Malliavin Calculus: Analysis on Gaussian spaces

Josef Teichmann

ETH Zürich

Oxford 2011

Isonormal Gaussian process

A *Gaussian space* is a (complete) probability space together with a Hilbert space of centered real valued Gaussian random variables defined on it. If the random variables of the given Hilbert space generate the underlying σ-algebra, we call the Gaussian space *irreducible*.

We speak about Gaussian spaces by means of a *coordinate space*.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, H a Hilbert space, and $W : H \to L^2[(\Omega, \mathcal{F}, P); \mathbb{R}]$ a linear isometry. Then W is called *isonormal Gaussian process* if $W(h)$ is a centered Gaussian random variable for all $h \in H$. In particular $(\Omega, \mathcal{F}, P)$ together with $W(H)$ is Gaussian.

Example

Given a d-dimensional Brownian motion $(W_t)_{t\geq 0}$ on its natural filtration $(\mathcal{F}_t)_{t\geq 0}$, then

$$W(h) := \sum_{k=1}^{d} \int_0^\infty h^k(s) dW_s^k$$

is an isonormal Gaussian process for $h \in H := L^2(\mathbb{R}_{\geq 0}; \mathbb{R}^d)$.

Notation

In the sequel we shall apply the following classes of functions on $\mathbb{R}^n$

$$C_0^\infty(\mathbb{R}^n) \subset C_b^\infty(\mathbb{R}^n) \subset C_p^\infty(\mathbb{R}^n),$$

which denote the functions with compact support, with bounded derivatives of all orders and with derivatives of all orders of polynomial growth.

Smooth random variables

Let W be an isonormal Gaussian process. We introduce random variables of the form

$$F := f(W(h_1), \ldots, W(h_n))$$

for $h_i \in H$ (mind the probabilistic notation, which would be bad style in analysis). If f belongs to one of the above classes of functions, the associated random variables are denoted by

$$\mathcal{S}_0 \subset \mathcal{S}_b \subset \mathcal{S}_p$$

and we speak of smooth random variables. The polynomials of elements $W(h)$ are denoted by $\mathcal{P}$.

Generation property

The algebra $\mathcal{P}$ is dense in $L^2(\Omega, \mathcal{F}_H, P)$, where $\mathcal{F}_H$ denotes the completed σ-algebra generated by the random variables $W(h)$ for $h \in H$.

Proof

Notice that it is sufficient to prove that every random variable F, which is orthogonal to all $\exp(W(h))$ for $h \in H$, vanishes. Choose now an ONB $(e_i)_{i \geq 1}$, then the entire function

$$(\lambda_1, \ldots, \lambda_n) \mapsto E(F \exp(\sum_{i=1}^{n} \lambda_i W(e_i)))$$

vanishes, which in turn means that $E(F \mid \bar{\sigma}(W(e_1), \ldots, W(e_n))) = 0$ by uniqueness of the Fourier transform, hence $F = 0$.

Therefore polynomials of Gaussians qualify as smooth test functions, since they lie in all L^p for $1 \leq p < \infty$ and are dense.

The representation of a smooth random variable is unique in the following sense: let

$$\begin{aligned} F &= f(W(h_1), \ldots, W(h_n)) \\ &= g(W(g_1), \ldots, W(g_m)), \end{aligned}$$

and denote the linear space $\langle h_1, \ldots, h_n, g_1, \ldots, g_n \rangle$ with orthonormal basis $(e_i)_{1 \leq i \leq k}$ and representations

$$h_i = \sum_{l=1}^{k} a_{il} e_l$$

$$g_j = \sum_{l=1}^{k} b_{jl} e_l.$$

Then the functions $f \circ A$ and $g \circ B$ coincide *everywhere*.

Notation

Notice the following natural isomorphisms

$$\begin{gathered} L^2[(\Omega,\mathcal{F},P);H] = L^2(\Omega,\mathcal{F},P)\otimes H \\ (\omega\mapsto F(\omega)h)\mapsto F\otimes h\,. \end{gathered}$$

If we are additionally given a concrete representation $H = L^2[(T,\mathcal{B},\mu);G]$, then

$$\begin{gathered} L^2[(\Omega\times T,\mathcal{F}\otimes\mathcal{B},P\otimes\mu);G] = L^2(\Omega,\mathcal{F},P)\otimes H \\ ((\omega,t)\mapsto F(\omega)h(t))\mapsto F\otimes h\,. \end{gathered}$$

The Malliavin Derivative

For $F \in \mathcal{S}_p$ we denote the *Malliavin derivative* by $DF \in L^2[(\Omega, \mathcal{F}, P); H]$ defined via

$$DF = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(W(h_1), \ldots, W(h_n)) \otimes h_i$$

for $F = f(W(h_1), \ldots, W(h_n))$. The definition does not depend on the particular representation of the smooth random variable $F = f(W(h_1), \ldots, W(h_n))$.

If we are given a concrete representation $H = L^2(T, \mathcal{B}, \mu)$, then we can identify

$$L^2(\Omega, \mathcal{F}, P) \otimes H = L^2(\Omega \times T, \mathcal{F} \otimes \mathcal{B}, P \otimes \mu)$$

and we obtain a measurable, not necessarily adapted process $(D_t F)_{t \in T}$ as *Malliavin derivative*.

Integration by parts 1

Let F be a smooth random variable and $h \in H$, then

$$E(\langle DF, h\rangle) = E(FW(h)).$$

Integration by parts 2

Let F, G be smooth random variables, then for $h \in H$

$$E(G \langle DF, h \rangle) + E(F \langle DG, h \rangle) = E(FG\, W(h)).$$

Proof

The first equation can be normalized such that $||h|| = 1$. Additionally there are by a variable transformation orthonormal elements e_i such that

$$F = f(W(e_1), ..., W(e_n))$$

with $f \in C_p^\infty(\mathbb{R}^n)$ and $h = e_1$. Then

$$\begin{aligned} E(\langle DF, h\rangle) &= \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}(x) \frac{1}{\sqrt{(2\pi)^n}} \exp(-\frac{||x||^2}{2}) dx \\ &\overset{\text{i.p.}}{=} \int_{\mathbb{R}^n} f(x) x_1 \frac{1}{\sqrt{(2\pi)^n}} \exp(-\frac{||x||^2}{2}) dx \\ &= E(F\, W(e_1)) = E(FW(h)). \end{aligned}$$

Proof

The second integration by parts formula follows directly from the Leibnitz rule

$$D(FG) = FDG + GDF$$

for $F, G \in \mathcal{S}_p$.

The Malliavin derivative is closable

We have already defined the derivative operator

$$D : \mathcal{S}_p \subset L^q[(\Omega, \mathcal{F}, P)] \to L^q[(\Omega, \mathcal{F}, P); H]$$

for $q \geq 1$. This linear operator is closable by integration by parts: given a sequence of smooth functionals $F_n \to 0$ in L^q and $DF_n \to G$ in $L^q[(\Omega, \mathcal{F}, P); H]$ as $n \to \infty$, then

$$\begin{aligned} E(\langle G, h\rangle_H F) &= \lim_{n\to\infty} E(\langle DF_n, h\rangle F) = \\ &= \lim_{n\to\infty} E(-F_n \langle DF, h\rangle) + \lim_{n\to\infty} E(F_n F\, W(h)) = 0 \end{aligned}$$

for $F \in \mathcal{S}_p$. Notice that $\mathcal{S}_p \subset \cap_{q\geq 1} L^q$. So $G = 0$ and therefore D is closeable. We denote the closure on each space by $\mathcal{D}^{1,q}$, respectively.

Operator norms

Given $q \geq 1$, then we denote by

$$||F||_{1,q} := (E(|F|^q) + E(||DF||_H^q))^{\frac{1}{q}}$$

the operator norm for any $F \in \mathcal{S}_p$. By closeability we know that the closure of this space is a Banach space, denoted by $\mathcal{D}^{1,q}$ and a Hilbert space for $q = 2$. We have the continuous inclusion

$$\mathcal{D}^{1,q} \hookrightarrow L^q[(\Omega, \mathcal{F}, P)]$$

which has as image the maximal domain of definition of $\mathcal{D}^{1,q}$ in L^q, where we shall write – by slight abuse of notation – again D for the Malliavin derivative.

Higher Derivatives

By tensoring the whole procedure we can define Malliavin derivative for smooth functionals with values in V, an additionally given Hilbert space,

$$\mathcal{S}_p \otimes V \subset L^p[(\Omega, \mathcal{F}, P)] \otimes V,$$

where we take the algebraic tensor products. We define the Malliavin derivative on this space by $D \otimes id$, and proceed as before showing that the operator is closable.

Consequently we can define higher derivatives via iteration

$$D^k F = DD^{k-1} F$$

for smooth functionals $F \in L^q[(\Omega, \mathcal{F}, P)] \otimes V$. Closing the spaces we get Malliavin derivatives D^k for elements of $L^q[(\Omega, \mathcal{F}, P); V]$ to $L^q[(\Omega, \mathcal{F}, P); V \otimes H^{\otimes k}]$ by induction.

Operator norms

We define the norms

$$||F||_{k,q} := (E(|F|^q) + \sum_{j=1}^{k} E(||D^j F||^q_{V \otimes H^{\otimes j}}))^{\frac{1}{q}}$$

for $k \geq 1$ and $q \geq 1$. The respective closed spaces $\mathcal{D}^{k,q}(V)$ are Banach spaces (Hilbert spaces), the maximal domains of D^k in $L^q(\Omega, \mathcal{F}, P; V)$. The Fréchet space $\cap_{p \geq 1} \cap_{k \geq 1} \mathcal{D}^{k,p}(V)$ is denoted by $\mathcal{D}^\infty(V)$.

Monotonicity

We see immediately the monotonicity

$$||F||_{k,p} \leq ||F||_{j,q}$$

for $p \leq q$ and $k \leq j$ by norm inequalities of the type

$$||f||_p \leq ||f||_q$$

for $1 \leq p \leq q$ for $f \in \cap_{p \geq 1} L^p[\Omega, \mathcal{F}, P]$.

Chain rule

Let $\phi \in C_b^1(\mathbb{R}^n)$ be given, such that the partial derivatives are bounded and fix $p \geq 1$. If $F \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, then $\phi(F) \in \mathcal{D}^{1,p}$ and

$$D(\phi(F)) = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(F) DF^i$$

Hence D^∞ is an C^∞-algebra.

A similar result holds true of ϕ is only globally Lipschitz, however, we cannot identify the derivative then anymore.

Proof

The proof is done by approximating F^i by smooth variables F_n^i and ϕ by $\phi * \psi_\epsilon$, where ψ_ϵ is a Dirac sequence of smooth functions. For the approximating terms the formula is satisfied, then we obtain

$$||\sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(F) DF^i - D((\phi * \psi_\epsilon) \circ F_n^i)||_p \to 0$$

as $\epsilon \to 0$ and $n \to \infty$, so by closedness we obtain the result since $(\phi * \psi_\epsilon) \circ F_n^i \to \phi \circ F$ in L^p as $\epsilon \to 0$ and $n \to \infty$.

Malliavin derivative as directional derivative

Consider the standard example $h \mapsto \sum_{k=1}^{d} \int_0^\infty h^k(s) dW_s^k$ with Hilbert space $H = L^2(\mathbb{R}_{\geq 0}; \mathbb{R}^d)$. Assume $\Omega = C(\mathbb{R}_{\geq 0}; \mathbb{R}^d)$, then we can define the Cameron-Martin directions

$$h \mapsto (t \mapsto \int_0^t h_s ds),$$

which embeds $H \hookrightarrow C(\mathbb{R}_{\geq 0}; \mathbb{R}^d)$. If we consider a smooth random variable $F = f(W_t)$, then

$$\langle DF, h \rangle = f'(W_t) \int_0^\infty 1_{[0,t]}(s) h(s) ds = \frac{d}{d\epsilon}|_{\epsilon=0} f(W_t + \epsilon \int_0^t h(s) ds),$$

so the Malliavin derivative evaluated in direction h appears as directional derivative in a Cameron-Martin direction, which are the only directions where directional derivatives make sense for P-almost surely defined random variables.

Malliavin derivative as directional derivative

Taking the previous consideration seriously we can replace h by a predictable strategy a such that the stochastic exponential of $\sum_{k=1}^{d}\int_0^t a_s^k dW_s^k$ is a closed martingale, then we obtain

$$\begin{aligned} E\left(\langle DF, a\rangle\right) &= E\left(\frac{d}{d\epsilon}|_{\epsilon=0} F(. + \epsilon\int_0^{.} a_s ds)\right) \\ &= \frac{d}{d\epsilon}|_{\epsilon=0} E\left(F(. + \epsilon\int_0^{.} a_s ds)\right) \\ &= \frac{d}{d\epsilon}|_{\epsilon=0} E\left(F(.)\exp(\epsilon\sum_{k=1}^{d}\int_0^{\infty} a_s^k dW_s^k - \frac{\epsilon^2}{2}\int_0^{\infty}|a_s|^2 ds)\right) \\ &= E(F(.)\sum_{k=1}^{d}\int_0^t a_s^k dW_s^k) \end{aligned}$$

for smooth bounded random variables F.

The adjoint

The adjoint operator $\delta : \mathrm{dom}_{1,2}(\delta) \subset L^p(\Omega) \otimes H \to L^2(\Omega)$ is a closed densely defined operator. We concentrate here on the case $p = 2$. By definition $u \in \mathrm{dom}_{1,2}(\delta)$ if and only if $F \mapsto E(\langle DF, u \rangle)$ for $F \in \mathcal{D}^{1,2}$ is a bounded linear functional on $L^2(\Omega)$.

If $u \in \mathrm{dom}_{1,2}(\delta)$, we have the following fundamental "integration by parts formula"

$$E(\langle DF, u \rangle) = E(F\delta(u))$$

for $F \in \mathcal{D}^{1,2}$. δ is called the *Skorohod integral* or *divergence operator* or simply *adjoint operator*.

We obtain immediately $H \subset \mathrm{dom}_{1,2}(\delta)$, the deterministic strategies, with $\delta(1 \otimes h) = \delta(h) = W(h)$.

A smooth elementary process is given by

$$u = \sum_{j=1}^{n} F_j \otimes h_j$$

with $F_j \in \mathcal{S}_p$ and $h_j \in H$. We shall denote the set of such processes by the (algebraic) tensor product $\mathcal{S}_p \otimes H$. By integration by parts we can conclude that $\mathcal{S}_p \otimes H \subset \mathrm{dom}_{1,2}(\delta)$ and

$$\delta(u) = \sum_{j=1}^{n} F_j \, W(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j \rangle_H \,,$$

since for all $G \in \mathcal{S}_p$

$$\begin{aligned} E(\langle u, DG\rangle_H) &= \sum_{j=1}^n E(F_j \langle h_j, DG\rangle_H) \\ &= \sum_{j=1}^n E(\langle h_j, D(F_j G)\rangle_H) - E(G \langle h_j, DF_j\rangle_H) \\ &= \sum_{j=1}^n E(F_j W(h_j) G) - E(G \langle h_j, DF_j\rangle_H). \end{aligned}$$

Skorohod meets Ito

Given an isonormal Gaussian process W and define a sub-σ-algebra $\mathcal{F}_G \subset \mathcal{F}_H$ by means of a closed subspace $G \subset H$. If $F \in \mathcal{D}^{1,2}$ is $\mathcal{F}_G$-measurable, then

$$\langle h, DF \rangle_H = 0$$

P-almost surely, for all $h \perp G$.

Skorohod meets Ito

The almost sure identity holds for smooth random variables $F = f(W(h_1), \dots, W(h_n))$, but every $F \in \mathcal{D}^{1,2}$ can be approximated by smooth random variables in L^2 such that also the derivatives are approximated (closedness!), hence the result follows.

Skorohod meets Ito

Given a d-dimensional Brownian motion $(W_t)_{t\geq 0}$ on its natural filtration $(\mathcal{F}_t)_{t\geq 0}$, then

$$W(h) := \sum_{k=1}^{d} \int_0^\infty h^k(s) dW_s^k$$

is an isonormal Gaussian process for $h \in H := L^2(\mathbb{R}_{\geq 0}; \mathbb{R}^d)$. Define $H_t \subset H$ those functions with support in $[0, t]$ for $t \geq 0$ and denote $\mathcal{F}_t := \mathcal{F}_{H_t}$. Hence for $\mathcal{F}_t$-measurable $F \in \mathcal{D}^{1,2}$ we obtain that $1_{[0,t]} DF = DF$ almost surely.

Skorohod meets Ito

Consequently for a simple, predictable strategy

$$u(s) = \sum_{j=1}^{n} F_j \otimes h_j$$

with $h_j = 1_{]t_j, t_{j+1}]} e_k$, for $0 = t_0 < t_1 < \cdots < t_{n+1}$ and $F_j \in L^2[(\Omega, \mathcal{F}_{t_j}, P)]$ for $j = 1, \ldots, n$, and $e_k \in \mathbb{R}^d$ a canonical basis vector, that

$$\begin{aligned} \delta(u) &= \sum_{j=1}^{n} F_j \, W(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j \rangle_H \\ &= \sum_{j=1}^{n} F_j (W_{j+1}^k - W_j^k). \end{aligned}$$

Skorohod meets Ito

Given a predictable strategy $u \in L^2_{pred}(\Omega \times \mathbb{R}_{\geq 0}; \mathbb{R}^d)$, then

$$\delta(u) = \sum_{k=1}^{d} \int_0^\infty u^k(s) dW_s^k.$$

Skorohod meets Ito

The Skorohod integral is a closed operator, the Ito integral is continuous on the space of predictable strategies. Both operators coincide on the dense subspace of simple predictable strategies, hence – by the fact that δ is closed – we obtain that they conincide on $L^2_{pred}(\Omega \times \mathbb{R}_{\geq 0}; \mathbb{R}^d)$.

The Clark-Ocone formula

Let $\mathcal{F}_H = \mathcal{F}$ and let $F \in \mathcal{D}^{1,2}$, then

$$F = E(F) + \sum_{i=1}^{d} \int_0^\infty E(D_t^i F \mid \mathcal{F}_t) dW_t^i.$$

Proof

By martingale representation we know that any $G \in L^2$ has a representation

$$G = E(G) + \sum_{i=1}^{d} \int_0^\infty \phi_t^i dW_t^i,$$

hence

$$E(FG) = E(F)E(G) + E\left(\sum_{i=1}^{d} \int_0^\infty D_t^i F \, \phi_t^i dt \right),$$

which yields the result.