Modelling energy forward prices
– Representation of ambit fields –

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Introduction

- Background: electricity forwards
- Study ambit fields as Volterra processes in Hilbert space
- Consider representations of ambit fields
  - Series representations as LSS processes
  - Solutions of SPDEs in Hilbert space
Background: electricity forwards
Power forwards: stylized facts of smoothed curves

- Example of power forward prices on NordPool
- Smoothed by fourth order polynomial spline
  - Imposed seasonal structure by industry spot prognosis
• Analysis of base load quarter/month/week contracts constructed from NordPool forward data
  • Daily forward curves 2001-2007
  • The "quarterly forward curve" 1 January, 2006
  • Andresen, Koekebakker and Westgaard (2010), B., Saltyte Benth and Koekebakker (2008)
• Correlation structure of quarterly contracts in NordPool
  • Correlation as a function of distance between start-of-delivery

[Graph showing observed and modeled correlation]

• High degree of ”idiosyncratic” risk
  • Quarterly contracts: 6 noise sources explain 96%, 7 explain 98%
- Observed Samuelson effect on (log-)returns
  - Volatility of forwards decrease with time to maturity
- Plot of Nordpool quarterly contracts, empirical volatility

![Observed and modeled volatility](image-url)
• Probability density of returns is non-Gaussian
• Example: weekly and monthly contracts
  • Fitted normal and NIG
  • ”True” and logarithmic frequency axis
  • NIG=normal inverse Gaussian distribution
Forward modelling by ambit processes

- Extension of the HJM approach
- Random field model for the smooth forward curve
  - by direct modelling rather than as the solution of some dynamic equation
- Simple arithmetic model could be (in the risk-neutral setting)

\[ F(t, x) = \int_{-\infty}^{t} \int_{0}^{\infty} g(t - s, x, y)\sigma(s, y)L(dy, ds) \]

- \( x \) is "time-to-maturity"
Definition of ”classical” ambit fields

\[ X(t, x) = \int_{-\infty}^{t} \int_{A} g(t - s, x, y) \sigma(s, y) L(ds, dy) \]

- \( L \) is a Lévy basis
- \( g \) non-negative deterministic function, \( g(u, x, y) = 0 \) for \( u < 0 \).
- Stochastic volatility process \( \sigma \) independent of \( L \), stationary
- \( A \) a Borel subset of \( \mathbb{R}^d \): ”ambit” set
• *L* is a *Lévy basis* on *R*^d* if*

1. the law of *L*(A) is infinitely divisible for all bounded sets *A*
2. if *A* ∩ *B* = ∅, then *L*(A) and *L*(B) are independent
3. if *A*₁, *A*₂, ... are disjoint bounded sets, then

\[
L(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} L(A_i), \text{ a.s}
\]

• We restrict to zero-mean, and square integrable Lévy bases *L*
• Use Walsh’s definition of the stochastic integral
• Our model: classical ambit field with \( d = 1 \) and \( \mathcal{A} = [0, \infty) \)

• Example I: exponential damping function

\[
g(u, x, y) = \exp(-\alpha(u + x + y))
\]

• Example II: the Musiela SPDE specification

  • \( L = W \), Brownian motion

\[
dF(t, x) = \frac{\partial F(t, x)}{\partial x} dt + g(x)\sigma(t) dW(t)
\]

• Solution of the SPDE

\[
F(t, x) = F_0(x + t) + \int_0^t g(x + (t-s))\sigma(s) dW(s)
\]
Hilbert-valued ambit fields

–Volterra processes in Hilbert space–
Recall definition of "classical" ambit fields

\[ X(t, x) = \int_{-\infty}^{t} \int_{A} g(t-s, x, y) \sigma(s, y) \, L(ds, dy) \]

- \( L \) is a \( \text{Lévy basis} \), \( g \) non-negative deterministic function, \( g(u, x, y) = 0 \) for \( u < 0 \), stochastic volatility process \( \sigma \) independent of \( L \) being stationary, \( A \) a Borel subset of \( \mathbb{R}^d \): "ambit" set
- **Our goals:**
  - Lift the ambit fields to processes in Hilbert space
  - ..and to analyse representations of such!
- Application of ambit fields: turbulence, tumor growth, energy finance, fixed-income markets
• Define $\mathcal{H}$-valued process $t \mapsto X(t)$

$$X(t) = \int_0^t \Gamma(t,s)(\sigma(s)) \, dL(s)$$

• $\mathcal{U}, \mathcal{V}, \mathcal{H}$ three separable Hilbert spaces
• $s \mapsto L(s)$ $\mathcal{V}$-valued Lévy process
  • Square integrable with mean zero ($L$ is $\mathcal{V}$-martingale)
  • Covariance operator $Q$ (symmetric, positive definite, trace class)
• $s \mapsto \sigma(s)$ predictable process with values in $\mathcal{U}$
  • Stochastic volatility or intermittency
• $(t, s) \mapsto \Gamma(t, s), s \leq t$, $\mathcal{L}(\mathcal{U}, \mathcal{L}(\mathcal{V}, \mathcal{H}))$-valued measurable mapping
  • Non-random kernel function
• Integrability condition for $\Gamma$ and $\sigma$:

$$\mathbb{E} \left[ \int_0^t \| \Gamma(t, s)(\sigma(s))Q^{1/2} \|^2_{\text{HS}} ds \right] < \infty$$

• We call $\mathcal{X}$ a *Hambit field*

• A sufficient integrability condition:

$$\int_0^t \| \Gamma(t, s) \|^2_{\text{op}} \mathbb{E} \left[ |\sigma(s)|^2_{\mathcal{U}} \right] \, ds < \infty$$
Proposition: Suppose that $\sigma$ is independent of $L$. For $h \in \mathcal{H}$ it holds

$$\mathbb{E} \left[ \exp(i(h, X(t))_\mathcal{H}) \right] = \mathbb{E} \left[ \exp \left( \int_0^t \Psi_L((\Gamma(t, s)(\sigma(s)))^* h) \right) ds \right]$$

where $\Psi_L$ is the characteristic exponent of $L(1)$.

Proof': Condition on $\sigma$, and use the independent increment property of $L$ along with the fact

$$(h, \Gamma(t, s)(\sigma(s))\Delta L(s))_\mathcal{H} = ((\Gamma(t, s)(\sigma(s)))^* h, \Delta L(s))_\mathcal{V}$$
• Example: $L = W$, $\mathcal{V}$-valued Wiener process
• For $\nu \in \mathcal{V}$, 
  
  \[
  \Psi_W(\nu) = -\frac{1}{2}(Q\nu, \nu)_{\mathcal{V}}
  \]

• Characteristic function of $X$ (Bochner $ds$-integral)

  \[
  \mathbb{E} \left[ \exp(i(h, X(t))_\mathcal{H}) \right] \\
  = \mathbb{E} \left[ \exp \left( -\frac{1}{2} (h, \int_0^t \Gamma(t, s)(\sigma(s))Q(\Gamma(t, s)(\sigma(s)))^* \, ds \, h)_{\mathcal{H}} \right) \right]
  \]

• $X$ is conditional Gaussian
### Examples
Example: from $\mathcal{H}$ambit to ambit

- Let $\mathcal{A} \subset \mathbb{R}^n$ Borel set, $\mathcal{U}$ a Hilbert space of real-valued functions on $\mathcal{A}$
- Let $(t, s, x, y) \mapsto g(t, s, x, y)$ be a measurable real-valued function for $0 \leq s \leq t \leq T$, $y \in \mathcal{A}$, $x \in \mathcal{B}$, $\mathcal{B} \subset \mathbb{R}^d$
- Suppose $\mathcal{V}$ is a Hilbert space of absolutely continuous functions on $\mathcal{A}$.
- Define for $\sigma \in \mathcal{U}$ the linear operator on $\mathcal{V}$

$$\Gamma(t, s)(\sigma) := \int_{\mathcal{A}} g(t, s, \cdot, y)\sigma(y)$$

acting on $f \in \mathcal{V}$ as

$$\Gamma(t, s)(\sigma)f = \int_{\mathcal{A}} g(t, s, \cdot, y)\sigma(s, y)f(dy).$$
• Let $\mathcal{H}$ be a Hilbert space of real-valued functions on $\mathcal{B}$
• Let $L$ be a $\mathcal{V}$-valued Lévy process, $\sigma$ $\mathcal{U}$-valued predictable process
  • Suppose integrability conditions on $s \mapsto \Gamma(t, s)(\sigma(s))$
• $X(t, x)$ is an ambit field

$$X(t, x) = \int_0^t \int_A g(t, s, x, y)\sigma(y) L(ds, dy)$$

• Example of Hilbert space?
Realization in Filipovic space

- Let $\mathcal{U} = \mathcal{V} = \mathcal{H}$, $n = d = 1$, $\mathcal{A} = \mathcal{B} = \mathbb{R}_+$
- Let $w \in C^1(\mathbb{R}_+)$ be non-decreasing, $w(0) = 1$ and $w^{-1} \in L^1(\mathbb{R}_+)$
- Let $\mathcal{U} := H_w$ be the space of absolutely continuous functions on $\mathbb{R}_+$ where

$$
|f|^2_w = f^2(0) + \int_{\mathbb{R}_+} w(y)|f'(y)|^2 \, dy < \infty
$$

- $H_w$ separable Hilbert space.
  - Introduced by Filipovic (2001)
  - Main application: realization of forward rate HJM models
Hilbert-valued OU with stochastic volatility

- Fix $\mathcal{V} = \mathcal{H}$, and let $A$ an unbounded operator on $\mathcal{H}$ with $C_0$-semigroup $S_t$.
- $\mathcal{W}$ $\mathcal{H}$-valued Wiener process with covariance operator $Q$.
- B. Rüdiger and Süss (2015): Let $\sigma(t)$ be a $\mathcal{U} := L_{HS}(\mathcal{H})$-valued predictable process,

$$dX(t) = AX(t) \, dt + \sigma(t) \, dW(t)$$

- Mild solution

$$X(t) = S_t X(0) + \int_0^t S_{t-s} \sigma(s) \, dW(s)$$
• $X$ as $\mathcal{H}$ambit field: define $\Gamma(t, s) \in \mathcal{L}(L_{HS}(\mathcal{H}), \mathcal{L}(\mathcal{H}))$

$$\Gamma(t, s) : \sigma \mapsto S_{t-s}\sigma$$

• A BNS SV model: $\sigma(t) = \mathcal{Y}^{1/2}(t)$

$$d\mathcal{Y}(t) = \mathbb{C}\mathcal{Y}(t) dt + d\mathcal{L}(t)$$

• $\mathbb{C} \in \mathcal{L}(L_{HS}(\mathcal{H}))$, with $C_0$-semigroup $S_t$

• $\mathcal{L}$ is a $L_{HS}(\mathcal{H})$-valued "subordinator"
• \( \mathcal{Y}(t) \) symmetric, positive definite, \( L_{HS}(\mathcal{H}) \)-valued process,

\[
\mathbb{E}[|\sigma(t)|^2_U] = \sum_{n=1}^{\infty} (\sigma(t)h_k, \sigma(t)h_k)_\mathcal{H} = \text{Tr}(\mathcal{Y}(t))
\]

• The trace is continuous, and hence the integrability condition for \( X \) holds

\[
\text{Tr}(\mathcal{Y}(t)) = \text{Tr}(\mathcal{S}_t\mathcal{Y}_0) + \text{Tr}(\int_0^t \mathcal{S}_s \, ds \mathbb{E}[\mathcal{L}(1)])
\]

• Infinite-dimensional extension of Barndorff-Nielsen and Stelzer (2007)
Hambit fields as Lévy semistationary (LSS) processes
• Let $\{u_n\}, \{v_m\}$ and $\{h_k\}$ be ONB in $\mathcal{U}, \mathcal{V}$ and $\mathcal{H}$ resp.
  • Recall separability of the Hilbert spaces
• $L_m := (L, v_m)_\mathcal{V}$ are $\mathbb{R}$-valued Lévy processes
  • zero mean, square integrable
  • but, not independent nor zero correlated
• Define LSS processes $Y_{n,m,k}(t)$ by

$$Y_{n,m,k}(t) = \int_0^t g_{m,n,k}(t, s)\sigma_n(s) \, dL_m(s)$$

$$g_{n,m,k}(t, s) := \langle \Gamma(t, s)(u_n)v_m, h_k \rangle_{\mathcal{H}} \quad \sigma_n(s) := \langle \sigma(s), u_n \rangle_{\mathcal{U}}$$
**Proposition:** Assume

\[ \int_0^t \| \Gamma(t, s) \|_{\text{op}}^2 \left( \sum_{n=1}^{\infty} \mathbb{E}[\sigma_n^2(s)]^{1/2} \right)^2 ds < \infty \]

then,

\[ X(t) = \sum_{n,m,k=1}^{\infty} Y_{n,m,k}(t) h_k \]

"Proof": Expand all elements along the ONB’s in their respective spaces. The integrability assumption ensures the commutation of an infinite sum and stochastic integral wrt. \( L_m \) (A stochastic Fubini theorem).
• Barndorff-Nielsen et al. (2013): energy spot price modeling using LSS processes
  • Finite factors
  • Implied forward prices become scaled finite sums of LSS processes
• Barndorff-Nielsen et al. (2014): energy forward prices as ambit fields
  • Infinite LSS factor models!
• B. Krühner (2014): HJM forward price dynamics representable as countable scaled sums of OU process
  • Possibly complex valued OU processes
• Integrability condition implies the sufficient condition for existence of $\mathcal{H}$ambit field:

• By Parseval’s identity

$$\mathbb{E}[|\sigma(s)|^2] = \sum_{n=1}^{\infty} \mathbb{E}[\langle \sigma(s), u_n \rangle^2]$$

• Sufficient condition for LSS representation: there exists $a_n > 0$ s.t. $\sum_{n=1}^{\infty} a_n^{-1} < \infty$ and

$$\sum_{n=1}^{\infty} a_n \int_0^t \|\Gamma(t,s)\|_\text{op}^2 \mathbb{E}[\langle \sigma(s), u_n \rangle^2] \, ds < \infty$$
Hambit fields and SPDEs
• Known connection between an LSS process and the boundary of a hyperbolic stochastic partial differential equation (SPDE):

\[
dZ(t, x) = \partial_x Z(t, x) \, dt + g(t + x, t)\sigma(t) \, dL(t)
\]

\[
Z_0(t) := Z(t, 0) = \int_0^t g(t, s)\sigma(s) \, dL(s)
\]

• \(L \, \mathbb{R}\)-valued Lévy process, \(x \geq 0\)

• **Goal**: show similar result for \(\mathcal{H}\)ambit fields!
  • Application: B. Eyjolfsson (2015+) devised iterative (finite difference) numerical schemes in the \(\mathbb{R}\)-valued case using this relationship
• Assume $\tilde{\mathcal{H}}$ a Hilbert space of strongly measurable $\mathcal{H}$-valued functions on $\mathbb{R}_+$

• Suppose $S_\xi$ right-shift operator is $C_0$-semigroup on $\tilde{\mathcal{H}}$

$$S_\xi f := f(\xi + \cdot), \quad f \in \tilde{\mathcal{H}}$$

• Generator is $\partial_\xi = \partial/\partial \xi$

• Consider hyperbolic SPDE in $\tilde{\mathcal{H}}$

$$X(t) = \partial_\xi X(t) \, dt + \Gamma(t + \cdot, t)(\sigma(t)) \, dL(t), \ X(0) \in \tilde{\mathcal{H}}$$
Predictable $\tilde{\mathcal{H}}$-valued unique solution

$$X(t) = S_t X(0) + \int_0^t S_{t-s} \Gamma(s + \cdot, s)(\sigma(s)) \, dL(s)$$

**Proposition**: Assume that the evaluation map $\delta_x : \tilde{\mathcal{H}} \to \mathcal{H}$ defined by $\delta_x f = f(x) \in \mathcal{H}$ for every $x \geq 0$ and $f \in \tilde{\mathcal{H}}$ is a continuous linear operator. If $X(0) = 0$, then $X(t) = \delta_0(X(t))$.

"Proof": Argue that

$$\delta_0 \int_0^t \Gamma(t + \cdot, s)(\sigma(s)) \, dL(s) = \int_0^t \Gamma(t, s)(\sigma(s)) \, dL(s)$$

• Need a space $\tilde{\mathcal{H}}$ with $\delta_x \in \mathcal{L}(\tilde{\mathcal{H}}, \mathcal{H})$
Abstract Filipovic space

- $f \in L^1_{loc}(\mathbb{R}_+, \mathcal{H})$ is \textit{weakly differentiable} if there exists $f' \in L^1_{loc}(\mathbb{R}_+, \mathcal{H})$ such that

$$\int_{\mathbb{R}_+} f(x)\phi'(x)\,dx = -\int_{\mathbb{R}_+} f'(x)\phi(x)\,dx, \forall \phi \in C^\infty_c(\mathbb{R}_+)$$

- Integrals interpreted in Bochner sense
- Let $w \in C^1(\mathbb{R}_+)$ be a non-decreasing function with $w(0) = 1$ and

$$\int_{\mathbb{R}_+} w^{-1}(x)\,dx < \infty$$
• Define $\mathcal{H}_w$ to be the space of $f \in L^1_{loc}(\mathbb{R}_+, \mathcal{H})$ for which there exists $f' \in L^1_{loc}(\mathbb{R}_+, \mathcal{H})$ such that

$$\|f\|^2_w = |f(0)|^2_{\mathcal{H}} + \int_{\mathbb{R}_+} w(x)|f'(x)|^2_{\mathcal{H}} \, dx < \infty.$$ 

• $\mathcal{H}_w$ is a separable Hilbert space with inner product

$$\langle f, g \rangle_w = (f(0), g(0))_{\mathcal{H}} + \int_{\mathbb{R}_+} w(x)(f'(x), g'(x))_{\mathcal{H}} \, dx$$
• Fundamental theorem of calculus: If $f \in \mathcal{H}_w$, then $f' \in L^1(\mathbb{R}_+, \mathcal{H})$, $\|f'\|_1 \leq c\|f\|_w$, and

$$f(x + t) - f(x) = \int_x^{x+t} f'(y) \, dy$$

• Shift-operator $S_\xi, \xi \geq 0$ is uniformly bounded

$$\|S_\xi f\|_w^2 \leq 2(1 + c^2)\|f\|_w^2$$

• Constant equal to $c^2 = \int_{\mathbb{R}_+} w^{-1}(x) \, dx$
Lemma: Evaluation map $\delta_x : \mathcal{H}_w \to \mathcal{H}$ is a linear bounded operator with

$$|\delta_x f|_{\mathcal{H}} \leq K \|f\|_w$$

"Proof": FTC, Bochner’s norm inequality and Cauchy-Schwartz inequality yield

$$|\delta_x f|_{\mathcal{H}}^2 = |f(x)|_{\mathcal{H}}^2 \leq 2|f(0)|_{\mathcal{H}}^2 + 2 \int_{\mathbb{R}^+} w^{-1}(y) \, dy \int_{\mathbb{R}^+} w(y)|f'(y)|_{\mathcal{H}}^2 \, dy$$

• We have an example $\tilde{\mathcal{H}} = \mathcal{H}_w$!
Classical and abstract Filipovcic space

**Proposition**: For \( L \in \mathcal{H}^* \), \( x \mapsto L \circ \delta_x(g) = L(g(x)) \in H_w \) for \( g \in \mathcal{H}_w \). Moreover, if \( h_x(y) = 1 + \int_0^{x \wedge y} w^{-1}(z) \, dz \) and \( \ell_x = L^*(h_x) \), then

\[
\mathcal{L}(g(x)) = \langle g, \ell_x \rangle_w
\]

"Proof": Follows from linearity of \( \mathcal{L} \), FTC and Bochner’s norm inequality. Further, if \( \tilde{\delta}_x \) is the evaluation map on \( H_w \), then \( \tilde{\delta}_x(\nu) = (\nu, h_x)_w \), \( \nu \in H_w \).
Wrapping up...

- Ambit fields: motivated from power forwards
- Hambit fields: general framework for
  - non-Gaussianity, stochastic volatility, Samuelson effect
- Representation in LSS processes
  - Spot price models
- Representation as boundary of solution of hyperbolic SPDE
  - Finite difference numerical schemes
- Outlook:
  - Pricing and hedging power forward options (B. Krühner (2015)).
  - Stochastic integration (B. Süß (2015))
Thank you for your attention!
References