Option pricing in a quadratic variance swap model

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Motivation:

- More and more instruments traded on variance
- Affine models fail to reproduce large increases in variance
- Quadratic models are an appealing alternative to jumps in variance
- Index options useful in portfolio allocation
- But not straightforward to price.

In this talk:

1. Provide a short introduction to quadratic variance swap models
2. Derive an option pricing method
3. Illustrate how this method can be applied in a portfolio allocation exercise.
Realized variance

- Filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})\).
- The \(\mathbb{Q}\)-dynamics of the index are specified as
  \[
  \frac{dS_t}{S_{t^-}} = r_t \, dt + \sigma_t \, dB_t + \int_{\mathbb{R}} \xi (\chi(dt, d\xi) - \nu_t^\mathbb{Q}(d\xi)dt),
  \]
- Let \(t = t_0 < t_1 < \cdots < t_n = T\). The annualized realized variance is given as:
  \[
  RV(t, T) = \frac{252}{n} \sum_{i=1}^{n} \left( \log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2
  \]
  \[\longrightarrow \quad QV(t, T) = \frac{252}{n} \sigma_t^2 \int_t^T ds + \int_t^T \int_{\mathbb{R}} (\log(1 + \xi))^2 \chi(ds, d\xi).\]
Variance swaps

A variance swap initiated at $t$ with maturity $T$, or term $T - t$, pays the difference between the annualized realized variance $RV(t, T)$ and the variance swap rate $VS(t, T)$ fixed at $t$. No arbitrage implies that

$$VS(t, T) = \frac{1}{T - t} \mathbb{E}^Q [QV(t, T) | \mathcal{F}_t] = \frac{1}{T - t} \mathbb{E}^Q \left[ \int_t^T \nu_s^Q \, ds \mid \mathcal{F}_t \right],$$

where the $Q$-spot variance process is

$$\nu_t^Q = \sigma^2_t + \int_{\mathbb{R}} (\log(1 + \xi))^2 \nu_t^Q (d\xi).$$
Quadratic variance swap models

- $X$ is a diffusion process in $\mathbb{R}^m$. Under $Q$, it satisfies:

$$dX_t = \mu(X_t)dt + \Sigma(X_t)dW_t.$$ 

- $X$ is quadratic if its drift and diffusion functions are linear and quadratic in the state variable:

$$\mu(x) = b + \beta x,$$

$$\Sigma(x)\Sigma(x)^T = a + \sum_{k=1}^{m} \alpha^k x_k + \sum_{k,l=1}^{m} A^{kl} x_k x_l.$$ 

- A quadratic variance swap model is obtained under the assumption that the $Q$-spot variance is a quadratic function of the latent state variable $X_t$:

$$\nu_t^Q = g(X_t) = \phi + \psi^T X_t + X_t^T \pi X_t$$

for $\phi \in \mathbb{R}$, $\psi \in \mathbb{R}^m$, and $\pi \in \mathbb{S}^m$. 

Option pricing in a quadratic variance swap model
Theorem: Quadratic term structure of variance

The quadratic variance swap model admits a quadratic term structure:

\[(T - t)VS(t, T) = \Phi(T - t) + \Psi(T - t)^T X_t + X_t^T \Pi(T - t) X_t = G(T - t, X_t)\]

where \(\Phi\), \(\Psi\) and \(\Pi\) satisfy the linear system of ODEs

\[
\frac{d\Phi(\tau)}{d\tau} = \phi + b^T \Psi(\tau) + tr(a\Pi(\tau)) \quad \Phi(0) = 0
\]

\[
\frac{d\Psi(\tau)}{d\tau} = \psi + \beta^T \Psi(\tau) + 2\Pi(\tau)b + \alpha \cdot \Pi(\tau) \quad \Psi(0) = 0
\]

\[
\frac{d\Pi(\tau)}{d\tau} = \pi + \beta^T \Pi(\tau) + \Pi(\tau)\beta + A \cdot \Pi(\tau) \quad \Pi(0) = 0
\]

where \((\alpha \cdot \Pi)_k = tr(\alpha^k \Pi)\) and \((A \cdot \Pi)_{kl} = tr(A^{kl} \Pi)\).
A Bivariate Model Specification

- Assume $X_t = (X_{1t}, X_{2t})^T$:

$$
\begin{align*}
    dX_{1t} &= (b_1 + \beta_{11}X_{1t} + \beta_{12}X_{2t})dt + \sqrt{a_1 + \alpha_1X_{1t} + A_1X_{1t}^2}dW_{1t}^* \\
    dX_{2t} &= (b_2 + \beta_{22}X_{2t})dt + \sqrt{a_2 + \alpha_2X_{2t} + A_2X_{2t}^2}dW_{2t}^*
\end{align*}
$$

- Spot variance quadratic in $X_{1t}$:

$$v_t = \phi_0 + \psi_0X_{1t} + X_{1t}\pi_0X_{1t}.$$  

- **Interpretation**: variance mean-reverts to a stochastic level

$$\frac{b_1 + \beta_{12}X_{2t}}{|\beta_{11}|}.$$  

- Explicit representation of forward variance $f(t, T)$.  

Option pricing in a quadratic variance swap model
Estimation results

Option pricing in a quadratic variance swap model
Modelling assumptions

- The price process jumps by a deterministic size $\xi > -1$. Jumps are driven by a Poisson process:

$$\frac{dS_t}{S_{t^-}} = r \, dt + \sigma(X_t) \, R(X_t) \, dW_t + \xi \, (dN_t - \nu^Q(X_t) \, dt).$$

The $\mathbb{Q}$-spot variance is given by:

$$\nu^Q_t = g(X_t) = \sigma(X_t)^2 + (\log(1 + \xi))^2 \, \nu^Q(X_t).$$

- $\nu^Q(x) = \nu^Q \sigma(x)^2$,

$$\sigma(x)^2 = \frac{g(x)}{1 + (\log(1 + \xi))^2 \, \nu^Q} = K_{\sigma^2} g(x).$$
Option pricing

Main steps:

- Derive the moments of the log price process $L_t = \log S_t$.

- Edgeworth expansion of the characteristic function of $L_T | \mathcal{F}_{t_0}$:

$$\mathbb{E}_{t_0}^Q \left[ e^{zL_T} \right] = \exp \left( \sum_{n=1}^{\infty} C_n \frac{z^n}{n!} \right) = \exp \left( C_1 z + C_2 \frac{z^2}{2} \right) \left( 1 + C_3 \frac{z^3}{3!} + O(z^4) \right).$$

Decomposition of $L_t = \log S_t$

For $t \geq t_0 \geq 0$, $L_t = Y_t + X_{3t}$ where

$$Y_t = \int_{t_0}^{t} \left( (\log(1 + \xi) - \xi)\nu^Q(X_s) - \frac{1}{2}\sigma(X_s)^2 \right) ds$$

$$= K_Y \int_{t_0}^{t} g(X_s) ds,$$

where

$$K_Y = -\frac{(\xi - \log(1 + \xi))\nu^Q + 1/2}{1 + (\log(1 + \xi))^2\nu^Q} < 0.$$ 

$$dX_{3t} = r\ dt + \sigma(X_t)R(X_t)^\top dW_t + \log(1 + \xi)(dN_t - \nu^Q(X_t)dt), \quad X_{3t_0} = \log S_{t_0}.$$
Analysis of $X_t$

- Define the jump-diffusion process $X_t = (X_{1t}, X_{2t}, X_{3t})^T$. Its diffusion matrix $A(x)$ is

$$A(x) = \begin{pmatrix} 1 + A_1 x_1^2 & 0 & R_1(x) \sigma(x) \sqrt{1 + A_1 x_1^2} \\ 0 & x_2 + A_2 x_2^2 & 0 \\ R_1(x) \sigma(x) \sqrt{1 + A_1 x_1^2} & 0 & \sigma(x)^2 \end{pmatrix}.$$ 

- We want $q_0$, $q_1$ and $q_2$ such that

$$R_1(x) \sigma(x) \sqrt{1 + A_1 x_1^2} = R_1(x_1) \sqrt{K_{\sigma^2} g(x_1) \sqrt{1 + A_1 x_1^2}} = q_0 + q_1 x_1 + q_2 x_1^2.$$ 

- To capture the leverage effect, $R_1(x_1) \approx -\text{sign}(\psi + 2\pi X_{1t}) \times 0.7$. Hence we choose $q_0$, $q_1$ and $q_2$ to match the highest order terms of:

$$(q_0 + q_1 x_1 + q_2 x_1^2)^2 \approx 0.7^2 K_{\sigma^2} g(x_1)(1 + A_1 x_1^2).$$

- With this specification, $X_t$ is a quadratic jump-diffusion process and hence is polynomial preserving.
Moments of $X_T$


**Conditional moments of $X_T$**

Let $D = \frac{(3+N)(2+N)(1+N)}{6}$ denote the dimension of the space of polynomials in $X_T$ of degree $N$ or less. The $D$-row vector of the mixed $\mathcal{F}_{t_0}$-conditional moments of $X_T$ of order $N$ or less with $T \geq t_0$ is given by

$$
\left(1, \mathbb{E}^Q[X_{1T}|\mathcal{F}_{t_0}], \ldots, \mathbb{E}^Q[X_{2T}X_{3T}^{N-1}|\mathcal{F}_{t_0}], \mathbb{E}^Q[X_{3T}^N|\mathcal{F}_{t_0}]\right)
$$

$$
= \left(1, X_{1t_0}, \ldots, X_{2t_0}X_{3t_0}^{N-1}, X_{3t_0}^N\right) e^{\tilde{B}(T-t_0)},
$$

where $\tilde{B}$ is an upper block triangular $D \times D$ matrix and $e^{\tilde{B}(T-t_0)}$ denotes the matrix exponential of $\tilde{B}(T-t_0)$. 

Option pricing in a quadratic variance swap model
Moments of $X_T$

**Proof:** The generator of $X_t$ is given by:

$$
Af(x) = \left( \begin{array}{c}
\beta_{11}x_1 + \beta_{12}x_2 \\
b_2 + \beta_{22}x_2 \\
r
\end{array} \right)^T \nabla_x f(x) + \frac{1}{2} \sum_{i,j=1}^3 A_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} 
+ (f(x + \log(1 + \xi)e_3) - f(x) - \nabla_x f(x)^T e_3 \log(1 + \xi)) \nu^Q(x),
$$

where $e_3 = (0, 0, 1)^T \Rightarrow A$ polynomial preserving.

The mixed conditional moments satisfy the backward Kolmogorov equation. Solve the PDE by guessing that the solution is a polynomial in $X_{t_0}$ of degree $N$. Apply $A$ to the mixed powers $1, x_1, ..., x_2x_3^{N-1}, x_3^N$ and collect terms.
Moments of $L_t$

- Powers of $L_T$ are obtained from

$$L_T^n = (Y_T + X_{3T})^n = \sum_{k=0}^{n} \binom{n}{k} Y_T^k X_{3T}^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} K_Y^k (k!) \int_{t_0}^{T} \int_{t_1}^{T} \cdots \int_{t_{k-1}}^{T} g(X_{t_1}) \cdots g(X_{t_k}) \, dt_k \cdots dt_1 X_{3T}^{n-k}.$$ 

- Moments of $L_T$ calculated using nested conditional expectations:

$$\mathbb{E}_{t_0}^{Q} \left[ g(X_{t_1}) \cdots g(X_{t_k}) X_{3T}^{n-k} \right] = \mathbb{E}_{t_0}^{Q} \left[ g(X_{t_1}) \cdots g(X_{t_k}) \underbrace{\mathbb{E}_{t_k}^{Q} \left[ X_{3T}^{n-k} \right]}_{P_0(t_k, X_{t_k})} \right]$$

$$= \mathbb{E}_{t_0}^{Q} \left[ g(X_{t_1}) \cdots g(X_{t_{k-1}}) \underbrace{\mathbb{E}_{t_{k-1}}^{Q} \left[ g(X_{t_k}) P_0(t_k, X_{t_k}) \right]}_{P_1(t_k, t_{k-1}, X_{t_{k-1}})} \right]$$

$$= P_k(t_k, t_{k-1}, \ldots, t_0, X_{t_0}).$$
Investing in variance swaps

- Variance swap issued at $t^*$ with maturity $T^* = t^* + \tau$.

- **Nominal spot value** $\Gamma_t$ at date $t \in [t^*, T^*]$ of a one dollar notional long position in this variance swap:

$$\Gamma_t = \mathbb{E}^Q \left[ e^{-r(T^*-t)} \frac{1}{\tau} \left( \int_{t^*}^{T^*} v_s^Q \, ds - \tau VS(t^*, T^*) \right) \mid \mathcal{F}_t \right]$$

$$= \frac{e^{-r(T^*-t)}}{\tau} \left( \int_{t^*}^{t} v_s^Q \, ds + (T^* - t) VS(t, T^*) - \tau VS(t^*, T^*) \right)$$

- In stochastic differential form: $d\Gamma_t = \Gamma_t \, r \, dt + dM_t$ with the $Q$-martingale increment excess return:

$$dM_t = \frac{e^{-r(T^*-t)}}{\tau} \left( \frac{v_t^Q \, dt + d \left( (T^* - t) VS(t, T^*) \right) \Sigma(X_t) \, dW_t}{\Sigma(X_t)} \right)$$

= \frac{e^{-r(T^*-t)}}{\tau} \nabla_x G(T^* - t, X_t) \Sigma(X_t) dW_t
Investing in variance swaps

Consider an investor with wealth $V_t$ who enters a position with relative notional exposure of $n_t$. The cost is $n_t V_t \Gamma_t$.

The remainder of the wealth, $V_t - n_t V_t \Gamma_t$, is invested in the risk free bond.

The resulting rate of return of this investment is

$$\frac{dV_t}{V_t} = (1 - n_t \Gamma_t) r \, dt + n_t \, d\Gamma_t = r \, dt + n_t \, dM_t.$$ 

Extends easily to $\tau$-variance swaps that are issued at a sequence of inception dates $0 = t_0^* < t_1^* < \cdots$, with $t_{k+1}^* - t_k^* \leq \tau$. At any date $t \in [t_k^*, t_{k+1}^*)$ the investor takes a position in the respective on-the-run $\tau$-variance swap with maturity $T^*(t) = t_k^* + \tau$. 

Option pricing in a quadratic variance swap model
Optimal portfolio problem

- One option needed to complete the market, with price $O_t = O(S_t, X_t)$. The $\mathbb{Q}$-dynamics of $O_t$ is

$$dO_t = r O_t \, dt + \left( \partial_s O_t \, S_t \sigma(X_t) R(X_t)^\top + \nabla_x O_t^\top \Sigma(X_t) \right) dW_t$$

$$+ \Delta O_t \left( dN_t - \nu^\mathbb{Q}(X_t) \, dt \right)$$

**Objective:** Find the **optimal dynamic portfolio allocation:**

- $w_t$ denotes the fraction of wealth invested in the stock index
- $\phi_t$ the fraction of wealth invested in the option
- $n_t = (n_{1t}, \ldots, n_{nt})^\top$ relative notional exposures to each on-the-run $\tau_i$-variance swap, $i = 1, \ldots, n$. 

Option pricing in a quadratic variance swap model
Optimal portfolio problem

- The resulting wealth process has dynamics

\[
\frac{dV_t}{V_{t-}} = n_t^T d\Gamma_t + w_t \frac{dS_t}{S_t} + \phi_t \frac{dO_t}{O_t} + (1 - n_t^T \Gamma_t - w_t - \phi_t) r \, dt \\
= r \, dt + \theta_t^W dW_t + \theta_t^N \xi (dN_t - \nu^\oplus (X_t) \, dt)
\]

- \(\theta_t^W\) and \(\theta_t^N\) are defined by

\[
\begin{pmatrix}
\theta_t^W \\
\theta_t^N \\
\phi_t
\end{pmatrix} = G_t
\begin{pmatrix}
n_t \\
w_t \\
\phi_t
\end{pmatrix}
\]

with

\[
G_t = \begin{pmatrix}
\Sigma(X_t)^T & \sigma(X_t)R(X_t) & 0_{d \times 1} \\
0_{1 \times m} & 0 & 1
\end{pmatrix}
\begin{pmatrix}
D_t & 0_{m \times 1} & \nabla_x O_t \\
0_{1 \times n} & \sigma(X_t) & \partial_s O_t S_t \sigma(X_t) \\
0_{1 \times n} & 1 & \Delta O_t \\
\xi & 0
\end{pmatrix}
\]

and \(D_t\) is the \(m \times n\) matrix whose \(i\)th column is given by

\[
\left( e^{-r(T_i^*(t) - t) / \tau_i} \right) \nabla_x G( T_i^*(t) - t, X_t )
\]
Optimal Investment

- **Power utility function**: \( u(V) = \frac{V^{1-\eta}}{1-\eta}, \ \eta > 0 \)

- **Optimization problem**: \( \max_{n, \omega} \mathbb{E}^P [u(V_T)] \).

- **Pricing kernel**:
  \[
  \frac{d\pi_t}{\pi_{t-}} = -r \, dt - \Lambda(X_t)^T dW_t^P + \left( \frac{\nu^Q(X_t)}{\nu^P(X_t)} - 1 \right) (dN_t - \nu^P(X_t) \, dt)
  \]

- **Assumption**: The market is complete with respect to the stock index, the index option, and the \( n \) on-the-run \( \tau_i \)-variance swaps. Specifically, we assume that \( n = m = d - 1 \), and that the \((d + 1) \times (d + 1)\) matrix \( \mathcal{G}_t \) is invertible \( dt \otimes dQ \)-a.s.

- **Indirect utility function**:
  \[
  J(t, v, x) = \max_{\{\theta_s^W, \theta_s^N, \ t \leq s \leq T\}} \mathbb{E}^P \left[ \frac{V_T^{1-\eta}}{1-\eta} \middle| V_t = v, X_t = x \right]
  \]
Optimal Investment

HJB equation:

\[
0 = \max_{\theta^W, \theta^N} \left\{ \frac{\partial J}{\partial t} + \frac{\partial J}{\partial v} \left( r + \theta^W \Lambda(x) - \theta^N \xi Q(x) \right) + \frac{1}{2} \frac{\partial^2 J}{\partial v^2} \sigma^2 \theta^W \theta^W \\
+ \nabla_x J^T (\mu(x) + \Sigma(x) \Lambda(x)) + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 J}{\partial x_i \partial x_j} \left( \Sigma(x) \Sigma(x)^T \right)_{ij}
+ \theta^W \sigma \Sigma(x) \nabla_x \left( \frac{\partial J}{\partial v} \right) + \left( J(t, \nu(1 + \theta^N \xi), x) - J(t, \nu, x) \right) \nu^\mathcal{P}(x) \right\}
\]

Optimal Allocation

There exists an optimal strategy \( n_t^*, w_t^*, \phi_t^* \) recovered from:

\[
\theta_t^{W^*} = \frac{1}{\eta} \Lambda(X_t) + \Sigma(X_t)^T \nabla_x h(T - t, X_t),
\]

\[
\theta_t^{N^*} = \frac{1}{\xi} \left( \left( \frac{\nu^\mathcal{P}(X_t)}{\nu^\mathcal{Q}(X_t)} \right)^{1/\eta} - 1 \right),
\]

where \( h \) is such that \( e^h \) satisfies a known PDE.
Short position in 2-year variance swap (earning variance risk premium) and long position in 3-month variance swaps (hedging volatility risk)

Periodic patterns

With $\lambda^Q/\lambda^P = 1.4$ and $\xi = -25\%$: Positive optimal weight in stock and option.
Wealth incurred by the optimal investment

- Smooth wealth growth with little volatility
- Suited for risk-averse investors.
The optimal portfolio exhibits larger fluctuations than the stock ⇒ The investor gains the risk premia.

Also suited for less risk-averse investors.
Conclusion

- We develop a quadratic variance swap model which is tractable and parsimonious in the number of parameters.

- Variance swap rates are available in closed-form, up to the resolution of ODEs.

- We derive an pricing methodology for European index options, which uses the polynomial preserving property of quadratic jump-diffusions to approximate the characteristic function of the log price.

- We show an application of our option pricing method to dynamic portfolio allocation.

Thank you for your attention!