Are American options European after all?

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based on joint work with
Sören Christensen (Göteborg) and Matthias Lenga (Kiel)

Zürich, September 18, 2015

Outline

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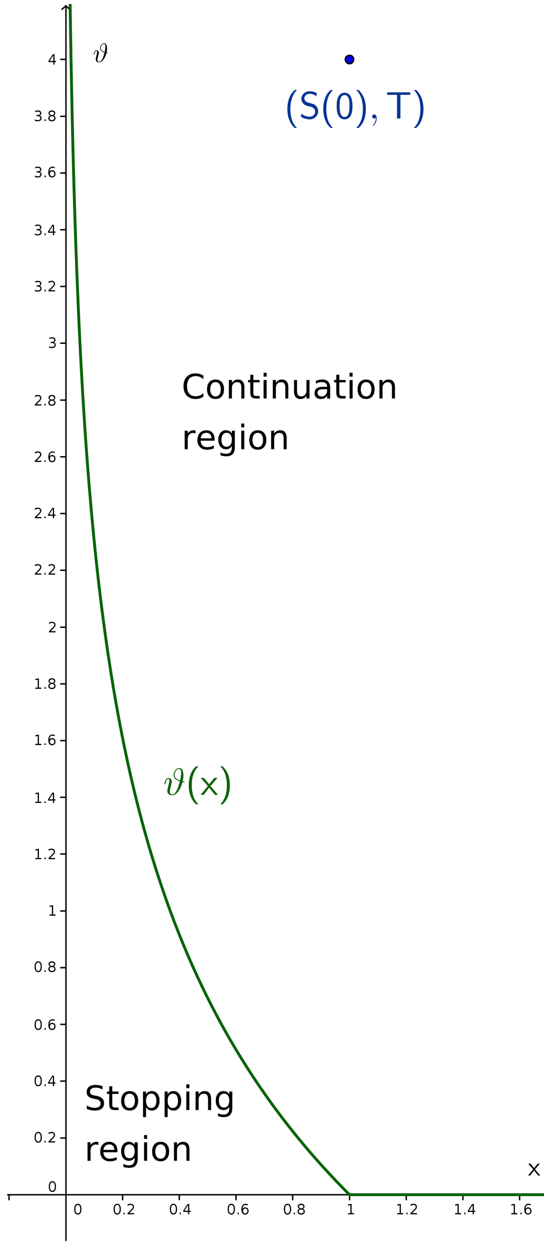
The basic question . . .

. . . which is still somewhat open

- Setup:
  - Black-Scholes model with positive interest rate
  - $v_{\mathrm{Am},g}(\vartheta, x)$: fair value of an American option with payoff function $g(x)$, time to maturity ϑ, stock price x
  - $v_{\mathrm{Eu},f}(\vartheta, x)$: fair value of a European option with payoff function $f(x)$, time to maturity ϑ, stock price x
- Consider the American put $g(x) := (K - x)^+$.
  Question: Is there a representing European payoff $f(x)$ in the sense that
  - $v_{\mathrm{Am},g} = v_{\mathrm{Eu},f}$ in the continuation region of g and
  - $g \leq v_{\mathrm{Eu},f}$ in the stopping region (and hence everywhere)?

  (Or at least for some g? Or even for all g?)
- This would imply that the American put allows for a static European hedge.



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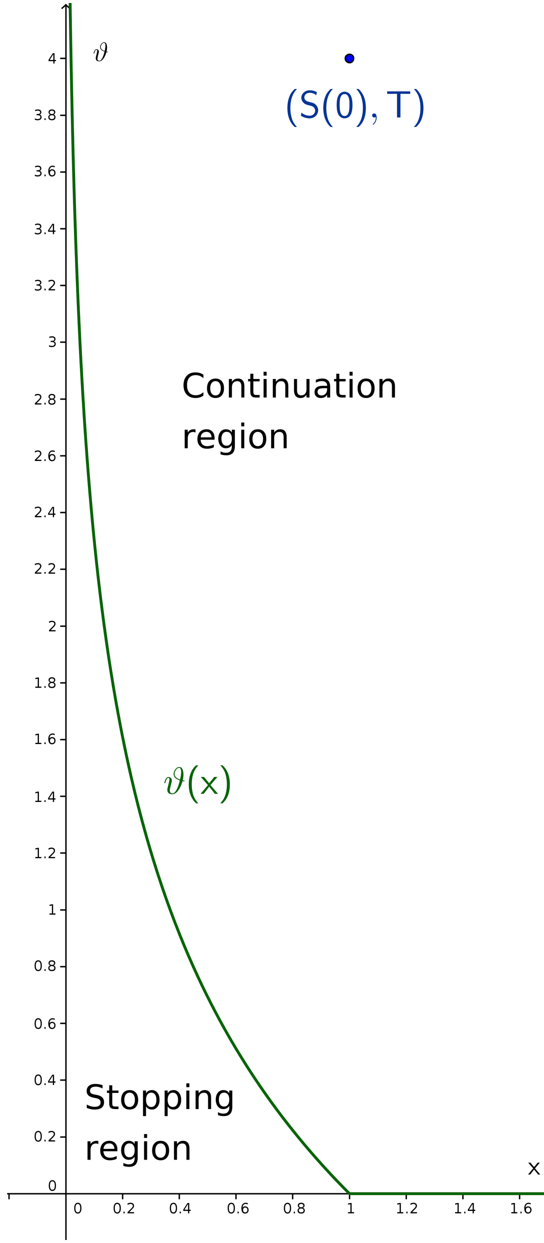

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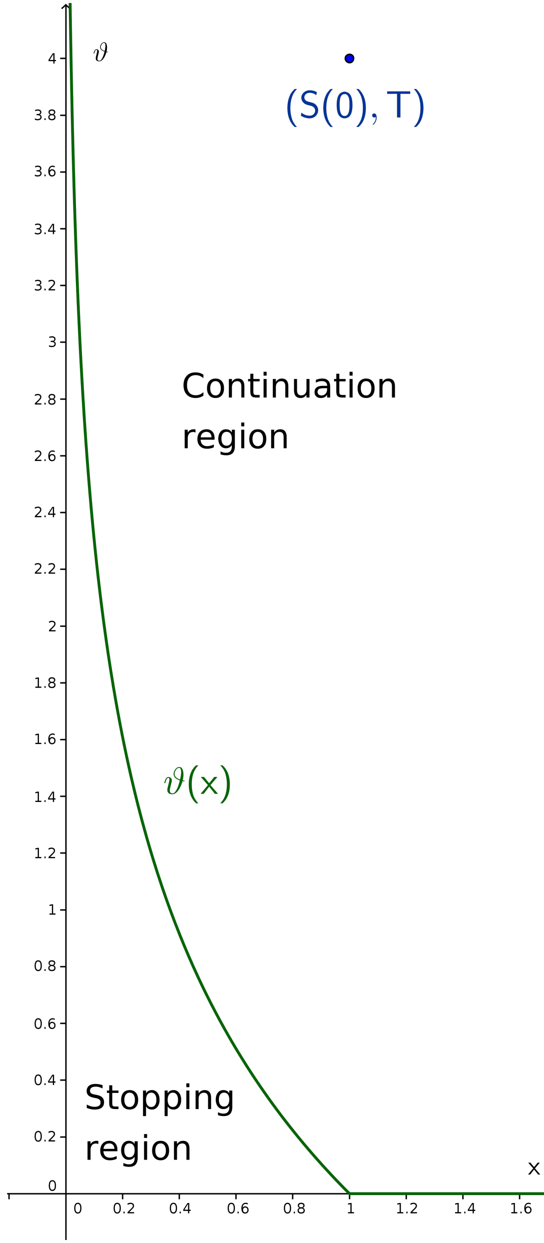

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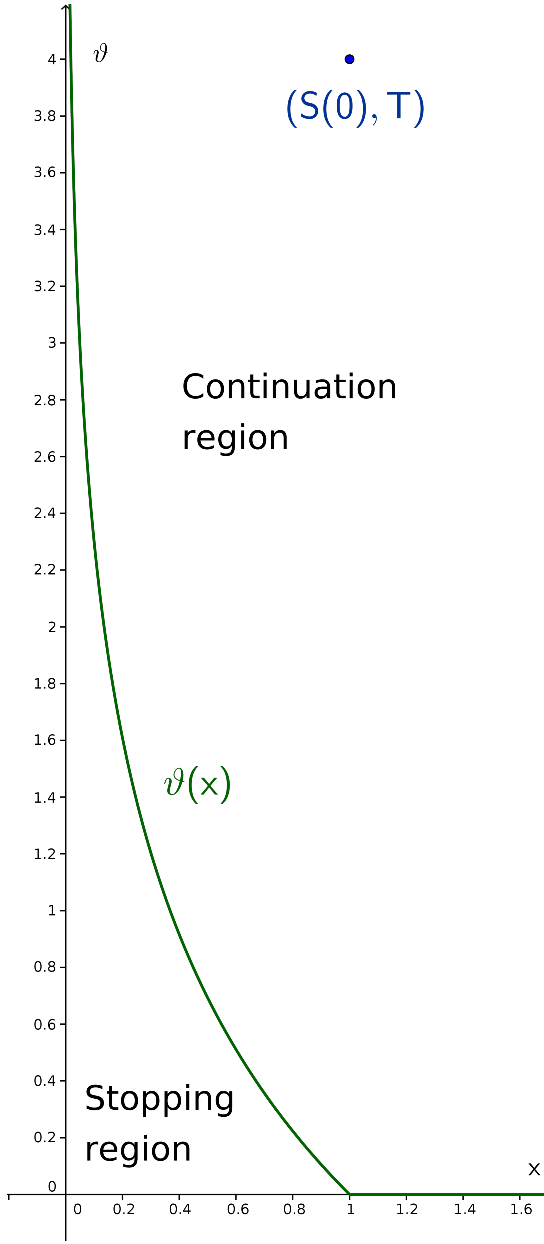

Outline

Cheapest dominating European options (CDEO)

Christensen (Math. Fin. 11)

- Black-Scholes model, American payoff $g(x)$, $T, S(0)$ given
- Solve

$$\min_f v_{\mathrm{Eu},f}(T, S(0))$$

  subject to $v_{\mathrm{Eu},f}(\vartheta, x) \geq g(x)$ for all $x > 0$ and all $\vartheta \leq T$
- CDEO: minimizer f if it exists
- semi-infinite linear programming
- upper bound for $\pi = v_{\mathrm{Am},g}(T, S(0))$, but surprisingly close
- implications of equality $v_{\mathrm{Eu},f}(T, S(0)) = v_{\mathrm{Am},g}(T, S(0))$ (if true):
  - new algorithm for American options
  - static European hedge for American options
  - interpretation of early exercise premium as payoff
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Computing American option prices by minimization

over sets of martingales

- Davis & Karatzas (94), Rogers (02), Haugh & Logan (04):

$$\pi = v_{\mathrm{Am},g}(T, S(0)) = \inf_{M \text{ mart., } M(0)=0} E_Q\left(\sup_{t\in[0,T]} (e^{-rt}g(S(t)) - M(t)) \right)$$

  "$\geq$" follows from the Doob-Meyer decomposition

$$v_{\mathrm{Am},g}(T-t, S(t))e^{-rt} = \pi + M^\star(t) - A^\star(t)$$

  with $M^\star(0) = 0 = A^\star(0)$, $M^\star$ martingale, $A^\star \geq 0$, $A^\star$ increasing.

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Embedded American options

Jourdain & Martini (Ann. IHP Anal. nonlin. 01, AAP 02)

- Black-Scholes model,
  given European payoff $f(x)$
- embedded American payoff

$$g(x) = \inf_{\vartheta} v_{\mathrm{Eu},f}(\vartheta, x) \quad \Big(= v_{\mathrm{Eu},f}(\vartheta(x), x) \Big)$$

  ($\vartheta \in [0, \infty)$ or $\vartheta \in [0, T]$)
- If curve $x \mapsto \vartheta(x)$ is nice:
  - $g \leq v_{\mathrm{Eu},f}$,
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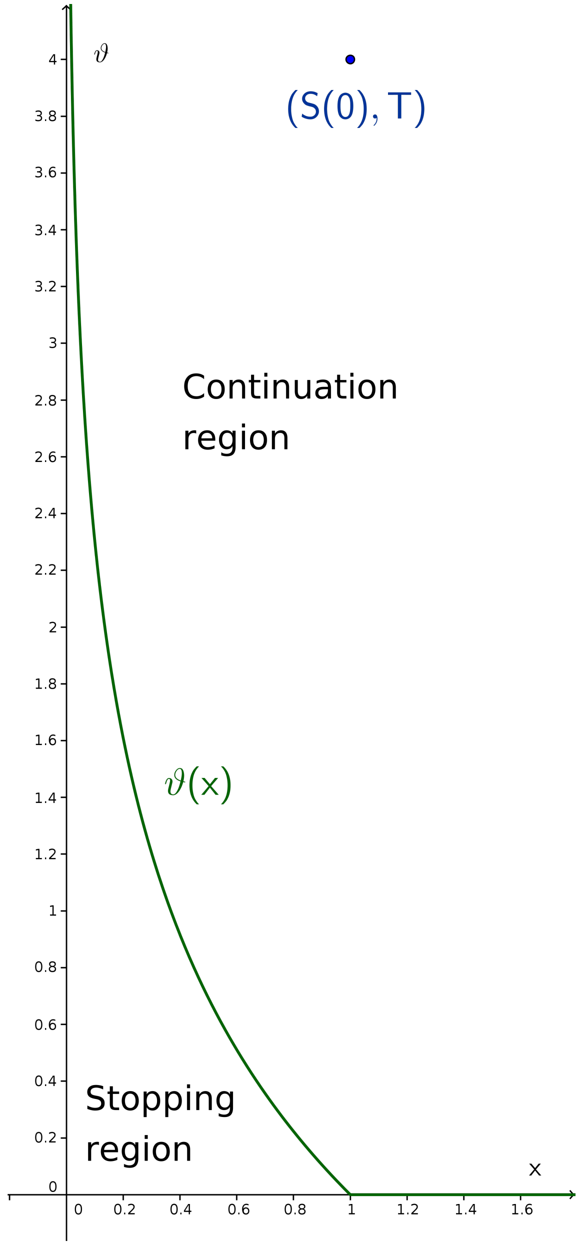

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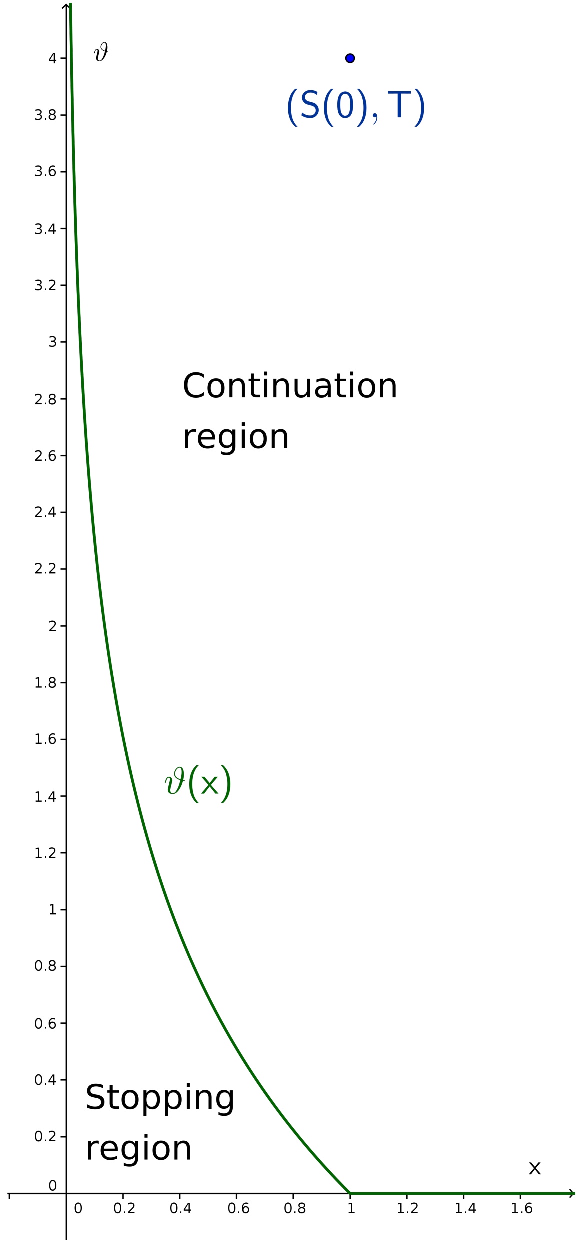

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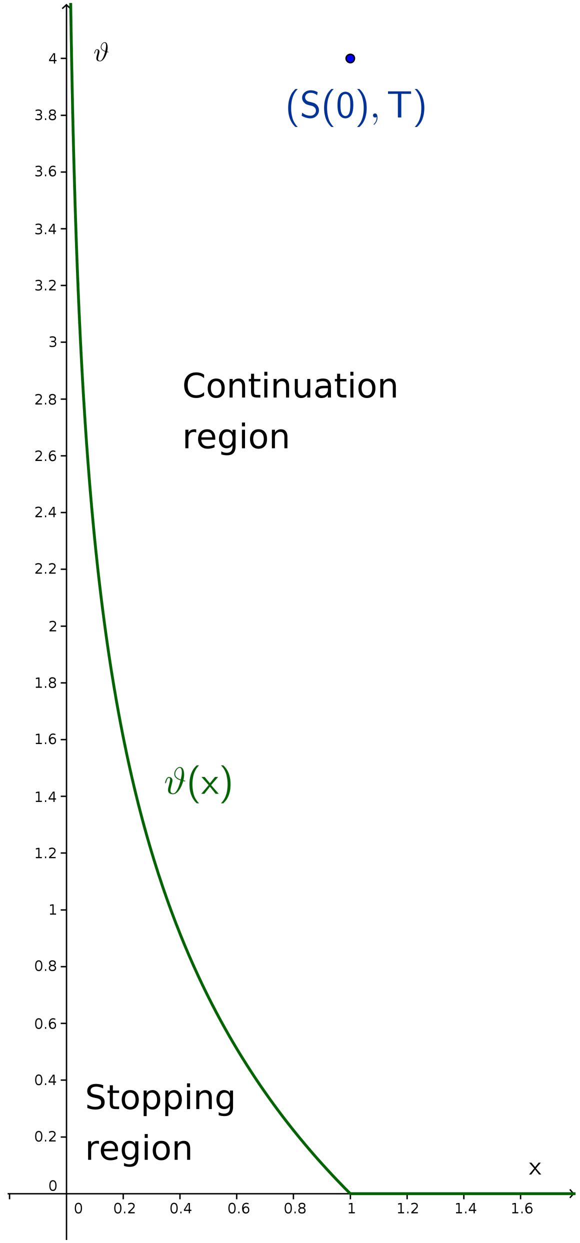

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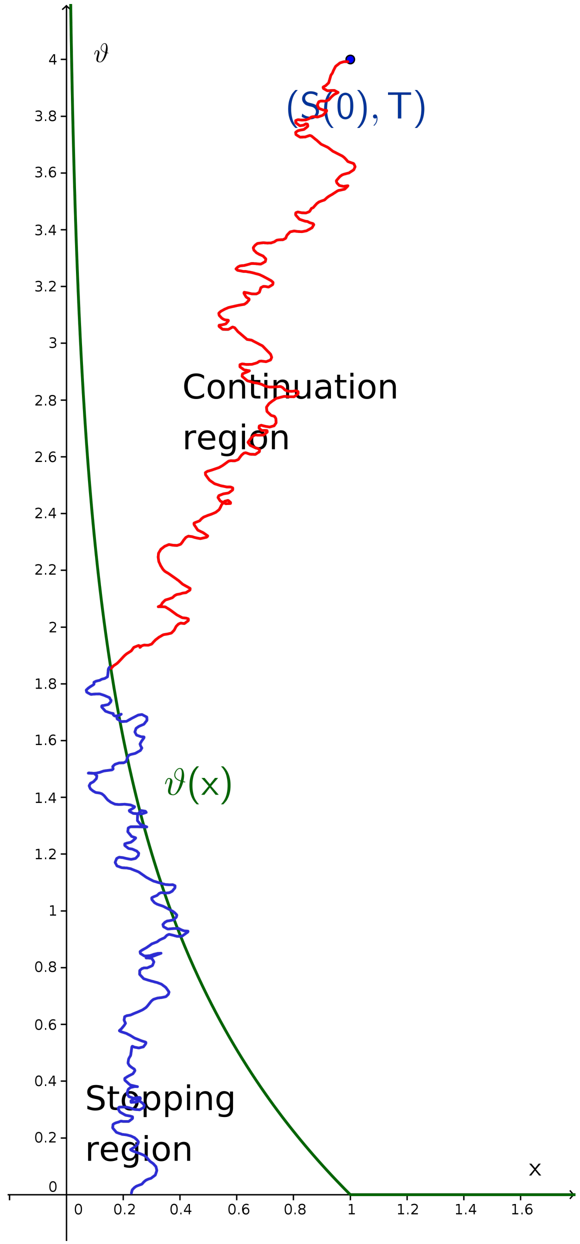

Examples

of embedded American payoffs

- $B(t) = 1$,
  $S(t) = \exp(\sqrt{2}W(t) - t)$
- European payoff
  $f(x) = 3x^{1/2} + x^{3/2}$
- American payoff
  $g(x) = 4x^{3/4}1_{\{x<1\}} + f(x)1_{\{x\geq 1\}}$
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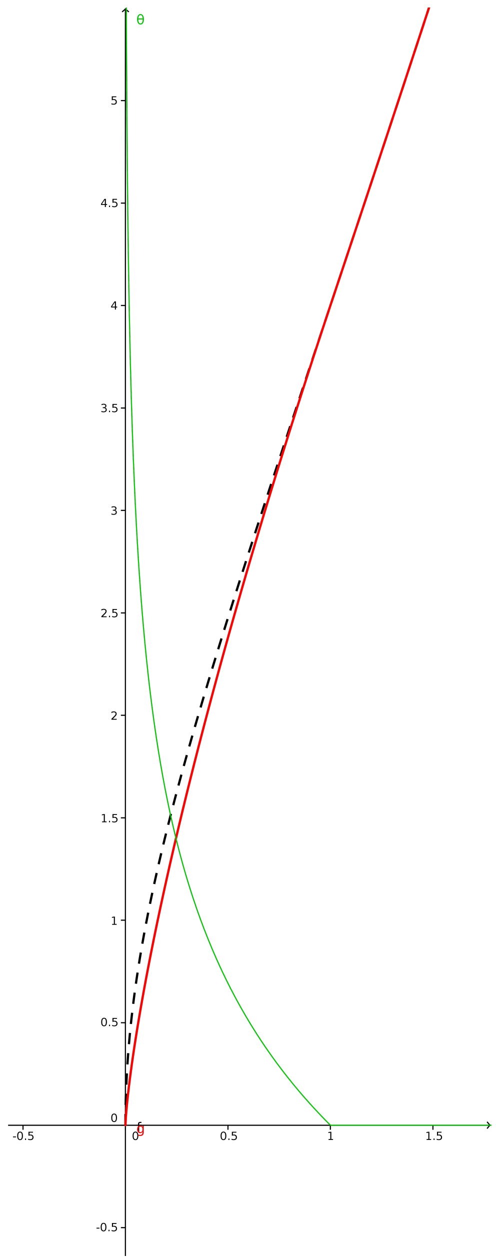
  $\vartheta(x) = -\log(x)1_{\{x<1\}}$
- Note that g is analytic on $(0, 1)$.



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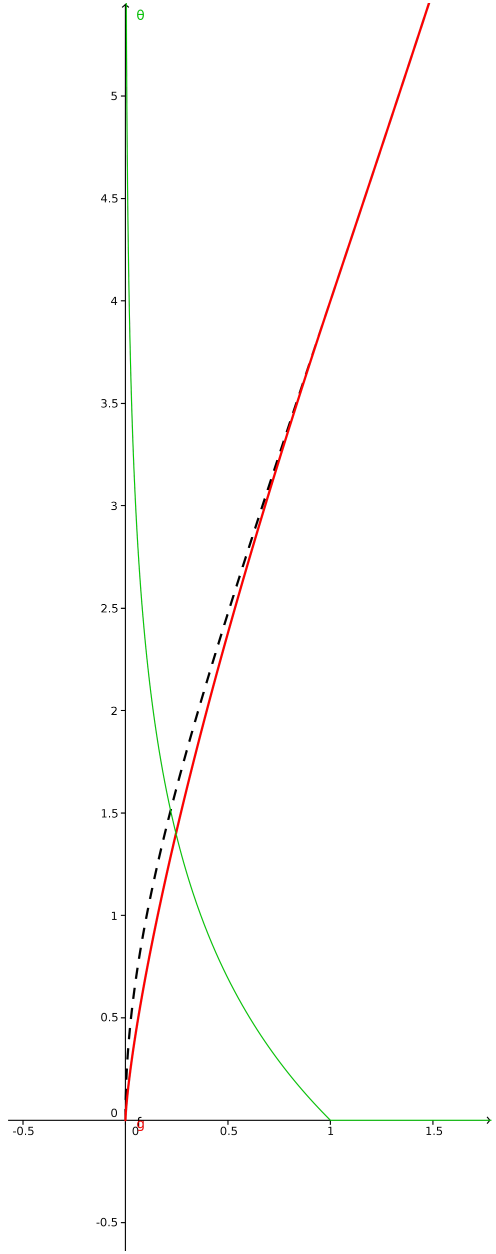
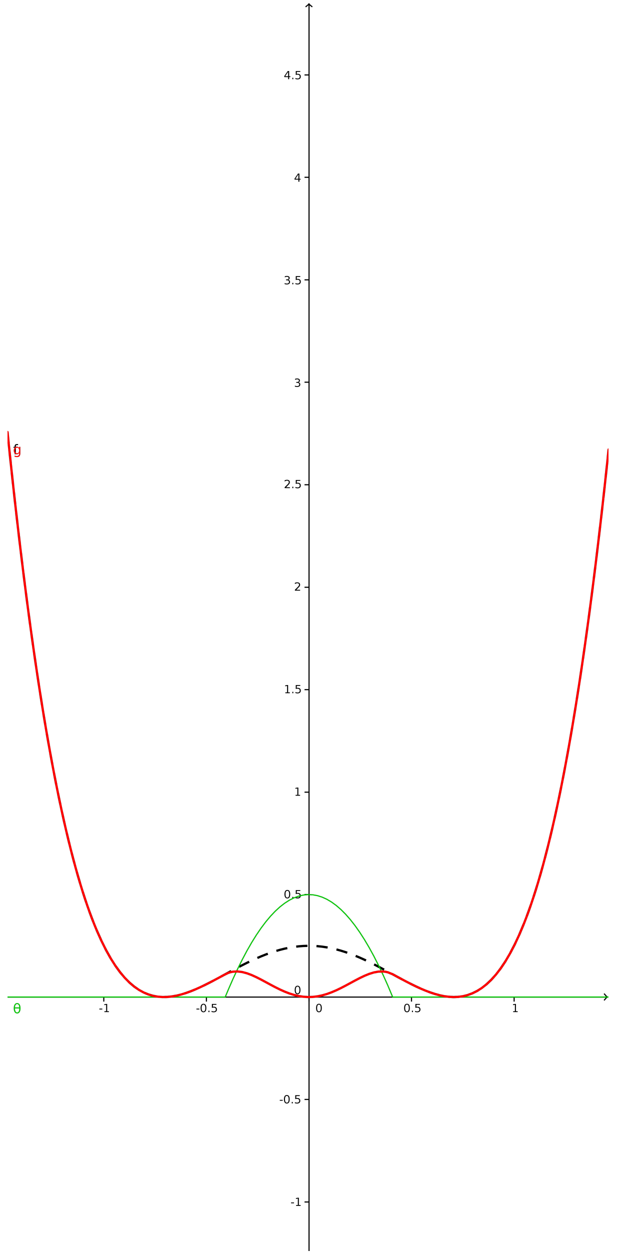


Examples

of embedded American payoffs ct'd

- $B(t) = 1$,
  $S(t) = W(t)$
- European payoff
  $f(x) = (x^2 - \frac{1}{2})^2$
- American payoff
  $g(x) = 2x^2(1 - 4x^2)1_{\{x^2 < 1/6\}} + f(x)1_{\{x^2 \geq 1/6\}}$
- early exercise curve
  $\vartheta(x) = (\frac{1}{2} - 3x^2)1_{\{x^2 < 1/6\}}$

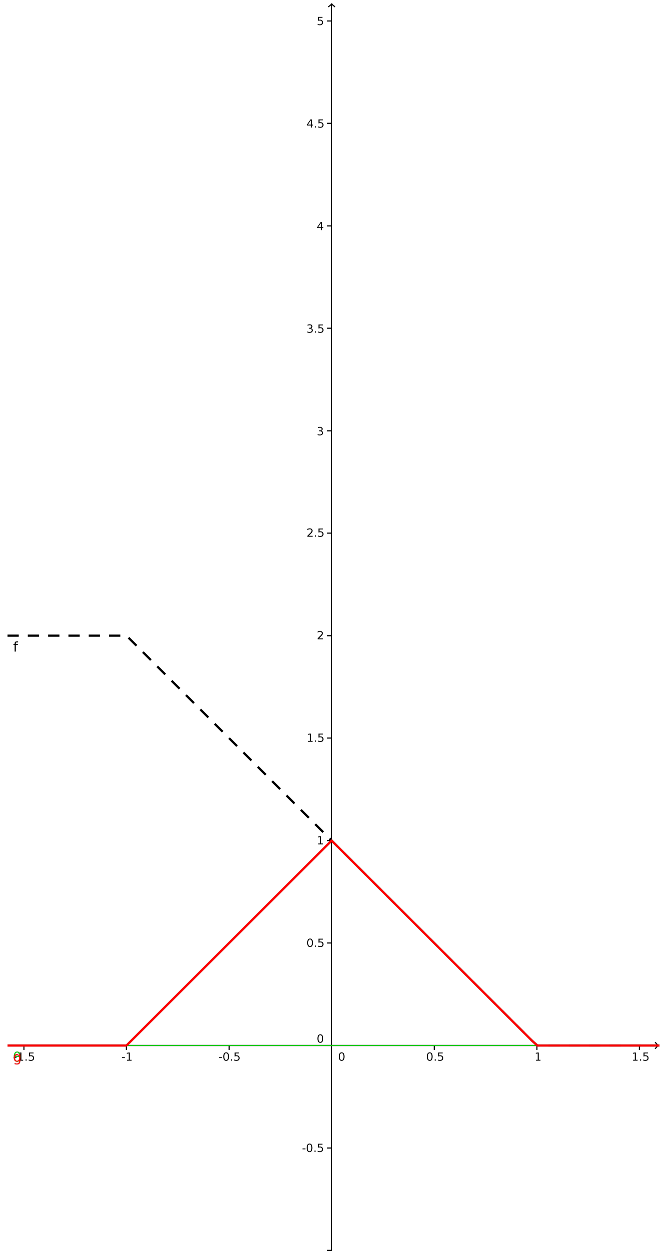


Examples

of embedded American payoffs ct'd

American butterfly in the *Bachelier model*:

- $B(t) = 1$,
  $S(t) = W(t)$
- European payoff
  $f(x) = 2 1_{\{x \leq -1\}} + (1 - x) 1_{\{-1 < x < 1\}}$
- American payoff
  $g(x) = (1 + x) 1_{\{-1 < x < 0\}} + (1 - x) 1_{\{0 \leq x < 1\}}$
- early exercise curve $\vartheta(x) = \infty 1_{\{x = 0\}}$

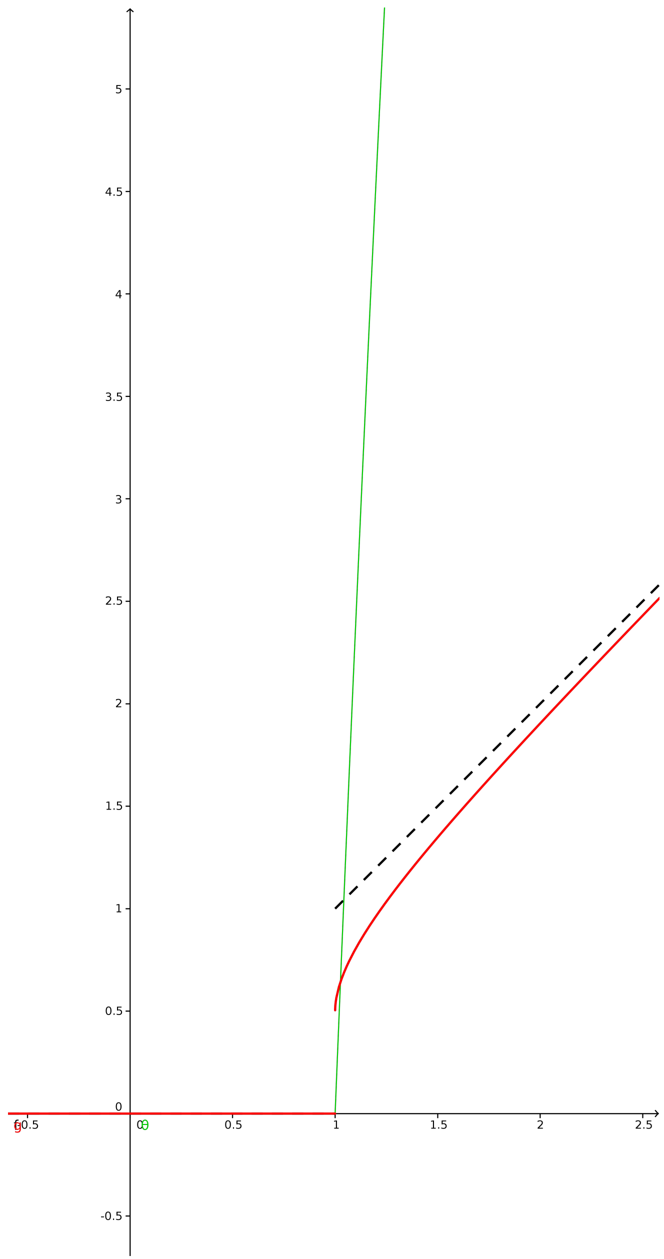


Examples

of embedded American payoffs ct'd

Jourdain & Martini (01):

- $B(t) = \exp(rt)$,
  $S(t) = S(0)\exp((r - \frac{\sigma^2}{2})t + \sigma W(t))$
- European payoff
  $f(x) = x1_{\{x>K\}}$
- American payoff
  $g(x) = f(x)\Phi\Big(\frac{2}{\sigma}\sqrt{(r + \frac{\sigma^2}{2})\log\frac{x}{K}}\Big)$
- early exercise curve
  $\vartheta(x) = \log(x)/(r + \frac{\sigma^2}{2})1_{\{x>K\}}$

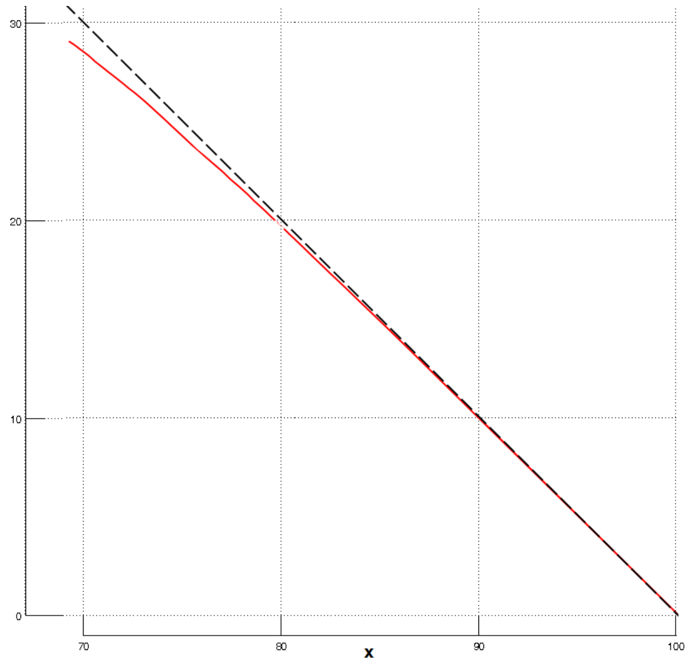


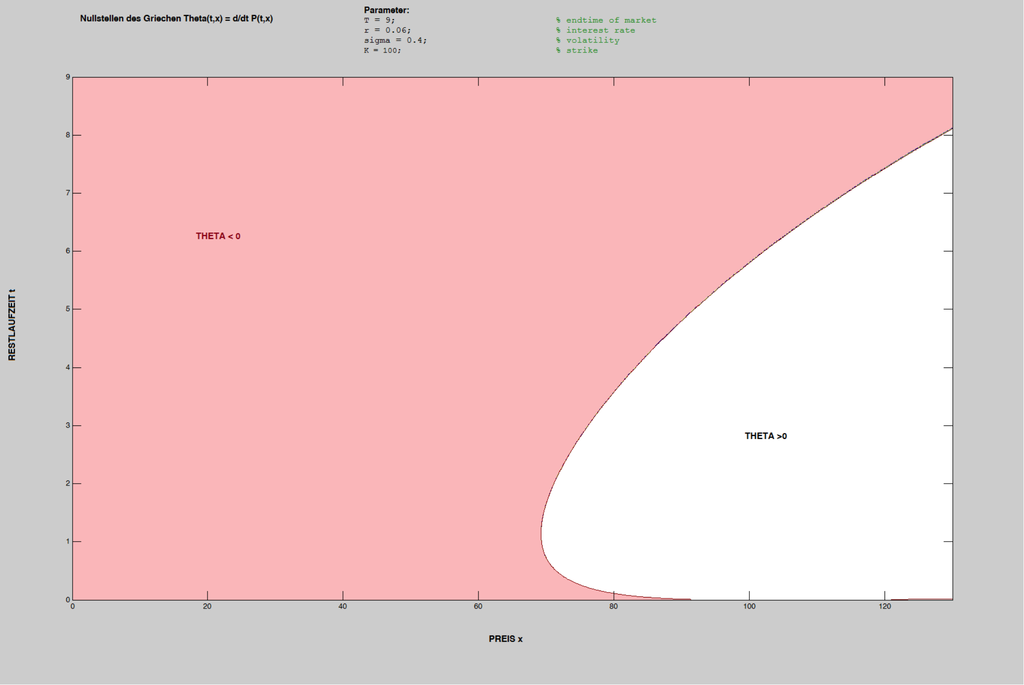
Examples

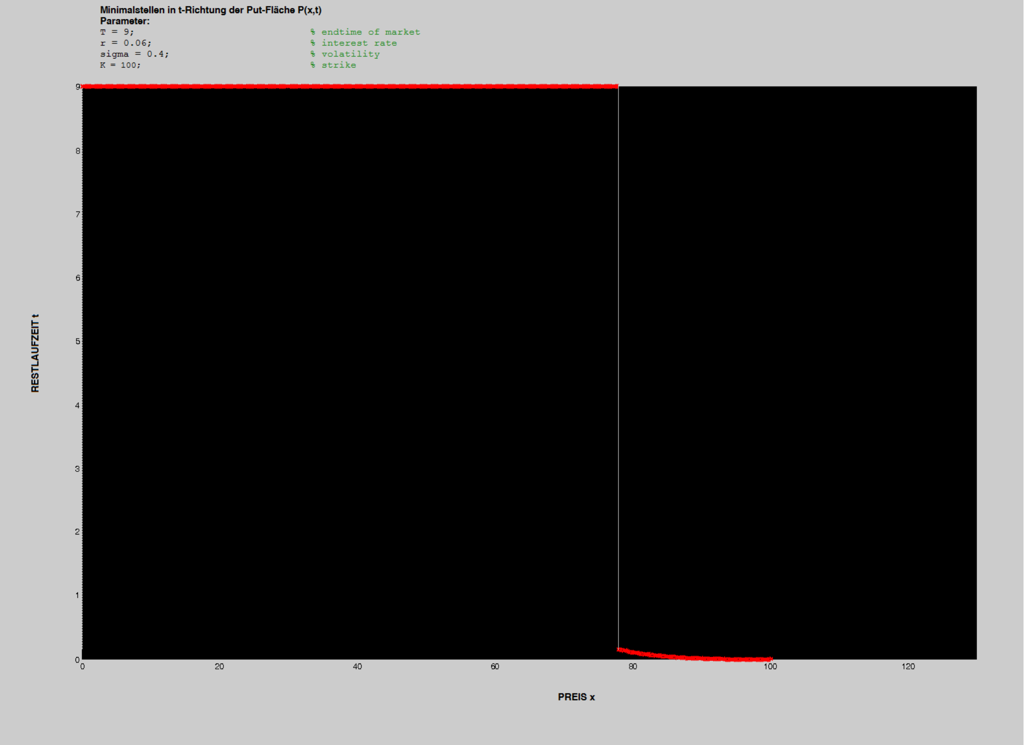
of embedded American payoffs ct'd

European put in the Black-Scholes model:

- $B(t) = \exp(rt)$,
  $S(t) = S(0)\exp((r - \frac{\sigma^2}{2})t + \sigma W(t))$
- European payoff $f(x) = (K - x)^+$
- yields an embedded American option, but only up to some maximal T




Nullstellen des Griechen Theta(t,x) = d/dt P(t,x)
Parameter:
T = 9;
r = 0.06;
sigma = 0.4;
K = 100;
% endtime of market
% interest rate
% volatility
% strike
THETA < 0
THETA >0
RESTLAUFZEIT t
PREIS x


Minimalstellen in t-Richtung der Put-Fläche P(x,t)
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RESTLAUFZEIT t
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Outline

Some bad news first . . .

. . . the second one making me nervous at some point

- Strehle (14): no representation for the American put in the Cox-Ross-Rubinstein model
- Jourdain & Martini (02): no generating European payoff exists for American put!?

First, in Section 2 we design a family of European payoffs which verify very crude necessary conditions for $\widehat{\varphi}(x) = (K - x)^+$ to have any chance to hold. This is the main step, it relies on the parameterization of φ by a measure h related to $\mathcal{A}\varphi$. Then we focus on the Continuation region. Among our family we find necessary and sufficient conditions which grant that the equation $\inf_{t\geq 0} v_\varphi(t,x) = v_\varphi(\widehat{t}(x), x)$ defines a curve which displays the same qualitative features as the free boundary of the American Put (Section 3).

Unfortunately, it is easy to see that for any function among our family $\widehat{\varphi}(x) = (K - K^*)(x/K^*)^{-\alpha}\mathbb{1}_{\{x\geq K^*\}}$ below K^*, which is not satisfactory. The third step is to prove that the price of the American option with modified payoff $(K - x)^+\mathbb{1}_{\{x\leq K^*\}} + \widehat{\varphi}(x)\mathbb{1}_{\{x>K^*\}}$, denoted by $\widehat{\varphi}_h$ to emphasize the dependence on the parameter h, and matching $(K - x)^+$ both for $x \geq K$ and for $x \leq K^*$ is still embedded in $v_\varphi(t,x)$: $v^{\mathrm{am}}_{\widehat{\varphi}_h}(t,x) = (K-x)^+\mathbb{1}_{\{x\leq K^*\}} + v_\varphi(t \vee \widehat{t}(x), x)\mathbb{1}_{\{x>K^*\}}$. This is done in Section 4.

Since we show that $\widehat{\varphi}_h$ cannot be equal to the Put payoff everywhere [indeed $\widehat{\varphi}''_h(K^{*+}) > 0$], we believe that at this stage there is little to get from further calculations. The last stage is to select among our family the point h^* so that,

A sufficient criterion

"for the engineer"

- American payoff: $g(x) = \varphi(x) 1_{\{x \le K\}}$,
- φ holomorphic, bounded, positive on $(0, K)$, and $\varphi(K) = 0$

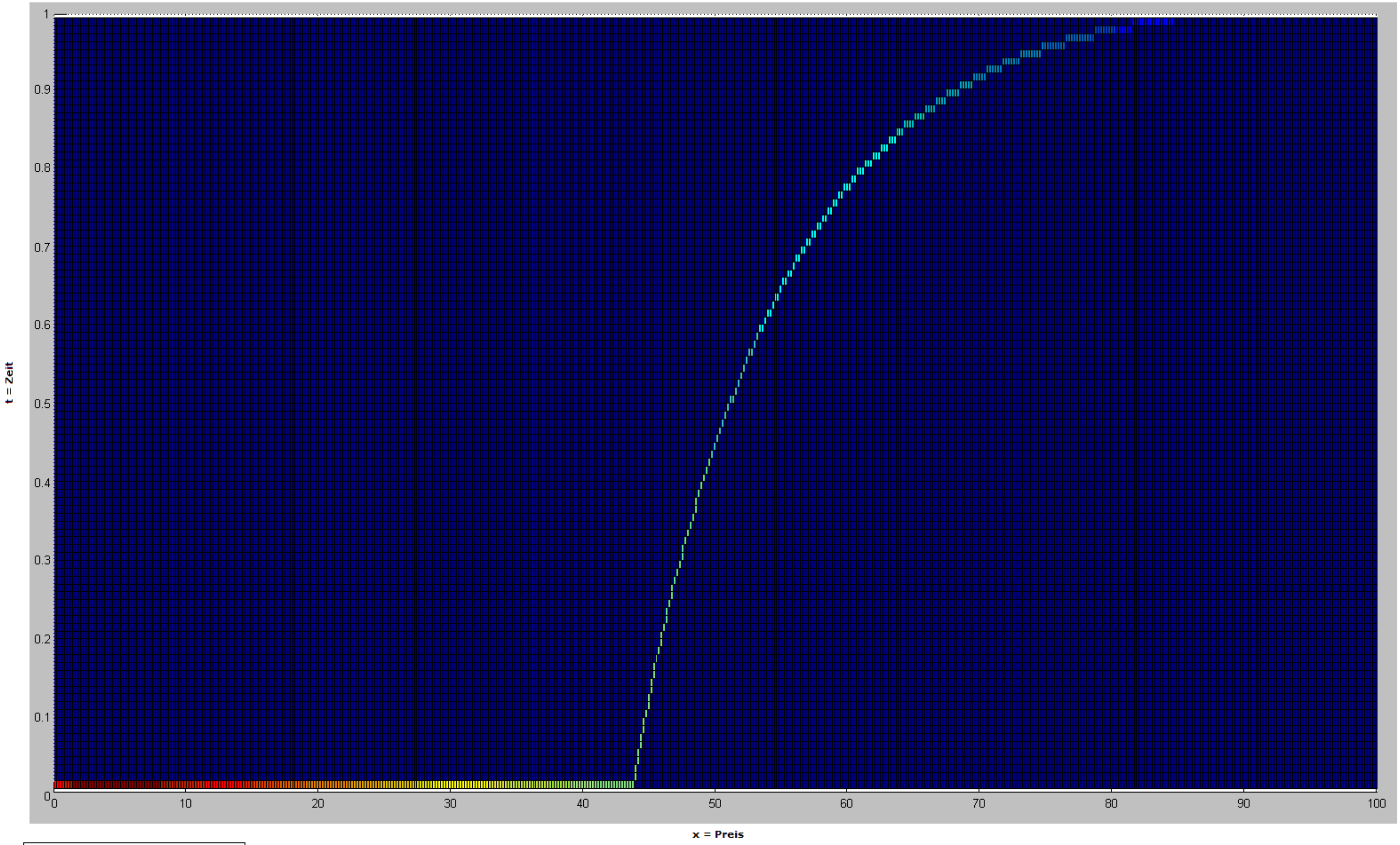
Theorem (Christensen, K., Lenga 15)

The CDEO f exists (as a generalized function). If

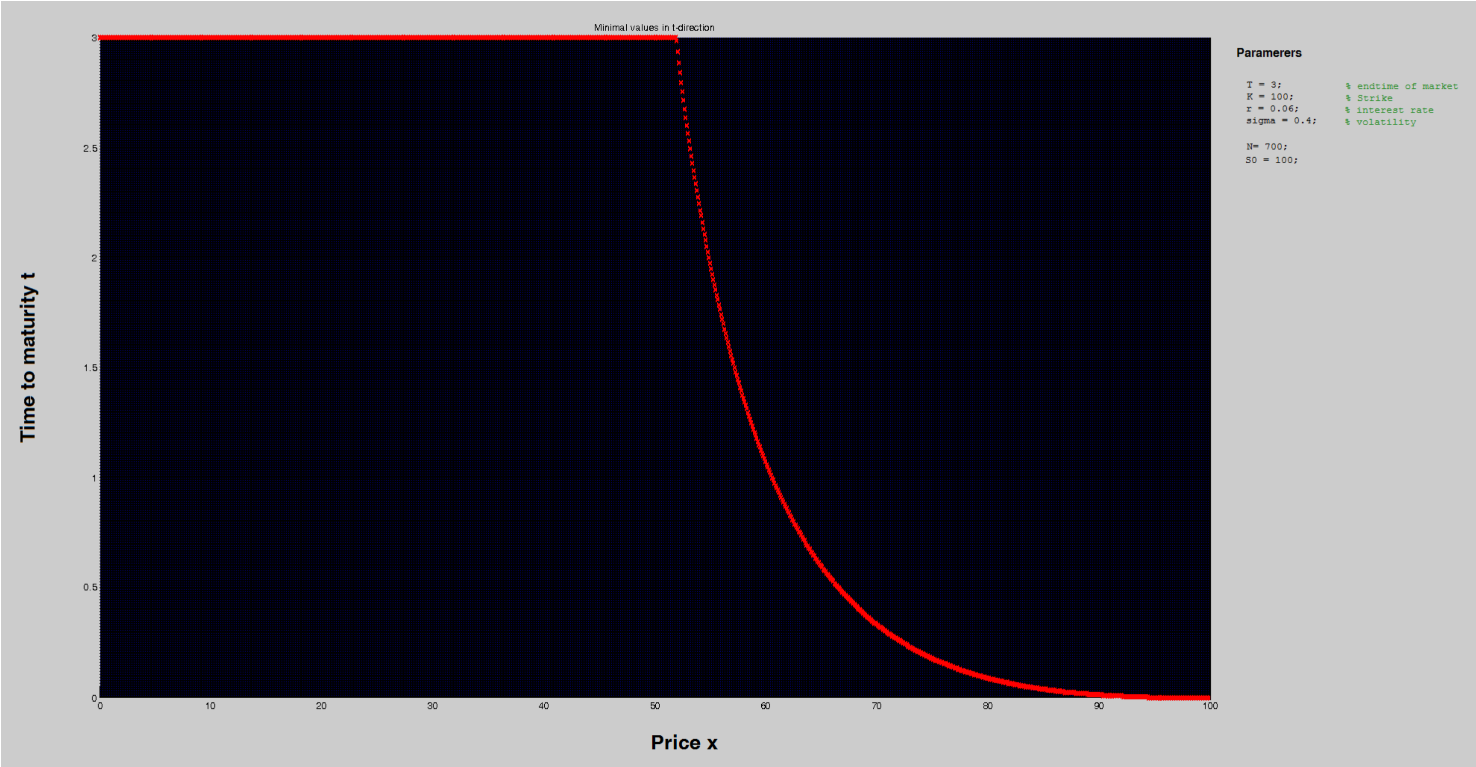
- $v_{Eu,f}(T + \epsilon, x) < \infty$ *for some* $\epsilon > 0, x < K$,
- $\lim_{\vartheta \to 0} v_{Eu,f}(\vartheta, x) > \varphi(x)$ *for any* $x < K$,
- *for any* $x \le K$*, function* $\vartheta \mapsto v_{Eu,f}(\vartheta, x)$ *has a unique minimum in some* $\vartheta(x)$ *(the embedded early exercise curve of the CDEO* f*),*
- *for some* x_0 *we have*
  - $\vartheta(x) = T$ *for* $x \le x_0$,
  - $\vartheta(x) \in (0, T)$ *for* $x \in (x_0, K)$,

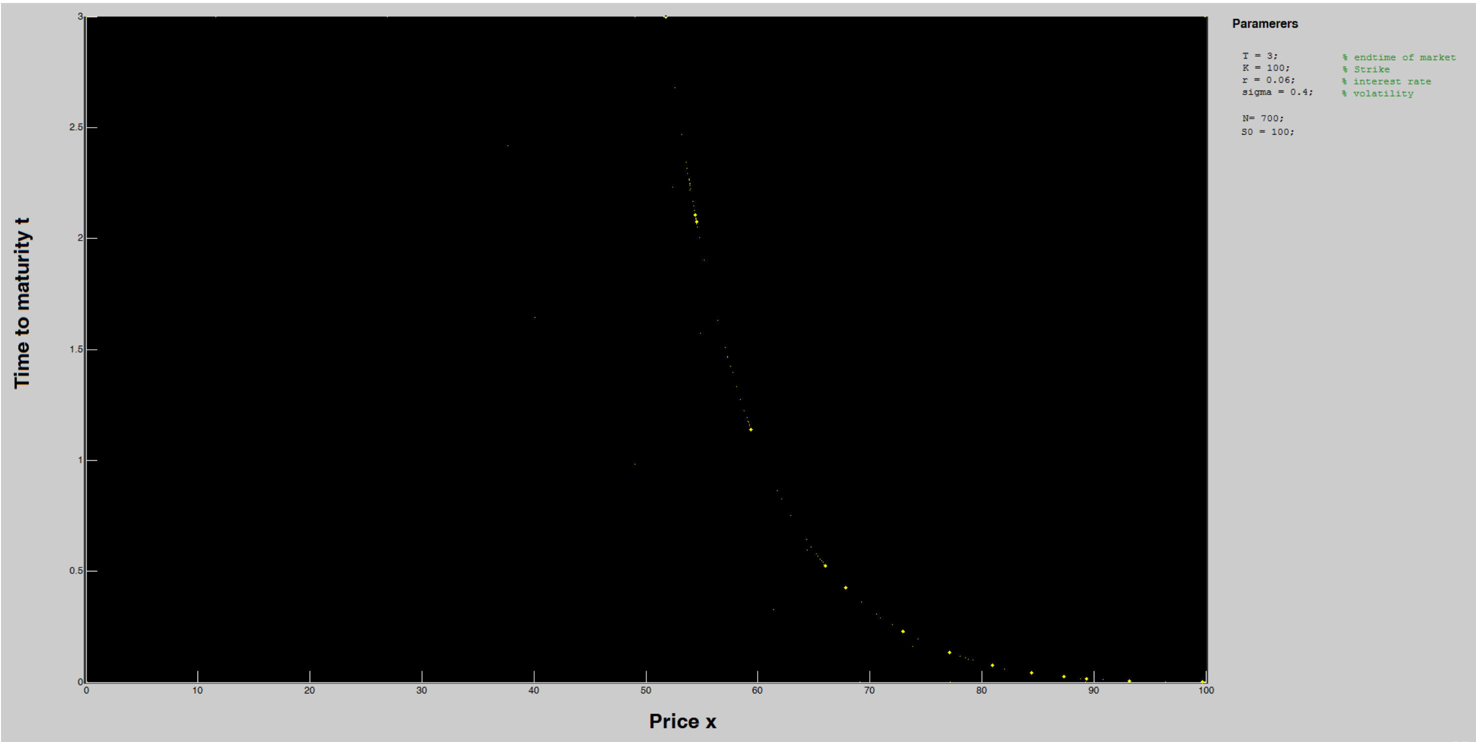
then the CDEO f represents g.

Numerical inspection for the American put



Parameter: T=1
Sigma = 0.6
r = 0.06
K = 100
S0 = 101


Minimal values in t-direction
Time to maturity t
Price x
Paramerers
T = 3; % endtime of market
K = 100; % Strike
r = 0.06; % interest rate
sigma = 0.4; % volatility
N= 700;
S0 = 100;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  - identity of analytic functions
- Primal problem: find CDEO (in space of generalized functions/ distributions/measures in order to warrant existence)
- Domain of dual problem: measures on $[0, T] \times \mathbb{R}_{++}$ (one Lagrange multiplier for each constraint $v_{\mathrm{Eu},f}(\vartheta, x) \geq g(x)$)
- Establish weak duality, existence of primal and dual optimizer, strong duality, complementary slackness condition
- Recall: Lagrange multiplier $\neq 0$ only if constraint is binding. Here: support of dual optimizer $\subset \{(\vartheta, x) : v_{\mathrm{Eu},f}(\vartheta, x) = g(x)\}$
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A bold conjecture

stated informally

- Necessary:
  - g analytic on the set where the optimal exercise time $\notin \{0, T\}$.
  - time horizon T "not too large"
- Conjecture: If
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Outline

Where are we now?

- Interesting relation between American and European options
- Several important implications of equality
- Verification theorem based on qualitative properties of the CDEO
- Not yet clear:
    - Rigorous proof for the American put?
    - How generally does equality hold?