EQUILIBRIA IN INCOMPLETE STOCHASTIC CONTINUOUS-TIME MARKETS: 
EXISTENCE AND UNIQUENESS UNDER “SMALLNESS”

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Stochastic Finance Economies

Financial Equilibrium: Discrete Time

- Walras 1874,
- Arrow-Debreu ’54, McKenzie ’59,
- Radner ’72 extends the classical Arrow-Debreu model.
- Hart ’75 gives a non-existence example.
- Duffie-Shafer ’85, ’86 show that an equilibrium exists for generic endowments
- Cass, Drèze, Geanakoplos, Magill, Mas-Colell, Polemarchakis, Stieglitz, and others.
Financial Equilibrium: Continuous Time

Complete Markets

- Merton ’73
- Duffie-Zame ’89, Araujo-Monteiro ’89,
- Karatzas-Lakner-Lehoczky-Shreve ’91.

Incomplete Markets

- Basak, Cheridito, Christensen, Choi, Cuoco, He, Hugonnier, Kupper, Larsen, Munk, Zhao, Žitković.
An Incomplete, Short-Lived-Asset Model
Our Problem

Setup: \( \{\mathcal{F}_t\}_{t \in [0,T]} \) generated by two independent BMs \( B \) and \( W \).

Price: \( dS_t^\lambda = \lambda_t \, dt + dB_t + 0 \, dW_t \).

Agents’ preferences: \( U^i(X) = -\delta^i \log \mathbb{E} \left[ \exp(-X/\delta^i) \right] \), \( X \in \mathbb{L}^0 \),
endowments: \( E^i \in \mathbb{L}^\infty(\mathcal{F}_T) \), \( i = 1, \ldots, I \).

Demand: \( \hat{\pi}^{\lambda, i} := \arg\max_{\pi \in A^\lambda} \mathbb{U}^i \left( \int_0^T \pi_u \, dS^\lambda_u + E^i \right) \).

Question: Is there an equilibrium market price of risk \( \lambda \)? That is, does there exist a process \( \lambda \) such that the clearing condition \( \sum_{i=1}^{I} \hat{\pi}^{\lambda, i} = 0 \) holds?

Answer: Yes, when endowments are close to Pareto-optimality.
Upon defining the risk-(tolerance-)denominated quantities

\[ G_i = \frac{1}{\delta_i} E_i, \quad \text{and} \quad \hat{\rho}_{\lambda,i} = \frac{1}{\delta_i} \hat{\pi}_{\lambda,i}, \]

the market clearing condition becomes

\[ \sum_i \alpha_i \hat{\rho}_{\lambda,i} = 0, \quad \text{where} \quad \alpha_i = \frac{\delta_i}{\sum_j \delta_j}. \]

The risk-denominated certainty equivalent processes are

\[ Y_{t}^{i,\lambda} = - \log E_t \left[ \exp \left( - \int_t^T \hat{\rho}_{u,i}^\lambda dS_u^\lambda - G_i^{\lambda} \right) \right], \quad t \in [0, T]. \]
BSDE Characterisation of Equilibrium

Define the aggregator

$$A[x] = \sum_i \alpha^i x^i, \text{ for } x = (x^i)_i.$$ 

**Theorem.** A process $\lambda \in \text{bmo}$ is an equilibrium if and only if

$$\lambda = A[\mu],$$

for some solution $(\mu, \nu, Y) \in \text{bmo} \times \text{bmo} \times S^\infty$ of the following nonlinear (quadratic) and fully-coupled BSDE system:

$$\begin{cases} 
    dY^i_t = \mu^i_t dB_t + \nu^i_t dW_t + \frac{1}{2} \left( (\nu^i_t)^2 - A[\mu_t]^2 + 2A[\mu_t] \mu^i_t \right) dt, \\
    Y^i_T = G^i, \quad i = 1, \ldots, I,
\end{cases}$$

where $\mu = (\mu^i)_i, \nu = (\nu^i)_i$ and $Y = (Y^i)_i$. 
Nonlinear Systems of BSDEs

- [Darling 95], [Blache 05, 06]: Harmonic maps.
- [Tang 03]: Riccati systems,
- [Tevzadze 08]: existence when terminal condition is small.
- [Delarue 02], [Cheridito-Nam 14]: generator $f + zg$, where both $f$ and $g$ are Lipschitz.
- [Hu-Tang 14]: diagonally quadratic, small-time existence.

Applications:

- [Bensoussan-Frehse 90], [El Karoui-Hamadène 03]: stochastic differential games.
- [Frei-dos Reis 11], [Frei 14]: relative performance.
  **Counterexample**: bounded terminal condition, no solution.
- [Cheridito-Horst-Kupper-Pirvu 12]: equilibrium pricing.
- [Kramkov-Pulido 14]: price impact problem.
**Existence and Uniqueness “with Cheating”**

**Theorem 0a.** An equilibrium exists and is unique if \((G^i)\) is an (unconstrained) Pareto-optimal allocation. Then \(\lambda \equiv 0\).

**Note:** \((G^i)\) is Pareto-optimal if and only if

\[ G^i - G^j \in \mathbb{R}, \text{ for all } i, j. \]

**Definition.** \((G^i)\) is **pre-Pareto** if there exists an equilibrium \(\lambda \in bmo\) such that the allocation

\[
\widetilde{G}^i = G^i + \int_0^T \hat{\rho}_{t,\lambda}^i dS_t^\lambda, \quad i = 1, \ldots, I, \text{ is Pareto optimal.}
\]

**Obviously . . .**

**Theorem 0b.** An equilibrium exists if \((G^i)\) is pre-Pareto.

**However, . . .**
Proposition. The following statements are equivalent:

1. \((G^i)_i\) is pre-Pareto.

2. There exists an equilibrium \(\lambda \in \text{bmo}\) such that

\[
\hat{Q}^{\lambda,i} = \hat{Q}^{\lambda,j}, \quad \text{for all } i, j,
\]

where \(\hat{Q}^{\lambda,i}, i = 1, \ldots, I\) denote the “dual optimizers”.

3. For \(\lambda, \nu\) defined by

\[
\exp(-\sum_i \alpha^i G^i) \propto \mathcal{E} \left( -\int_0^T \lambda_t dB_t - \int_0^T \nu_t dW_t \right)_T,
\]

there exist \((y^i)_i \in \mathbb{R}^I\) and \((\varphi^i)_i \in \text{bmo}^I\) such that

\[
G^i - G^j = y^i - y^j + \int_0^T (\varphi^i_t - \varphi^j_t) dS^\lambda_t, \quad \text{for all } i, j.
\]

In each of these cases, \(\lambda\) as above is the unique equilibrium.
Certainty Equivalents and BMO

Let $G \in \mathbb{L}^\infty$. Define

$$X_t^G = -\log \mathbb{E}_t[\exp(-G)], \quad t \in [0, T],$$

and note the dynamics

$$dX_t^G = m_t^G \, dB_t + n_t^G \, dW_t + \frac{(m_t^G)^2 + (n_t^G)^2}{2} \, dt, \quad X_T^G = G.$$

Define also the bmo$^2$-norm:

$$\|(m, n)\|_{\text{bmo}^2(\tilde{\mathbb{P}})} = \left\| \text{ess sup} \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \int_\tau^T (m_t^2 + n_t^2) \, dt \right] \right\|_{\mathbb{L}^\infty}^{1/2}.$$
The General “Smallness” Result

For an allocation \((G^i)_{i}\), with \(G^i \in \mathbb{L}^\infty\) for \(i = 1, \ldots, I\), we define its distance to Pareto optimality \(H((G^i)_{i})\) via

\[
H((G^i)_{i}) := \inf_{G \in \mathbb{L}^\infty} \max_i \left\| \left( m^{G^i} - m^G, n^{G^i} - n^G \right) \right\|_{\text{bmo}^2(P^G)},
\]

where \(dP^G/dP \propto \exp(-G)\) for \(G \in \mathbb{L}^\infty\).

Theorem. An equilibrium \(\lambda \in \text{bmo}\) exists and is unique if

\[
H((G^i)_{i}) < \frac{3}{2} - \sqrt{2} \approx 0.0858.
\]

NB: A similar result with “distance-to-Pareto” replaced by “distance-to-pre-Pareto” holds (mutadis mutandis), with a different proof technique.
**Corollaries**

**Corollary 1.** A unique equilibrium exists if

\[(1/\delta^i)\|E^i\|_{L^\infty} \text{ is sufficiently small for each } i,\]

**Corollary 2.** A unique equilibrium exists if

there are sufficiently many sufficiently homogeneous agents,

i.e., if \(I \geq I(\|\sum_i E^i\|_{L^\infty}, \min_i \delta^i, \chi^E)\), where the **endowment heterogeneity index** \(\chi^E \in [0, 1]\) is defined via

\[
\chi^E = \max_{i,j} \frac{\|E^i - E^j\|_{L^\infty}}{\|E^i\|_{L^\infty} + \|E^j\|_{L^\infty}}.
\]
Corollary 3. (Small time existence and uniqueness.) A unique equilibrium exists if

\[ T < T^* = \frac{(3/2 - \sqrt{2})^2}{\max_i \left( \| D^b(G^i) \|^2_{S_{\infty}} + \| D^w(G^i) \|^2_{S_{\infty}} \right)}, \]

provided all \( E^i \) have bounded Malliavin derivatives.
**Future Work**

1. General global existence and uniqueness (?)

2. Sensitivity analysis around Pareto optimality.

3. Long-lived securities.

4. Endowments depending also on prices.
Thanks for your attention!

P.S. Preprint available on the arXiv.