$
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- By semi-static completeness,
  \[ \mathcal{H}^2 = \text{span}\{1\} \oplus \text{span}\{M\} \oplus S(S) \]
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- By semi-static completeness,
  $$\mathcal{H}^2 = \text{span}\{1\} \oplus \text{span}\{M\} \oplus S(S)$$

- Consequently,
  $$S(M) = \text{span}\{M\},$$

  which is one-dimensional!
A curious consequence of semi-static completeness

We will use the following result on \( \psi \):

Lemma

Let \( N \) be a square-integrable martingale null at zero. The following are equivalent:

(i) \( S(N) = \text{span}\{N\} \)

(ii) \( N = N_T \mathbf{1}_{B \times [t^*, T]} \) for some \( t^* \in (0, T] \) and some atom \( B \) of \( \mathcal{F}_{t^*} \)
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(ii) $N = N_T^1_{B \times [t^*, T]}$ for some $t^* \in (0, T]$ and some atom $B$ of $\mathcal{F}_{t^*}$

And the following one on $S$, when $S$ is continuous:

**Lemma**

Let $N$ be a continuous local martingale, and let $B$ be an atom of $\mathcal{F}_{t^*}$ for some $t^* \in (0, T]$. Then $N_t = N_0$ on $B$ for all $t < t^*$. 
A curious consequence of semi-static completeness

Recall: $\Psi = \{\psi\}$, $Q \in \text{SSC}(\mathcal{F})$. Now, for $S$ continuous we have

$$M = M_T 1_{B \times [t^*, T]} \quad \text{and} \quad S_t = S_0 \quad \text{on} \quad B \quad \text{for} \quad t \leq t^*$$
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By semi-static completeness,

\[
1_B = Q(B) + aM_T + (H \cdot S)_T
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$$1_B = \mathbb{E}_Q \left[ Q(B) + aM_T + (H \cdot S)_T | \mathcal{F}_{t^*} \right] = Q(B) + (H \cdot S)_{t^*}$$
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$$= Q(B) \mathbf{1}_B \quad \Rightarrow \quad Q(B) = 1.$$
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Semi-static completeness and filtration structure
Atomic tree

- Fix $Q \in \mathcal{M}(\mathcal{F})$
- For $A \in \mathcal{F}_T$, denote by $t(A)$ the first time $A$ becomes measurable, 
  $$t(A) = \inf\{t \in [0, T] : A \in \mathcal{F}_t\}.$$

**Definition**

An **atomic tree** is a finite collection $T$ of events in $\mathcal{F}_T$ s.t.:

(i) every $A \in T$ is a non-null atom of $\mathcal{F}_{t(A)}$;
(ii) $\forall A, A' \in T$ s.t. $t(A) < t(A')$, either $A \supseteq A'$ or $A \cap A' = \emptyset$;
(iii) $\forall A, A' \in T$ such that $A \supseteq A'$, $Q(A \setminus A') > 0$;
(iv) the leaves form a partition of $\Omega$ (up to nullsets), and $A$ is an atom of $\mathcal{F}_{t(A')}$ whenever $A'$ is a child of $A$.

**leaf:** $A \in T$ s.t. there is no $A' \in T$ s.t. $A' \subset A$;  
**dim $T$:** # leaves

**child:** $A'$ is a child of $A$ if $A, A' \in T$ satisfy $A' \subset A$ and there is no $A'' \in T$ such that $A' \subset A'' \subset A$
Atomic tree
Atomic tree

Remarks.

- $\sigma(T)$ is well-defined. It can be described as $\sigma(T) = F_{\zeta(T)}$, where the stopping time $\zeta(T)$ is the “end” of the tree:

$$\zeta(T) = \sum_{A \in T \text{ is a leaf}} t(A)1_A.$$

- Note that $\dim T = \dim L^2(\sigma(T))$. 
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- Note that $\dim T = \dim L^2(\sigma(T))$.

Definition

We say that $S$ is complete on $A \times [t, T]$ for given $t \in [0, T]$ and $A \in \mathcal{F}_t$ if any $X \in L^2(\mathcal{F}_T)$ can be dynamically replicated there:

$$X = x + (H \cdot S)_T \quad \text{on} \quad A$$

for some $x \in \mathbb{R}$ and some $H \in L^2(S)$ with $H = 0$ on $[0, t]$. 
Recall: $Q \in \mathcal{M}(\mathbb{F})$ is fixed.

**Theorem**

Let $S$ be continuous. Then $Q \in \text{SSC}(\mathbb{F})$ IFF $\exists$ an atomic tree $T$ s.t.

1. $\{ \mathbb{E}_Q[\psi_i | \sigma(T)] : i = 1, \ldots, n \}$ has dim $T - 1$ lin. indep. elements,
2. $S$ is complete on $A \times [t(A), T]$ for each leaf $A \in T$.

In this case, $S$ is constant on $[0, \zeta(T)]$ and

$$L^2(\mathcal{F}_T) = \text{span}\{1, \Psi\} + S(S) = L^2(\sigma(T)) \oplus S(S).$$

**Remark:** $\psi_i = \mathbb{E}_Q[\psi_i | \sigma(T)] + (H^i \cdot S)_T$, $i = 1, \ldots, n$. orthog. proj.
Semi-static completeness for continuous price processes

The filtration $\mathcal{F}$ under $Q \in SSC(\mathcal{F})$. Each set of lines emanating from the leaves of $T$ corresponds to a dynamically complete stock price model.
### Semi-static completeness for continuous price processes

#### Example (Semi-statically complete continuous model)

One static claim $\psi = \langle S, S \rangle_T - K$ with zero value at $t = 0$.

- Pick $t^* \in (0, T)$, $\sigma_1, \sigma_2 > 0$ with $\sigma_1 \neq \sigma_2$.
- Set $Q = \lambda Q_1 + (1 - \lambda) Q_2$ where
  \[ S_t = \sigma_i W_{t-t^*} 1_{\{t \geq t^*\}} \] under $Q^i$,
  where $W$ is Brownian motion, and $\lambda$ is determined by calibration:
  \[ 0 = E_Q[\psi | F_0] = \lambda \sigma_1^2 (T - t^*) + (1 - \lambda) \sigma_2^2 (T - t^*) - K. \]

- Define $A_i = \{ \partial^+ \langle S, S \rangle_{t^*} = \sigma_i^2 \}$ and set $T = \{ \Omega, A_1, A_2 \}$.

- $T$ is an atomic tree with dim $T = 2$ and
  \[ E_Q[\psi | \sigma(T)] = \sigma_1^2 (T - t^*) 1_{A_1} + \sigma_2^2 (T - t^*) 1_{A_2} - K \neq 0. \]

- By the theorem, $Q \in SSC(\mathbb{F})$. 
The leaves $A_1, A_2$ correspond to Bachelier models with volatilities $\sigma_1 > \sigma_2$. Thus the “variance swap” $\psi = \langle S \rangle_T$ is priced differently under the two models, and can be used to hedge against $A_1$ or $A_2$. 
Example (Semi-statically complete jump model, but no atomic tree)

- $\psi = [S, S]_T - K$
- $S_t = \begin{cases} 
-t & t < \theta \land t^* \\
1 - \theta + f(\theta) W_{t-\theta} & t \geq \theta, \ \theta < t^* \\
-t^* + 1_{A_1} \sigma_1 W_{t-t^*} + 1_{A_2} \sigma_2 W_{t-t^*} & t \geq t^*, \ t^* \leq \theta 
\end{cases}$

with $\theta \sim \text{Exp}(1)$, $W$, $t^*$, $\sigma_1$, $\sigma_2 > 0$ as above, $f(t) : [0, t^*) \rightarrow \mathbb{R}_+$. 

**Conclusion:** When the asset is allowed to jump, we do not have anymore the tree structure.
Pricing by informed investors
\[ \mathcal{G} = (\mathcal{G}_t)_{0 \leq t \leq T} : \text{right-continuous filtration (of the informed agent) with} \]

\[ \mathcal{F}_t \subseteq \mathcal{G}_t, \quad 0 \leq t \leq T. \]

- Access to the same trading instruments: risk-free asset, \( S \), \( \Psi \).
\[ G = (G_t)_{0 \leq t \leq T} \text{: right-continuous filtration (of the informed agent) with} \]
\[ \mathcal{F}_t \subseteq G_t, \quad 0 \leq t \leq T. \]

- Access to the same trading instruments: risk-free asset, \( S \), \( \Psi \)

- Consider a payoff \( \Phi \). The robust super-hedging price of the informed agent:
\[
\sup_{Q \in \mathcal{M}(G)} \mathbb{E}_Q[\Phi]
\]

- As before, we want to study \( \text{ext} \mathcal{M}(G) \equiv \text{SSC}(G) \).
Setup

- $\mathcal{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$: right-continuous filtration (of the informed agent) with
  \[ \mathcal{F}_t \subseteq \mathcal{G}_t, \quad 0 \leq t \leq T. \]
- Access to the same trading instruments: risk-free asset, $S$, $\Psi$
- Consider a payoff $\Phi$. The robust super-hedging price of the informed agent:
  \[ \sup_{Q \in \mathcal{M}(\mathcal{G})} \mathbb{E}_Q[\Phi] \]
- As before, we want to study $\text{ext} \mathcal{M}(\mathcal{G}) \equiv \text{SSC}(\mathcal{G})$.

**Question:** How are $\text{SSC}(\mathcal{G})$ and $\text{SSC}(\mathcal{F})$ related?
Progressive filtration enlargement

**Specification of G**: Progressive enlargement of $\mathcal{F}$ with $\mathcal{H}$

$$
\mathcal{G}_t = \bigcap_{u > t} \mathcal{F}_u \vee \mathcal{H}_u.
$$

Smallest right-continuous filtration that contains both $\mathcal{F}$ and $\mathcal{H}$.

- $\mathcal{H}$ generated by a collection of single-jump processes $X 1_{[\tau, T]}$, where $X$ is a non-negative bounded random variable and $\tau$ is a random time (that is, $[0, T] \cup \{\infty\}$-valued random variable). (W.l.g., suppose $\tau = \infty$ on $\{X = 0\}$.)

- Remark: **special cases** are the classical progressive enlargement with a random time and initial enlargement with a random variable.
Progressive filtration enlargement

**Specification of \( \mathcal{G} \):** Progressive enlargement of \( \mathcal{F} \) with \( \mathcal{H} \)

\[
\mathcal{G}_t = \bigcap_{u > t} \mathcal{F}_u \lor \mathcal{H}_u.
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Smallest right-continuous filtration that contains both \( \mathcal{F} \) and \( \mathcal{H} \).

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- Remark: **special cases** are the classical progressive enlargement with a random time and initial enlargement with a random variable.

- For this kind of filtration enlargement there are clear-cut results between \( \text{SSC}(\mathcal{G}) \) and \( \text{SSC}(\mathcal{F}) \).
Progressive filtration enlargement

Let $\sigma$ be the first time $S$ starts to move: $\sigma = \inf\{t \in [0, T] : S_t \neq 0\}$.

**Theorem**

Let $S$ be continuous and $\mathbb{H}$ generated by $X_k 1_{[\tau_k, T]}$, $k = 1, \ldots, p$. Assume $\tau_k > \sigma$ on $\{0 < \tau_k < \infty\}$ for all $k$. Then

$$\text{SSC}(G) = \{ Q \in \text{SSC}(F) : F = G \text{ under } Q \}$$
Progressive filtration enlargement

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**Theorem**

Let $S$ be continuous and $\mathbb{H}$ generated by $X_k 1_{[\tau_k, T]}$, $k = 1, \ldots, p$. Assume $\tau_k > \sigma$ on $\{0 < \tau_k < \infty\}$ for all $k$. Then

$$\text{SSC}(\mathbb{G}) = \{ Q \in \text{SSC}(\mathbb{F}): \mathbb{F} = \mathbb{G} \text{ under } Q \}$$

In the proof we use an extension of the classical Jeulin-Yor theorem.

- Fix $Q \in \text{SSC}(\mathbb{G})$
- Let $Z$ be the Azéma supermartingale: $Z_t = Q(\tau > t | \mathbb{F}_t)$
- Let $A$ be is the dual predictable projection of $X 1_{[\tau, \infty]}$

**Theorem (Jeulin-Yor (1978))**

The following process is a $\mathbb{G}$-martingale w.r.to $Q$:

$$M_t = X 1_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \frac{1}{Z_s} dA_s.$$
Progressive filtration enlargement

Sketch of the proof of “⊆” (for \( p = 1, X \equiv 1 \))

- Fix \( Q \in \text{SSC}(\mathcal{G}) \)
- Consider the process \( M_t = 1_{\{\tau \leq t\}} - \int_0^{t \land \tau} \frac{1}{Z_{s-}} dA_s \) \hspace{1cm} (1)
Progressive filtration enlargement

Sketch of the proof of “⊆” (for $p = 1$, $X \equiv 1$)

- Fix $Q \in \text{SSC}(\mathcal{G})$
- Consider the process $M_t = 1_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \frac{1}{Z_s} dA_s$ \hspace{1cm} (1)
- By semi-static completeness,

$$M = M_0 + V + H \cdot S,$$

for some $H \in L(S)$ and martingale $V$ with $V_T \in L^2(\sigma(T))$
Progressive filtration enlargement

Sketch of the proof of “⊆” (for $p = 1$, $X \equiv 1$)

- Fix $Q \in \text{SSC}(\mathcal{G})$
- Consider the process $M_t = 1_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \frac{1}{Z_{s^-}} dA_s$ (1)

By semi-static completeness,

$$M = M_0 + V + H \cdot S,$$ (2)

for some $H \in L(S)$ and martingale $V$ with $V_T \in L^2(\sigma(T))$

- By (1), (2) and continuity of $S$, by considering the jumps of $M$:

$$\tau = \inf \left\{ t \in [0, T] : \frac{1}{Z_{t^-}} \Delta A_t + \Delta V_t = 1 \right\}.$$
Progressive filtration enlargement

Sketch of the proof of “⊂” (for $p = 1$, $X \equiv 1$)

- Fix $Q \in \text{SSC}(\mathcal{G})$
- Consider the process $M_t = 1_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} dA_s$ \hspace{1cm} (1)
- By semi-static completeness,
  $$M = M_0 + V + H \cdot S,$$ \hspace{1cm} (2)
  for some $H \in L(S)$ and martingale $V$ with $V_T \in L^2(\sigma(T))$
- By (1), (2) and continuity of $S$, by considering the jumps of $M$:
  $$\tau = \inf \left\{ t \in [0, T] : \frac{1}{Z_{t-}} \Delta A_t + \Delta V_t = 1 \right\}.$$  
- By assumption, $\tau > \sigma = \inf\{t > 0 : S_t \neq S_0\}$
- And $V$ is constant on $[\sigma, \infty[$ by our characterization Theorem
- Therefore $\tau = \inf\{t \in [0, T] : \frac{1}{Z_{t-}} \Delta A_t = 1\}$ $\mathbb{F}$-stopping time.
Remarks.

- From the proof it is clear that the set equivalence still holds true without any assumption on $S$ when $\Psi = \emptyset$. 
Remarks.

- From the proof it is clear that the set equivalence still holds true without any assumption on $S$ when $\Psi = \emptyset$.

- We can generalize the theorem for filtration enlargements with countably many single-jump processes.

**Theorem**

Let $S$ be continuous and $\mathcal{H}$ generated by $X_k 1_{[\tau_k, T]}$, $k \in \mathbb{N}$. Assume $\tau_k > \sigma$ on $\{0 < \tau_k < \infty\}$ for all $k$, and $|\{k : \tau_k(\omega) \leq T\}| < \infty \ \forall \ \omega$.

Then

$$\text{SSC}(\mathcal{G}) = \{ Q \in \text{SSC}(\mathcal{F}) : \mathcal{F} = \mathcal{G} \ \text{under} \ Q \}$$
Motivated by robust super-hedging price computation, we study extreme calibrated martingale measures.

We obtain:

- Semi-static version of the Jacod-Yor theorem.
- Description of semi-statically complete models in terms of dynamically complete models glued together by means of an atomic tree.
- Application to robust pricing by informed agents: under structural assumptions, informed agents price using only those models that render the additional information uninformative.

Lots of things remain to be done and appear to be within reach:

- Infinitely many static claims \( \rightarrow \) case \( S_T \sim \mu \)
- Better understanding of price processes with jumps
- More general filtration enlargements
- \ldots
Thank you for your attention!

© Walter: have a great year in Zurich!