Option pricing in a quadratic variance swap model

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Motivation:

- More and more instruments traded on variance
- Affine models fail to reproduce large increases in variance
- Quadratic models are an appealing alternative to jumps in variance
- Index options useful in portfolio allocation
- But not straightforward to price.

In this talk:

1. Provide a short introduction to quadratic variance swap models
2. Derive an option pricing method
3. Illustrate how this method can be applied in a portfolio allocation exercise.

Realized variance

- Filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$.
- The $\mathbb{Q}$-dynamics of the index are specified as

$$\frac{dS_t}{S_{t-}} = r_t\, dt + \sigma_t\, dB_t + \int_{\mathbb{R}} \xi \left(\chi(dt, d\xi) - \nu_t^{\mathbb{Q}}(d\xi) dt \right),$$

- Let $t = t_0 < t_1 < \cdots < t_n = T$. The annualized realized variance is given as:

$$\begin{aligned}
\mathrm{RV}(t, T) = \quad & \frac{252}{n} \sum_{i=1}^{n} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \\
\longrightarrow \quad & \frac{252}{n} \mathrm{QV}(t, T) \\
& = \frac{252}{n} \int_t^T \sigma_s^2\, ds + \int_t^T \int_{\mathbb{R}} (\log(1+\xi))^2\, \chi(ds, d\xi).
\end{aligned}$$

Variance swaps

- A variance swap initiated at t with maturity T, or term $T-t$, pays the difference between the annualized realized variance $\mathrm{RV}(t,T)$ and the variance swap rate $\mathrm{VS}(t,T)$ fixed at t. No arbitrage implies that

$$\mathrm{VS}(t,T) = \frac{1}{T-t}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{QV}(t,T) \mid \mathcal{F}_t\right] = \frac{1}{T-t}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T v_s^{\mathbb{Q}}\, ds \mid \mathcal{F}_t\right],$$

where the $\mathbb{Q}$-spot variance process is

$$v_t^{\mathbb{Q}} = \sigma_t^2 + \int_{\mathbb{R}} (\log(1+\xi))^2\, \nu_t^{\mathbb{Q}}(d\xi).$$

Quadratic variance swap models

- X is a diffusion process in $\mathbb{R}^m$. Under $\mathbb{Q}$, it satisfies:

$$dX_t = \mu(X_t)dt + \Sigma(X_t)dW_t.$$

- X is quadratic if its drift and diffusion functions are linear and quadratic in the state variable:

$$\mu(x) = b + \beta x,$$

$$\Sigma(x)\Sigma(x)^T = a + \sum_{k=1}^{m} \alpha^k x_k + \sum_{k,l=1}^{m} A^{kl} x_k x_l.$$

- A **quadratic variance swap model** is obtained under the assumption that the $\mathbb{Q}$-spot variance is a quadratic function of the latent state variable X_t:

$$v_t^{\mathbb{Q}} = g(X_t) = \phi + \psi^\top X_t + X_t^\top \pi X_t$$

for $\phi \in \mathbb{R}$, $\psi \in \mathbb{R}^m$, and $\pi \in \mathbb{S}^m$.

Quadratic variance swap model

Theorem: Quadratic term structure of variance

The quadratic variance swap model admits a quadratic term structure:

$$(T-t)VS(t,T) = \Phi(T-t) + \Psi(T-t)^T X_t + X_t^T \Pi(T-t) X_t = G(T-t, X_t)$$

where Φ, Ψ and Π satisfy the linear system of ODEs

$$\begin{aligned}
\frac{d\Phi(\tau)}{d\tau} &= \phi + b^T\Psi(\tau) + \mathrm{tr}(a\Pi(\tau)) && \Phi(0) = 0 \\
\frac{d\Psi(\tau)}{d\tau} &= \psi + \beta^T\Psi(\tau) + 2\Pi(\tau)b + \alpha\cdot\Pi(\tau) && \Psi(0) = 0 \\
\frac{d\Pi(\tau)}{d\tau} &= \pi + \beta^T\Pi(\tau) + \Pi(\tau)\beta + A\bullet\Pi(\tau) && \Pi(0) = 0
\end{aligned}$$

where $(\alpha\cdot\Pi)_k = \mathrm{tr}(\alpha^k\Pi)$ and $(A\bullet\Pi)_{kl} = \mathrm{tr}(A^{kl}\Pi)$.

A Bivariate Model Specification

- Assume $X_t = (X_{1t}, X_{2t})^T$:

$$dX_{1t} = (b_1 + \beta_{11}X_{1t} + \beta_{12}X_{2t})dt + \sqrt{a_1 + \alpha_1 X_{1t} + A_1 X_{1t}^2}\, dW_{1t}^*$$
$$dX_{2t} = (b_2 + \beta_{22}X_{2t})dt + \sqrt{a_2 + \alpha_2 X_{2t} + A_2 X_{2t}^2}\, dW_{2t}^*$$

- Spot variance quadratic in X_{1t}:

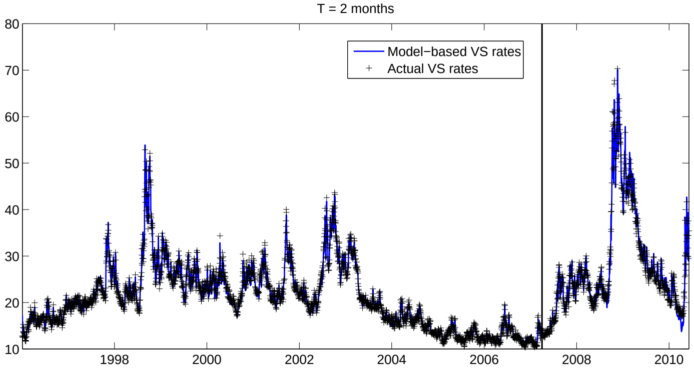
$$v_t = \phi_0 + \psi_0 X_{1t} + X_{1t}\pi_0 X_{1t}.$$

- **Interpretation:** variance mean-reverts to a stochastic level

$$\frac{b_1 + \beta_{12}X_{2t}}{|\beta_{11}|}.$$

- Explicit representation of forward variance $f(t, T)$.

Estimation results



Modelling assumptions

- The price process jumps by a deterministic size $\xi > -1$. Jumps are driven by a Poisson process:

$$\frac{dS_t}{S_{t-}} = r\,dt + \sigma(X_t)\,\mathbf{R}(X_t)^\top dW_t + \xi\,(dN_t - \nu^{\mathbb{Q}}(X_t)\,dt).$$

  The $\mathbb{Q}$-spot variance is given by:

$$v_t^{\mathbb{Q}} = g(X_t) = \sigma(X_t)^2 + (\log(1+\xi))^2\,\nu^{\mathbb{Q}}(X_t).$$

- $\nu^{\mathbb{Q}}(x) = \nu^Q \sigma(x)^2$,

$$\sigma(x)^2 = \frac{g(x)}{1 + (\log(1+\xi))^2\,\nu^Q} = K_{\sigma^2} g(x).$$

Option pricing

Main steps:

- Derive the moments of the log price process $L_t = \log S_t$.
- Edgeworth expansion of the characteristic function of $L_T | \mathcal{F}_{t_0}$:

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{zL_T}\right] = \exp\left(\sum_{n=1}^{\infty} C_n \frac{z^n}{n!}\right) = \exp\left(C_1 z + C_2 \frac{z^2}{2}\right)\left(1 + C_3 \frac{z^3}{3!} + O(z^4)\right).$$

- Option price recovered by Fourier inversion (Chen and Scott (1992), Heston (1993), Bates (1996), Scott (1997), Bakshi and Chen (1997), Carr and Madan (1999), Duffie, Pan and Singleton (2000) etc.).

Decomposition of $L_t = \log S_t$

For $t \geq t_0 \geq 0$, $L_t = Y_t + X_{3t}$ where

$$
\begin{aligned}
Y_t &= \int_{t_0}^{t} \left((\log(1+\xi) - \xi)\nu^{\mathbb{Q}}(X_s) - \frac{1}{2}\sigma(X_s)^2 \right) ds \\
&= K_Y \int_{t_0}^{t} g(X_s)\, ds,
\end{aligned}
$$

where

$$
K_Y = -\frac{(\xi - \log(1+\xi))\,\nu^Q + 1/2}{1 + (\log(1+\xi))^2 \nu^Q} < 0.
$$

$$
dX_{3t} = r\,dt + \sigma(X_t)\mathbf{R}(X_t)^\top dW_t + \log(1+\xi)(dN_t - \nu^{\mathbb{Q}}(X_t)dt), \quad X_{3t_0} = \log S_{t_0}.
$$

Analysis of X_t

- Define the jump-diffusion process $X_t = (X_{1t}, X_{2t}, X_{3t})^T$. Its diffusion matrix $A(x)$ is

$$A(x) = \begin{pmatrix} 1 + A_1 x_1^2 & 0 & \mathbf{R_1(x)\sigma(x)\sqrt{1 + A_1 x_1^2}} \\ 0 & x_2 + A_2 x_2^2 & 0 \\ \mathbf{R_1(x)\sigma(x)\sqrt{1 + A_1 x_1^2}} & 0 & \sigma(x)^2 \end{pmatrix}.$$

- We want q_0, q_1 and q_2 such that

$$R_1(x)\sigma(x)\sqrt{1 + A_1 x_1^2} = R_1(x_1)\sqrt{K_{\sigma^2} g(x_1)}\sqrt{1 + A_1 x_1^2} = q_0 + q_1 x_1 + q_2 x_1^2.$$

- To capture the leverage effect, $R_1(x_1) \approx -\mathrm{sign}\,(\psi + 2\pi X_{1t}) \times 0.7$. Hence we choose q_0, q_1 and q_2 to match the highest order terms of:

$$(q_0 + q_1 x_1 + q_2 x_1^2)^2 \approx 0.7^2 K_{\sigma^2} g(x_1)(1 + A_1 x_1^2).$$

- With this specification, X_t is a quadratic jump-diffusion process and hence is **polynomial preserving**.

Moments of X_T

- Literature on polynomial preserving processes include Wong (1964), Mazet (1997), Zhou (2003), Forman and Sørensen (2008), Cuchiero (2011), Cuchiero, Keller-Ressel and Teichmann (2012), Filipović, Mayerhofer and Schneider (2013), Filipović, Larsson and Trolle (2014), Filipović and Larsson (2015) etc.

Conditional moments of X_T

Let $D = \frac{(3+N)(2+N)(1+N)}{6}$ denote the dimension of the space of polynomials in X_T of degree N or less. The D-row vector of the mixed $\mathcal{F}_{t_0}$-conditional moments of X_T of order N or less with $T \geq t_0$ is given by

$$
\begin{aligned}
&\left(1,\, \mathbb{E}^{\mathbb{Q}}\left[X_{1T}|\mathcal{F}_{t_0}\right], \ldots, \mathbb{E}^{\mathbb{Q}}\left[X_{2T}X_{3T}^{N-1}|\mathcal{F}_{t_0}\right],\, \mathbb{E}^{\mathbb{Q}}\left[X_{3T}^{N}|\mathcal{F}_{t_0}\right]\right) \\
&= \left(1,\, X_{1t_0}, \ldots, X_{2t_0}X_{3t_0}^{N-1},\, X_{3t_0}^{N}\right) e^{\widetilde{B}(T-t_0)},
\end{aligned}
$$

where $\widetilde{B}$ is an upper block triangular $D \times D$ matrix and $e^{\widetilde{B}(T-t_0)}$ denotes the matrix exponential of $\widetilde{B}(T-t_0)$.

Moments of X_T

Proof: The generator of X_t is given by:

$$\mathcal{A}f(x) = \begin{pmatrix} \beta_{11}x_1 + \beta_{12}x_2 \\ b_2 + \beta_{22}x_2 \\ r \end{pmatrix}^\top \nabla_x f(x) + \frac{1}{2}\sum_{i,j=1}^{3} A_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j}$$
$$+ \big(f\left(x + \log(1+\xi)e_3\right) - f(x) - \nabla_x f(x)^\top e_3 \log(1+\xi)\big)\,\nu^{\mathbb{Q}}(x),$$

where $e_3 = (0,0,1)^\top \Rightarrow \mathcal{A}$ polynomial preserving.

The mixed conditional moments satisfy the backward Kolmogorov equation. Solve the PDE by guessing that the solution is a polynomial in X_{t_0} of degree N. Apply $\mathcal{A}$ to the mixed powers $1, x_1, ..., x_2 x_3^{N-1}, x_3^N$ and collect terms.

Moments of L_t

- Powers of L_T are obtained from

$$
\begin{aligned}
L_T^n = (Y_T + X_{3T})^n &= \sum_{k=0}^{n} \binom{n}{k} Y_T^k X_{3T}^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} K_Y^k (k!) \int_{t_0}^{T} \int_{t_1}^{T} \dots \int_{t_{k-1}}^{T} g(X_{t_1}) \cdots g(X_{t_k})\, dt_k \dots dt_1\, X_{3T}^{n-k}.
\end{aligned}
$$

- Moments of L_T calculated using nested conditional expectations:

$$
\begin{aligned}
\mathbb{E}_{t_0}^{\mathbb{Q}} \left[g(X_{t_1}) \cdots g(X_{t_k}) X_{3T}^{n-k} \right] &= \mathbb{E}_{t_0}^{\mathbb{Q}} \Big[g(X_{t_1}) \cdots g(X_{t_k}) \underbrace{\mathbb{E}_{t_k}^{\mathbb{Q}} \left[X_{3T}^{n-k} \right]}_{P_0(t_k, X_{t_k})} \Big] \\
&= \mathbb{E}_{t_0}^{\mathbb{Q}} \Big[g(X_{t_1}) \cdots g(X_{t_{k-1}}) \underbrace{\mathbb{E}_{t_{k-1}}^{\mathbb{Q}} \Big[g(X_{t_k}) P_0(t_k, X_{t_k}) \Big]}_{P_1(t_k, t_{k-1}, X_{t_{k-1}})} \Big] \\
&= P_k(t_k, t_{k-1}, .., t_0, X_{t_0}).
\end{aligned}
$$

Investing in variance swaps

- Variance swap issued at t^* with maturity $T^* = t^* + \tau$.
- **Nominal spot value** Γ_t at date $t \in [t^*, T^*]$ of a one dollar notional long position in this variance swap:

$$\begin{aligned}\Gamma_t &= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T^*-t)}\frac{1}{\tau}\left(\int_{t^*}^{T^*} v_s^{\mathbb{Q}}\,ds - \tau VS(t^*, T^*)\right) \mid \mathcal{F}_t\right] \\ &= \frac{\mathrm{e}^{-r(T^*-t)}}{\tau}\left(\int_{t^*}^{t} v_s^{\mathbb{Q}}\,ds + (T^* - t)VS(t, T^*) - \tau VS(t^*, T^*)\right)\end{aligned}$$

- In stochastic differential form: $d\Gamma_t = \Gamma_t\, r\, dt + dM_t$ with the $\mathbb{Q}$-martingale increment excess return:

$$\begin{aligned}dM_t &= \frac{\mathrm{e}^{-r(T^*-t)}}{\tau}\left(v_t^{\mathbb{Q}}\,dt + d\left((T^* - t)VS(t, T^*)\right)\right) \\ &= \frac{\mathrm{e}^{-r(T^*-t)}}{\tau}\nabla_x G(T^* - t, X_t)^\top \Sigma(X_t)\,dW_t\end{aligned}$$

Investing in variance swaps

- Consider an investor with wealth V_t who enters a position with relative notional exposure of n_t. The cost is $n_t V_t \Gamma_t$.

- The remainder of the wealth, $V_t - n_t V_t \Gamma_t$, is invested in the risk free bond.

- The resulting rate of return of this investment is

$$\frac{dV_t}{V_t} = (1 - n_t \Gamma_t)\, r\, dt + n_t\, d\Gamma_t = r\, dt + n_t\, dM_t.$$

- Extends easily to τ-variance swaps that are issued at a sequence of inception dates $0 = t_0^* < t_1^* < \cdots$, with $t_{k+1}^* - t_k^* \leq \tau$. At any date $t \in [t_k^*, t_{k+1}^*)$ the investor takes a position in the respective on-the-run τ-variance swap with maturity $T^*(t) = t_k^* + \tau$.

Optimal portfolio problem

- One option needed to complete the market, with price $O_t = O(S_t, X_t)$. The $\mathbb{Q}$-dynamics of O_t is

$$dO_t = r\,O_t\,dt + \left(\partial_s O_t\,S_t\sigma(X_t)\mathbf{R}(X_t)^\top + \nabla_x O_t^\top \Sigma(X_t)\right) dW_t + \Delta O_t\,(dN_t - \nu^{\mathbb{Q}}(X_t)\,dt)$$

Objective: Find the **optimal dynamic portfolio allocation**:

- w_t denotes the fraction of wealth invested in the stock index
- ϕ_t the fraction of wealth invested in the option
- $\mathbf{n}_t = (n_{1t}, \ldots, n_{nt})^\top$ relative notional exposures to each on-the-run τ_i-variance swap, $i = 1, \ldots, n$.

Optimal portfolio problem

- The resulting wealth process has dynamics

$$\begin{aligned}\frac{dV_t}{V_{t-}} &= \mathbf{n}_t^\top d\Gamma_t + w_t \frac{dS_t}{S_{t-}} + \phi_t \frac{dO_t}{O_{t-}} + (1 - \mathbf{n}_t^\top \Gamma_t - w_t - \phi_t)\, r\, dt \\ &= r\, dt + \theta_t^{W\top} dW_t + \theta_t^N \xi \left(dN_t - \nu^{\mathbb{Q}}(X_t)\, dt\right)\end{aligned}$$

- θ_t^W and θ_t^N are defined by $\begin{pmatrix}\theta_t^W \\ \theta_t^N\end{pmatrix} = \mathcal{G}_t \begin{pmatrix}\mathbf{n}_t \\ w_t \\ \phi_t\end{pmatrix}$ with

$$\mathcal{G}_t = \begin{pmatrix}\Sigma(X_t)^\top & \sigma(X_t)\mathbf{R}(X_t) & \mathbf{0}_{d\times 1} \\ \mathbf{0}_{1\times m} & 0 & 1\end{pmatrix} \begin{pmatrix}\mathcal{D}_t & \mathbf{0}_{m\times 1} & \frac{\nabla_x O_t}{O_{t-}} \\ \mathbf{0}_{1\times n} & \sigma(X_t) & \frac{\partial_s O_t S_t \sigma(X_t)}{O_{t-}} \\ \mathbf{0}_{1\times n} & 1 & \frac{\Delta O_t}{\xi O_{t-}}\end{pmatrix}$$

and $\mathcal{D}_t$ is the $m \times n$ matrix whose ith column is given by $\left(\mathrm{e}^{-r(T_i^*(t)-t)}/\tau_i\right) \nabla_x G(T_i^*(t) - t, X_t)$.

Optimal Investment

- **Power utility function:** $u(V) = \frac{V^{1-\eta}}{1-\eta}, \quad \eta > 0$
- **Optimization problem:** $\max_{\mathbf{n},\omega} \mathbb{E}^{\mathbb{P}}[u(V_T)]$.
- **Pricing kernel:**

$$\frac{d\pi_t}{\pi_{t-}} = -r\,dt - \Lambda(X_t)^\top dW_t^{\mathbb{P}} + \left(\frac{\nu^{\mathbb{Q}}(X_t)}{\nu^{\mathbb{P}}(X_t)} - 1\right)(dN_t - \nu^{\mathbb{P}}(X_t)dt)$$

- **Assumption:** The market is complete with respect to the stock index, the index option, and the n on-the-run τ_i-variance swaps. Specifically, we assume that $n = m = d - 1$, and that the $(d+1) \times (d+1)$ matrix $\mathcal{G}_t$ is invertible $dt \otimes d\mathbb{Q}$-a.s.
- Indirect utility function:

$$J(t, v, x) = \max_{\{\theta_s^W, \theta_s^N, t \le s \le T\}} \mathbb{E}^{\mathbb{P}}\left[\frac{V_T^{1-\eta}}{1-\eta} \middle| V_t = v, X_t = x \right]$$

Optimal Investment

HJB equation:

$$0 = \max_{\theta^W,\,\theta^N} \Bigg\{ \frac{\partial J}{\partial t} + \frac{\partial J}{\partial v} v \left(r + \theta^{W\top} \Lambda(x) - \theta^N \xi \nu^{\mathbb{Q}}(x) \right) + \frac{1}{2} \frac{\partial^2 J}{\partial v^2} v^2 \theta^{W\top} \theta^W$$

$$+ \nabla_x J^\top (\mu(x) + \Sigma(x)\Lambda(x)) + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 J}{\partial x_i \partial x_j} \left(\Sigma(x)\Sigma(x)^\top \right)_{ij}$$

$$+ \theta^{W\top} v \Sigma(x)^\top \nabla_x \left(\frac{\partial J}{\partial v} \right) + \left(J(t, v(1+\theta^N \xi), x) - J(t, v, x) \right) \nu^{\mathbb{P}}(x) \Bigg\}$$

Optimal Allocation

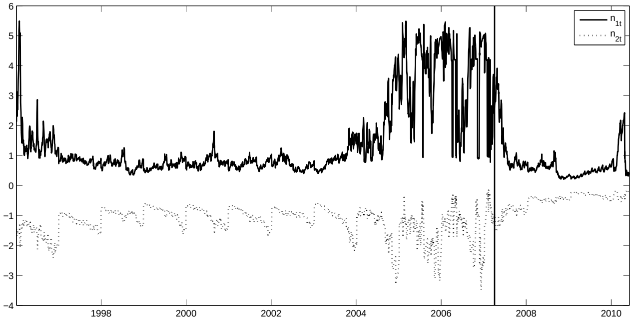
There exists an optimal strategy $\mathbf{n}_t^*$, w_t^*, ϕ_t^* recovered from:

$$\theta_t^{W*} = \frac{1}{\eta} \Lambda(X_t) + \Sigma(X_t)^\top \nabla_x h(T-t, X_t),$$

$$\theta_t^{N*} = \frac{1}{\xi} \left(\left(\frac{\nu^{\mathbb{P}}(X_t)}{\nu^{\mathbb{Q}}(X_t)} \right)^{1/\eta} - 1 \right),$$

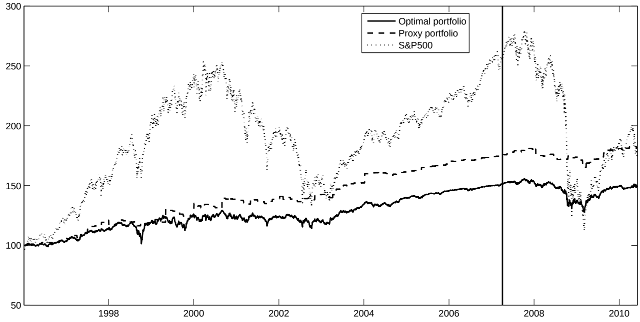
where h is such that e^h satisfies a known PDE.

Example: bivariate model, $\eta = 5$



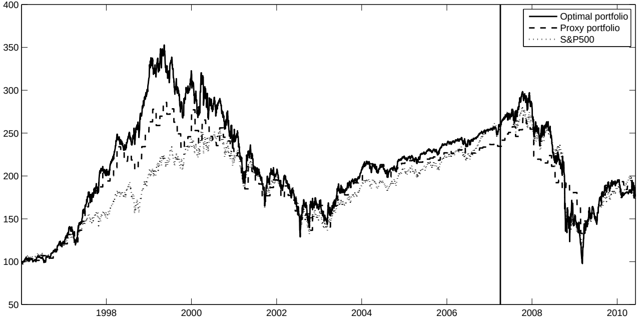
- **Short position in 2-year variance swap** (earning variance risk premium) and **long position in 3-month variance swaps** (hedging volatility risk)
- Periodic patterns
- With $\lambda^Q/\lambda^P = 1.4$ and $\xi = -25\%$: Positive optimal weight in stock and option.

Wealth incurred by the optimal investment



- Smooth wealth growth with little volatility
- Suited for risk-averse investors.

Investor with logarithmic utility



- The optimal portfolio exhibits larger fluctuations than the stock $\Rightarrow$ The investor gains the risk premia.
- Also suited for less risk-averse investors.

Conclusion

- We develop a quadratic variance swap model which is tractable and parsimonious in the number of parameters.
- Variance swap rates are available in closed-form, up to the resolution of ODEs.
- We derive an pricing methodology for European index options, which uses the polynomial preserving property of quadratic jump-diffusions to approximate the characteristic function of the log price.
- We show an application of our option pricing method to dynamic portfolio allocation.

Thank you for your attention!