On the Relation between Linearity-Generating Processes and Linear-Rational Models

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Outline

Linearity-Generating (LG) Processes

Linear-Rational (LR) Models

Relation between LG processes and LR models

State Price Density Decomposition
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Ingredients

- FPS ($\Omega, \mathcal{F}_t, \mathcal{F}, \mathbb{P}$)
- State price density process

\[ \zeta_t = \zeta_0 e^{-\int_0^t r_s \, ds} \mathcal{E}_t(L) \]

- Risk-neutral measure \( \frac{dQ}{dP}|_{\mathcal{F}_t} = \mathcal{E}_t(L) \)
- \( m \)-dimensional semimartingale \( X_t \)
Definition LG Process (Gabaix 2009)

\((\zeta_t, X_t)\) forms \((m + 1)\)-dimensional **linearity-generating (LG)** process if

\[
\mathbb{E}_t \left[ \frac{\zeta_T}{\zeta_t} \right] = \mathcal{A}(T - t) + \mathcal{B}(T - t)X_t
\]

\[
\mathbb{E}_t \left[ \frac{\zeta_T}{\zeta_t} X_T \right] = \mathcal{C}(T - t) + \mathcal{D}(T - t)X_t
\]

for some continuously differentiable functions \(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\).

\(\Rightarrow\) Linear \(T\)-claims in \(X_T\) have linear time-\(t\) prices in \(X_t\)

- E.g. zero-coupon bond price

\[
P(t, T) = \mathcal{A}(T - t) + \mathcal{B}(T - t)X_t
\]
Hidden Non-degeneracy Assumption

Support of $X_{t^*}$ / $\zeta_{t^*} X_{t^*}$ / $Z_{t^*}$ affinely spans $\mathbb{R}^m$ for some $t^* \geq 0$
Characterization Theorem

The following statements are equivalent:

1. \((\zeta_t, X_t)\) forms an LG process;
2. short rate \(r_t\), \(\mathbb{Q}\)-drift \(\mu_t^{X,\mathbb{Q}}\) of \(X_t\) are linear, quadratic in \(X_t\),
   \[ r_t = -A - BX_t \]
   \[ \mu_t^{X,\mathbb{Q}} = C + (r_t + D)X_t = C + (D - A)X_t - (BX_t)X_t \]
3. drift of \(Y_t = (\zeta_t, \zeta_t X_t)\) is strictly linear in \(Y_t\),
   \[ dY_t = K Y_t \, dt + dM_t^Y \]

In either case,
\[ K = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \begin{pmatrix} A(\tau) & B(\tau) \\ C(\tau) & D(\tau) \end{pmatrix} = e^{K\tau} \]
Sketch of Proof

LG condition holds if and only if either

- The processes
  \[ M_t = e^{-\int_0^t r_s ds}(A(T - t) + B(T - t)X_t) \]
  \[ N_t = e^{-\int_0^t r_s ds}(C(T - t) + D(T - t)X_t) \]
  are \( \mathbb{Q} \)-martingales (\( \rightarrow \) set drift zero)

- \( Y_t = (\zeta_t, \zeta_t X_t) \) satisfies
  \[ \mathbb{E}_t[Y_T] = e^{K(T-t)} Y_t \]
Remarks

- Part 3 is definition of LG process given in Gabaix (2009)
- Gabaix (2009) refers to \((BX_t)X_t\) in

\[
\mu_{t}^{X,Q} = C + (r_t + D)X_t = C + (D - A)X_t - (BX_t)X_t
\]

as “linearity-generating twist of an AR(1) process”
Discussion

- Existence of LG processes \((\zeta_t, X_t)\)?

- Carr, Gabaix, Wu (2009) specify \(Y_t\),

\[
dY_t = KY_t \, dt + dM_t^Y,
\]

and set \(\zeta_t = Y_{1t}\) and \(X_t = Y_{2..m+1,t}/Y_{1,t}\)

- Problem: \(Y_t\) is not stationary: \(Y_{1t} > 0\) and \(\mathbb{E}[Y_{1t}] \to 0\)

- \(X_t = Y_{2..m+1,t}/Y_{1,t}\) is stationary, but . . .
  - no functional relation between \(\zeta_t\) and \(X_t\) (e.g. \(\bar{\zeta}_t = N_t \zeta_t\))
  - nontrivial viability conditions for \(X_t\) in view of

\[
0 < P(t, T) = A(T - t) + B(T - t)X_t \leq 1
\]

- quadratic \(\mathbb{Q}\)-drift and highly nonlinear \(\mathbb{P}\)-drift of \(X_t\)
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State Price Density Decomposition
Definition (Filipović, Larsson, Trolle 2014)

An $m$-dimensional **linear-rational (LR) model** consists of an $m$-dimensional semimartingale $Z_t$ with linear drift,

$$dZ_t = (b + \beta Z_t) \, dt + dM^Z_t,$$

and parameters $\alpha, \phi \in \mathbb{R}$ and $\psi \in \mathbb{R}^m$ such that

$$\zeta_t = e^{-\alpha t} \left( \phi + \psi^\top Z_t \right) > 0.$$
LR model implies linear-rational bond prices

\[ P(t, T) = \mathbb{E}_t \left[ \frac{\zeta_T}{\zeta_t} \right] = e^{-\alpha(T-t)} \phi + \psi^\top e_{\beta(T-t)} \int_0^{T-t} e^{-\beta s} b \, ds + \psi^\top e_{\beta(T-t)} Z_t \phi + \psi^\top Z_t \]

and short rate

\[ r_t = -\partial_T \log P(t, T) \big|_{T=t} = \alpha - \frac{\psi^\top (b + \beta Z_t)}{\phi + \psi^\top Z_t}. \]
Representation as LG Process

- Define normalized factor

$$X_t = \frac{Z_t}{\phi + \psi^T Z_t}$$

- Simple algebraic fact (if $\phi \neq 0$):

$$\frac{p + q^T Z_t}{\phi + \psi^T Z_t} = \frac{p}{\phi} + \left(q - \frac{p\psi}{\phi}\right)^T X_t$$

$$\Rightarrow$$ Bond price and short rate become linear in $X_t$
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- Relation between LG processes and LR models
- State Price Density Decomposition
**Representation Theorem:** \( m \)-dim LR as \((m+1)\)-dim LG

An \( m \)-dimensional LR model

\[
dZ_t = (b + \beta Z_t) \, dt + dM_t^Z, \quad \zeta_t = e^{-\alpha t} \left( \phi + \psi^T Z_t \right)
\]

can be represented as \((m+1)\)-dimensional LG process \((\zeta_t, X_t)\) through

\[
X_t = \frac{Z_t}{\phi + \psi^T Z_t}
\]

if and only if \( b = C\phi \).

The respective \( Y_t = (\zeta_t, \zeta_t X_t) \) in Characterization Theorem is

\[
Y_t = e^{-\alpha t} (\phi + \psi^T Z_t, Z_t)
\]

and the matrix \( K \) in \( dY_t = K Y_t \, dt + dM_t^Y \) is given by

\[
A = -\alpha + \psi^T C, \quad B = \psi^T (-C\psi^T + \beta),
\]
\[
C = \frac{b}{\phi}, \quad D = -\alpha \text{Id} - C\psi^T + \beta
\]  

\((*)\)
Representation Corollary 1: $m$-dim LR as $(m + 2)$-dim LG

By increasing dimension can always assume $b = 0$:

\[
\bar{Z}_t = \begin{pmatrix} Z_t \\ 1 \end{pmatrix}, \quad \bar{b} = 0, \quad \bar{\beta} = \begin{pmatrix} \beta \\ 0 \end{pmatrix}, \quad M_t^{\bar{Z}} = \begin{pmatrix} M_t^{Z} \\ 0 \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} \psi \\ 0 \end{pmatrix}
\]

is econ equivalent $(m + 1)$-dim LR model with strictly linear drift

\[
d\bar{Z}_t = \bar{\beta}\bar{Z}_t \, dt + dM_t^{\bar{Z}}, \quad \zeta_t = e^{-\alpha t} \left( \phi + \bar{\psi}^\top \bar{Z}_t \right)
\]

**Corollary 3.1.**

$m$-dim LR model can always be represented as $(m + 2)$-dim LG process through

\[
\bar{X}_t = \frac{(Z_t, 1)}{\phi + \psi^\top \bar{Z}_t}.
\]

The respective $\bar{Y}_t = (\zeta_t, \zeta_t \bar{X}_t) = e^{-\alpha t} (\phi + \psi^\top Z_t, Z_t, 1) \ldots$
For given parameters $A, B, C, D$ condition (*) holds if and only if

$$(1 \ -\psi^\top) \begin{pmatrix} A & B \\ C & D \end{pmatrix} = -\alpha (1 \ -\psi^\top)$$

**Corollary 3.2.**

The functions $A, B, C, D$ of an $(m + 1)$-dimensional LG process can be obtained from an $m$-dimensional LR model if and only if the respective matrix $K$ admits a left-eigenvector $v^\top$ with $v_1 \neq 0$. 
Counterexample

For $B \neq 0$, $C = 0$, $D = A\text{Id}$ there exists no such left-eigenvector.

$\Rightarrow$ not every $(m + 1)$-dimensional LG process $(\zeta_t, X_t)$ can be represented as LR model of dimension $m$ or lower.

Characterization Theorem $\Rightarrow$ $(m + 1)$-dim LG process $(\zeta_t, X_t)$ can always be represented as $(m + 1)$-dim LR model

$$Z_t \equiv Y_t = (\zeta_t, \zeta_t X_t), \quad \zeta_t = Z_{1,t}$$

Next step: characterize those $(m + 1)$-dim LG processes that can be represented as $m$-dim LR model
Representation Theorem: \((m + 1)\)-dim LG as \(m\)-dim LR

Consider \((m + 1)\)-dim LG process \((\zeta_t, X_t)\) and let \(Y_t = (\zeta_t, \zeta_t X_t)\).

The following statements are equivalent:

1. \((\zeta_t, X_t)\) can be represented as \(m\)-dim LR model
2. there exist parameters \(\alpha, \phi, \psi\) such that
   \[
   (1 - \psi^\top) Y_t = \phi e^{-\alpha t}
   \]
3. there exist nonzero \(v \in \mathbb{R}^{m+1}\) and function \(f(t)\) such that
   \[
   v^\top Y_t = f(t) \tag{**}
   \]

Note: \((**)) \Rightarrow M_t^Y - M_0^Y \perp v
Semimartingale $S_t$ is **mean-reverting** to **mean-reversion level** $\theta$ if 
\[
\frac{1}{T-t} \int_t^T \mathbb{E}_t[S_u] \, du \to \theta \text{ as } T \to \infty \text{ almost surely for all } t \geq 0.
\]
Consider \((m + 1)\)-dim LG process \((\zeta_t, X_t)\) and let \(Y_t = (\zeta_t, \zeta_t X_t)\).

The following statements are equivalent:

1. \((\zeta_t, X_t)\) can be represented as \(m\)-dim LR model \(Z_t\) and \(Z_t\) is mean-reverting to level \(\theta \in \mathbb{R}^m\) satisfying \(\phi + \psi^\top \theta > 0\);

2. \(e^{\alpha t} Y_t\) is mean-reverting to level \(\tilde{\theta} \in \mathbb{R}^{m+1}\) satisfying \(\tilde{\theta}_1 > 0\) for some \(\alpha\).

Mean-reversion levels are related by \(\tilde{\theta} = (\phi + \psi^\top \theta, \theta)\).
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Markov Valuation


- Economy described by Markov state $X_t$
- State price density forms positive multiplicative functional:

$$\frac{\zeta_T(X)}{\zeta_t(X)} = \frac{\zeta_{T-t}(X \circ \theta_t)}{\zeta_0(X \circ \theta_t)}$$

$\Rightarrow$ Pricing semigroup $S_t$:

$$S_t f(x) = \mathbb{E}_x \left[ \frac{\zeta_t}{\zeta_0} f(X_t) \right]$$
Multiplicative Decomposition Theorem

Let \( \varphi(x) \) be positive eigenfunction of pricing semigroup \( S_t \) with eigenvalues \( e^{\rho t} \) then \( \zeta_t \) admits the multiplicative decomposition

\[
\zeta_t = e^{\rho t} \frac{1}{\varphi(X_t)} \hat{M}_t
\]

where \( \hat{M}_t \) is a positive martingale with \( \hat{M}_0 = 1 \).

If \( X_t \) is recurrent and stationary under \( \mathbb{A} \) given by \( \frac{d\mathbb{A}}{dP}|_{\mathcal{F}_t} = \hat{M}_t \) then this decomposition is unique.

HS (2009) also provide conditions for existence of positive ef \( \varphi(x) \)
LR Models Revisited

An $m$-dimensional LR model

$$dZ_t = (b + \beta Z_t) \, dt + dM_t^Z, \quad \zeta_t = e^{-\alpha t} \left( \phi + \psi^\top Z_t \right)$$

satisfies multiplicative decomposition for

$$\rho = -\alpha, \quad \varphi(x) = \frac{1}{\phi + \psi^\top z}, \quad \hat{M}_t = 1$$

and can be (part of) recurrent and stationary Markov process!
LR Models Revisited cont’d

- $\mathbb{A}$ is long forward measure:

$$\frac{\zeta_t P(t, T)}{\zeta_0 P(0, T)} = \frac{\phi + \mathbb{E}_t[\psi^\top Z_T]}{\phi + \mathbb{E}[\psi^\top Z_T]} \to 1 \quad \text{as} \quad T \to \infty$$

Hence deflating by $\zeta_t/\zeta_0$ amounts to discounting by gross return on long-term bond $\lim_{T \to \infty} \frac{P(t, T)}{P(0, T)}$

It also implies that the long-term bond is growth optimal under $\mathbb{A}$ (Qin, Linetsky 2015)

- Flexible market price of risk specification: free to modify

$$\zeta_t \sim \zeta_t \hat{M}_t$$

for some auxiliary density process $\hat{M}_t$
Conclusion

- LG processes are related to LR models
- \( \{m\text{-dim LR models}\} \subset \{(m + 1(2))\text{-dim LG processes}\} \)
- \( \{(m + 1)\text{-dim LG processes}\} \subset \{(m + 1)\text{-dim LR models}\} \)
- \((m + 1)\text{-dim LG process} \in \{\text{mean-rev. } m\text{-dim LR models}\} \) if and only if mean-reverting after exponential scaling
- HS decomposition theorem favors mean-reverting LR model specification

LR models = “reasonable” specifications of LG processes
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LR models = “reasonable” specifications of LG processes