Are American options European after all?

Jan Kallsen

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based on joint work with
Sören Christensen (Göteborg) and Matthias Lenga (Kiel)

Zürich, September 18, 2015
Outline

1. Are American options European after all?
2. Cheapest dominating European option
3. Embedded American options
4. A new result
5. Conclusion
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The basic question . . .

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Setup:

▶ Black-Scholes model with positive interest rate
▶ $v_{Am,g}(\vartheta, x)$: fair value of an American option with payoff function $g(x)$, time to maturity $\vartheta$, stock price $x$
▶ $v_{Eu,f}(\vartheta, x)$: fair value of a European option with payoff function $f(x)$, time to maturity $\vartheta$, stock price $x$

Consider the American put $g(x) := (K - x)^+$. Question: Is there a representing European payoff $f(x)$ in the sense that

▶ $v_{Am,g} = v_{Eu,f}$ in the continuation region of $g$ and
▶ $g \leq v_{Eu,f}$ in the stopping region (and hence everywhere)?

(Or at least for some $g$? Or even for all $g$?)

This would imply that the American put allows for a static European hedge.
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Cheapest dominating European options (CDEO)
Christensen (Math. Fin. 11)

- Black-Scholes model, American payoff \( g(x), T, S(0) \) given
- Solve

\[
\min_f \nu_{Eu,f}(T, S(0))
\]

subject to \( \nu_{Eu,f}(\vartheta, x) \geq g(x) \) for all \( x > 0 \) and all \( \vartheta \leq T \)
- **CDEO**: minimizer \( f \) if it exists
- semi-infinite linear programming
- upper bound for \( \pi = \nu_{Am,g}(T, S(0)) \), but surprisingly close
- implications of equality \( \nu_{Eu,f}(T, S(0)) = \nu_{Am,g}(T, S(0)) \) (if true):
  - new algorithm for American options
  - static European hedge for American options
  - interpretation of early exercise premium as payoff
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Computing American option prices by minimization over sets of martingales

- Davis & Karatzas (94), Rogers (02), Haugh & Logan (04):

\[ \pi = \nu_{Am,g}(T, S(0)) = \inf_{M \text{ mart.}, M(0)=0} E_Q \left( \sup_{t \in [0,T]} (e^{-rt} g(S(t)) - M(t)) \right) \]

“≥” follows from the Doob-Meyer decomposition

\[ \nu_{Am,g}(T - t, S(t)) e^{-rt} = \pi + M^*(t) - A^*(t) \]

with \( M^*(0) = 0 = A^*(0) \), \( M^* \) martingale, \( A^* \geq 0 \), \( A^* \) increasing.

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Embedded American options
Jourdain & Martini (Ann. IHP Anal. nonlin. 01, AAP 02)

- Black-Scholes model, given European payoff $f(x)$
- Embedded American payoff

\[ g(x) = \inf_{\vartheta} v_{\text{Eu},f}(\vartheta, x) \quad \left( = v_{\text{Eu},f}(\vartheta(x), x) \right) \]

($\vartheta \in [0, \infty)$ or $\vartheta \in [0, T]$)

- If curve $x \mapsto \vartheta(x)$ is nice:
  - $g \leq v_{\text{Eu},f}$,
  - $v_{\text{Am},g} = v_{\text{Eu},f}$ in continuation region of $g$,
  - The embedded early exercise curve $x \mapsto \vartheta(x)$ is the early exercise curve of $g$.  

![Graph showing the continuation region and stopping region](image)
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Examples
of embedded American payoffs

- $B(t) = 1,$
  $S(t) = \exp(\sqrt{2} W(t) - t)$
- European payoff
  $f(x) = 3x^{1/2} + x^{3/2}$
- American payoff
  $g(x) = 4x^{3/4} 1_{\{x<1\}} + f(x) 1_{\{x\geq 1\}}$
- early exercise curve
  $\vartheta(x) = - \log(x) 1_{\{x<1\}}$
- Note that $g$ is analytic on $(0, 1)$. 
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Examples
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- $B(t) = 1,$
  $S(t) = W(t)$

- European payoff
  $f(x) = \left(x^2 - \frac{1}{2}\right)^2$

- American payoff
  $g(x) = 2x^2(1 - 4x^2)1_{\{x^2 < 1/6\}} + f(x)1_{\{x^2 \geq 1/6\}}$

- early exercise curve
  $\vartheta(x) = \left(\frac{1}{2} - 3x^2\right)1_{\{x^2 < 1/6\}}$
Examples
of embedded American payoffs ct'd

American butterfly in the Bachelier model:

- \( B(t) = 1, \quad S(t) = W(t) \)
- European payoff
  \[ f(x) = 21\{x \leq -1\} + (1 - x)1\{-1 < x < 1\} \]
- American payoff
  \[ g(x) = (1 + x)1\{-1 < x < 0\} + (1 - x)1\{0 \leq x < 1\} \]
- early exercise curve \( \vartheta(x) = \infty 1\{x = 0\} \)
Examples
of embedded American payoffs ct’d

Jourdain & Martini (01):
- $B(t) = \exp(rt)$,
  
  $S(t) = S(0) \exp((r - \frac{\sigma^2}{2})t + \sigma W(t))$
- European payoff
  $f(x) = x 1_{\{x > K\}}$
- American payoff
  $g(x) = f(x) \Phi\left(\frac{2}{\sigma} \sqrt{(r + \frac{\sigma^2}{2}) \log \frac{x}{K}}\right)$
- early exercise curve
  $\vartheta(x) = \log(x)/(r + \frac{\sigma^2}{2}) 1_{\{x > K\}}$
European put in the Black-Scholes model:

- $B(t) = \exp(rt)$,
- $S(t) = S(0) \exp((r - \frac{\sigma^2}{2})t + \sigma W(t))$
- European payoff $f(x) = (K - x)^+$
- yields an embedded American option, but only up to some maximal $T$
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Some bad news first . . .

. . . the second one making me nervous at some point

- Strehle (14): no representation for the American put in the Cox-Ross-Rubinstein model
- Jourdain & Martini (02): no generating European payoff exists for American put!?

First, in Section 2 we design a family of European payoffs which verify very crude necessary conditions for \( \Phi(x) = (K - x)^+ \) to have any chance to hold. This is the main step, it relies on the parameterization of \( \varphi \) by a measure \( h \) related to \( A\varphi \). Then we focus on the Continuation region. Among our family we find necessary and sufficient conditions which grant that the equation \( \inf_{t \geq 0} v_{\varphi}(t, x) = v_{\varphi}(\tilde{T}(x), x) \) defines a curve which displays the same qualitative features as the free boundary of the American Put (Section 3).

Unfortunately, it is easy to see that for any function among our family \( \Phi(x) = (K - K^*)(x/K^*)^{-\alpha} \mathbb{1}_{x \geq K^*} \) below \( K^* \), which is not satisfactory. The third step is to prove that the price of the American option with modified payoff \( (K - x)^+ \mathbb{1}_{x \leq K^*} + \Phi(x) \mathbb{1}_{x > K^*} \), denoted by \( \Phi_h \) to emphasize the dependence on the parameter \( h \), and matching \( (K - x)^+ \) both for \( x \geq K \) and for \( x \leq K^* \) is still embedded in \( v_{\varphi}(t, x) : v_{\Phi_h}(t, x) = (K - x)^+ \mathbb{1}_{x \leq K^*} + v_{\varphi}(t \sqrt{\tilde{T}(x)}, x) \mathbb{1}_{x > K^*} \). This is done in Section 4.

Since we show that \( \Phi_h \) cannot be equal to the Put payoff everywhere [indeed \( \Phi_h''(K^*+) > 0 \)], we believe that at this stage there is little to get from further calculations. The last stage is to select among our family the point \( h^* \) so that,
A sufficient criterion
“for the engineer”

- American payoff: \( g(x) = \varphi(x)1_{\{x \leq K\}} \),
- \( \varphi \) holomorphic, bounded, positive on \((0, K)\), and \( \varphi(K) = 0 \)

**Theorem (Christensen, K., Lenga 15)**

*The CDEO \( f \) exists (as a generalized function). If*

- \( v_{Eu,f}(T + \epsilon, x) < \infty \) for some \( \epsilon > 0, x < K \),
- \( \lim_{\vartheta \to 0} v_{Eu,f}(\vartheta, x) > \varphi(x) \) for any \( x < K \),
- for any \( x \leq K \), function \( \vartheta \mapsto v_{Eu,f}(\vartheta, x) \) has a unique minimum in some \( \vartheta(x) \) (the embedded early exercise curve of the CDEO \( f \)),
- for some \( x_0 \) we have
  - \( \vartheta(x) = T \) for \( x \leq x_0 \),
  - \( \vartheta(x) \in (0, T) \) for \( x \in (x_0, K) \),

*then the CDEO \( f \) represents \( g \).*
Numerical inspection for the American put

Parameter: 
- $T_{m}$
- $\Sigma = 6.6$
- $r = 0.05$
- $K = 100$
- $S_{0} = 101$
Key steps of the proof

- **Key ingredients:**
  - convex duality in locally convex spaces
  - identity of analytic functions

- **Primal problem:** find CDEO (in space of generalized functions/distributions/measures in order to warrant existence)

- **Domain of dual problem:** measures on \([0, T] \times \mathbb{R}^{++}\)
  (one Lagrange multiplier for each constraint \(\nu_{Eu,f}(\vartheta, x) \geq g(x)\))

- **Establish weak duality, existence of primal and dual optimizer, strong duality, complementary slackness condition**

- **Recall:** Lagrange multiplier \(\neq 0\) only if constraint is binding.
  Here: support of dual optimizer \(\subset \{(\vartheta, x) : \nu_{Eu,f}(\vartheta, x) = g(x)\}\)

- **Slackness condition:** gBM started on support of dual optimizer has lognormal law at \(T\).

- **Using assumptions and identity of analytic functions:** support of dual optimizer must be nice connected curve.

- **Consequence:** Am. payoff \(g\) is represented by its CDEO.
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  Here: support of dual optimizer $\subset \{(\vartheta, x) : v_{Eu,f}(\vartheta, x) = g(x)\}$

- Slackness condition: gBm started on support of dual optimizer
  has lognormal law at $T$.

- Using assumptions and identity of analytic functions: support of dual optimizer must be nice connected curve.

- Consequence: Am. payoff $g$ is represented by its CDEO.
Key steps of the proof

- Key ingredients:
  - convex duality in locally convex spaces
  - identity of analytic functions

- Primal problem: find CDEO (in space of generalized functions/distributions/measures in order to warrant existence)

- Domain of dual problem: measures on $[0, T] \times \mathbb{R}_{++}$
  (one Lagrange multiplier for each constraint $v_{Eu,f}(\vartheta, x) \geq g(x)$)

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stated informally

Necessary:
- \( g \) analytic on the set where the optimal exercise time \( \notin \{0, T\} \).
- time horizon \( T \) “not too large”

Conjecture: If
- \( g \) is analytic on \( \{x \in \mathbb{R}^+ : \frac{\partial}{\partial \theta} v_{E \approx},g(0, x) < 0\} \) and
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Outline

1. Are American options European after all?
2. Cheapest dominating European option
3. Embedded American options
4. A new result
5. Conclusion
Where are we now?

- Interesting relation between American and European options
- Several important implications of equality
- Verification theorem based on qualitative properties of the CDEO
- Not yet clear:
  - Rigorous proof for the American put?
  - How generally does equality hold?