SMOOTH PERFECTNESS FOR THE GROUP OF DIFFEOMORPHISMS

STEFAN HALLER AND JOSEF TEICHMANN

Abstract. Given the result of Herman, we provide a new elementary proof of the fact that the group of diffeomorphisms is a perfect and hence simple group. Moreover, we show that every diffeomorphism can be represented as a product of commutators $f = [g_1, h_1] \cdots [g_m, h_m]$ where the factors g_i and h_i can be chosen to depend smoothly on f. The elegance of the approach relies on the fact that we can choose the factors smoothly. If the underlying manifold is of dimension $n \geq 2$ then one can take $m = 9(n+1)$ provided f is sufficiently close to the identity.

1. Introduction and statement of the result

Definition 1. A possibly infinite dimensional Lie group G is called *locally smoothly perfect of order at most m* if the smooth map $K : G^{2m} \to G$

$$K(g_1, h_1, \dots, g_m, h_m) := [g_1, h_1] \circ \cdots \circ [g_m, h_m]$$

has a local smooth right inverse at the identity. More precisely, we ask for an open neighborhood $\mathcal{V}$ of $e \in G_o$ and a smooth map $\sigma : \mathcal{V} \to (G_o)^{2m}$ such that $K \circ \sigma = \mathrm{id}_{\mathcal{V}}$. Here G_o denotes the connected component of $e \in G$. Be aware that this concept is slightly different from the one used in [HT03].

For a smooth manifold M without boundary we denote by $\mathrm{Diff}^\infty_c(M)$ the group of compactly supported diffeomorphisms of M. This is a Lie group modeled on the convenient vector space $\mathfrak{X}_c(M)$ of compactly supported vector fields, see [KM97] for all necessary details. For compact M this smooth structure coincides with the well known Fréchet–Lie group structure on $\mathrm{Diff}^\infty(M)$.

The aim of this paper is to establish the following

Theorem 1. *Let M be a smooth manifold without boundary of dimension $n \geq 2$. Then $\mathrm{Diff}^\infty_c(M)$ is locally smoothly perfect of order at most $9(n+1)$.*

Since any open neighborhood of $e \in \mathrm{Diff}^\infty_c(M)_o$ generates the group, Theorem 1 implies that $\mathrm{Diff}^\infty_c(M)_o$ is a perfect group. This was already proved by Epstein [E84] using ideas of Mather, see [M74] and [M75], who dealt with the C^r–case, $1 \leq r < \infty$, $r \neq n+1$. The Epstein–Mather proof is elementary, but tricky. It is based on the Schauder–Tychonov fixed point theorem, and guaranties the existence

---

Date: Draft from October 4, 2004.

2000 *Mathematics Subject Classification.* 58D05.

Key words and phrases. diffeomorphism group; perfect group; simple group; fragmentation; convenient calculus; foliation.

The first author is supported by the *Fonds zur Förderung der wissenschaftlichen Forschung* (Austrian Science Fund), project number P14195-MAT.

of a presentation $f = [g_1, h_1] \cdots [g_m, h_m]$, but without any further control on the factors g_i and h_i.

Theorem 1 actually implies the stronger statement that the universal covering of $\mathrm{Diff}_c^\infty(M)_o$ is a perfect group. This result is known too, see [T74] and [M84]. Thurston's proof is based on a result of Herman for the torus, see [H71], [H73]. It also does not imply that the factors g_i and h_i can be chosen to depend smoothly on f.

Note that the perfectness of $\mathrm{Diff}_c^\infty(M)_o$ implies that this group is simple, see [E70]. The methods used in [E70] are elementary and actually work for a rather large class of groups of homeomorphisms.

Perfectness and simplicity results have been obtained for several other groups of diffeomorphisms. For instance, the group of volume preserving diffeomorphisms is not perfect. However, there is a well understood normal subgroup, the kernel of the flux homomorphism, which is simple according to unpublished work of Thurston, see [B97] for a write-up, and the series of papers [MD82], [MD83], [MD84].

Using Thurston's methods Banyaga clarified the algebraic structure of the group of compactly supported symplectomorphisms, see [B78] and [B97]. Again, the group itself is not perfect but the normal subgroup of Hamiltonian diffeomorphisms is simple for compact M. For non-compact M the group of compactly supported Hamiltonian diffeomorphisms contains a normal subgroup, the kernel of the Calabi homomorphism, which turns out to be a simple group. In [HR99] Banyaga's results have been generalized to the slightly more general situation of locally conformal symplectic manifolds.

To our knowledge the case of contact diffeomorphisms is still open, though it appears reasonable to expect that the group of compactly supported contact diffeomorphisms is perfect and hence [E70] simple. Thurston's approach should work in this case. However, a model case playing the role of Herman's torus is lacking. The Lie algebra of compactly supported contact vector fields certainly is perfect in view of [O97].

Rybicki [R95] proved that the group of compactly supported diffeomorphisms, which preserve the leaves of a foliation of constant rank, is a perfect group — though not simple of course.

Note, however, that none of the results above guaranties that the factors in a presentation $f = [g_1, h_1] \cdots [g_m, h_m]$ can be chosen to depend smoothly on f, except Herman's result for the torus. Nor are there any estimates available on how many commutators one actually needs. In [HT03] we provided a class of closed manifolds, including compact Lie groups and odd dimensional spheres, for which the factors can be chosen smoothly, and estimates on the number of commutators are available.

Our proof of Theorem 1 rests on Herman's result too but is otherwise elementary and completely different from Thurston's approach. The drawback of our methods is that they do not cover the manifold $M = \mathbb{R}$.

The idea of our approach is to consider the foliation $\mathcal{F}$ on $\mathbb{R} \times T^{n-1}$ with leaves $\{\mathrm{pt}\} \times T^{n-1}$. Parametrizing, see Section 4, Herman's result for the torus, see Section 5, one obtains that the group of leave preserving diffeomorphisms $\mathrm{Diff}_c^\infty(\mathbb{R} \times T^{n-1}, \mathcal{F})$ is locally smoothly perfect. Perturbing $\mathcal{F}$, we obtain finitely many diffeomorphic foliations $\mathcal{F}_1, \dots, \mathcal{F}_k$ on $\mathbb{R} \times T^{n-1}$ which span the tangent bundle. Actually $k = 3$ suffices, see Section 6. Decomposing a diffeomorphism into leaf preserving ones, see Section 3, we show that $\mathrm{Diff}_c^\infty(\mathbb{R} \times T^{n-1})$ is locally smoothly perfect, see Proposition 4. Since $\mathbb{R} \times T^{n-1}$ embeds into $\mathbb{R}^n$, we can cover M with open sets diffeomorphic to $\mathbb{R} \times T^{n-1}$. Using a classical tool known as fragmentation, see Section 2, we conclude that $\mathrm{Diff}^\infty_c(M)$ is locally smoothly perfect.

2. Smooth fragmentation

Let M be a smooth manifold without boundary. For an open subset $U \subseteq M$ we denote by $\mathrm{Diff}^\infty_U(M) = \mathrm{Diff}^\infty_c(U)$ the group of diffeomorphisms with compact support contained in U. This is a Lie group modeled on the convenient vector space $\mathfrak{X}_U(M) = \mathfrak{X}_c(U)$ of vector fields with compact support contained in U.

The following is a folklore statement which can be found all over the literature on diffeomorphism groups.

Proposition 1. *Let M be a smooth manifold without boundary, and suppose $\mathcal{U} = \{U_1, \dots, U_k\}$ is a finite open covering of M. Then the smooth map*

$$P : \mathrm{Diff}^\infty_{U_1}(M) \times \cdots \times \mathrm{Diff}^\infty_{U_k}(M) \to \mathrm{Diff}^\infty_c(M),$$

$$P(g_1, \dots, g_k) := g_1 \circ \cdots \circ g_k$$

has a local smooth right inverse at the identity. More precisely, there exist an open neighborhood $\mathcal{V}$ of $e \in \mathrm{Diff}^\infty_c(M)_o$ and a smooth map $\sigma : \mathcal{V} \to \mathrm{Diff}^\infty_{U_1}(M)_o \times \cdots \times \mathrm{Diff}^\infty_{U_k}(M)_o$ with $\sigma(e) = (e, \dots, e)$ such that $P \circ \sigma = \mathrm{id}_{\mathcal{V}}$.

Proof. Let $\pi : TM \to M$ denote the projection of the tangent bundle. Choose a Riemannian metric on M and let exp denote the corresponding exponential map. Choose an open neighborhood of the zero section $W \subseteq TM$ such that $(\pi, \exp) : W \to M \times M$ is a diffeomorphism onto an open neighborhood of the diagonal. Let $W' \subseteq T^*M \otimes TM$ be an open neighborhood of the zero section, and set

$$\mathcal{W} := \big\{ X \in \mathfrak{X}_c(M) \,\big|\, \mathrm{img}(X) \subseteq W, \mathrm{img}(\nabla X) \subseteq W' \big\},$$

where ∇ denotes the Levi–Civita connection associated to the Riemannian metric. This is a C^1–open neighborhood of $0 \in \mathfrak{X}_c(M)$. For $X \in \mathcal{W}$ define $f_X := \exp \circ X \in C^\infty_c(M, M)$. We choose W and W' sufficiently small such that every f_X is a diffeomorphism. The map

$$\mathcal{W} \to \mathrm{Diff}^\infty_c(M), \quad X \mapsto f_X \tag{1}$$

provides a chart of $\mathrm{Diff}^\infty_c(M)$ centered at the identity. This is the standard way to put a smooth structure on $\mathrm{Diff}^\infty_c(M)$, see [KM97].

Choose a smooth partition of unity $\lambda_1, \dots, \lambda_k$ with $\mathrm{supp}(\lambda_i) \subseteq U_i$ for all $1 \leq i \leq k$. Define

$$\mathcal{W}' := \big\{ X \in \mathcal{W} \,\big|\, (\lambda_1 + \cdots + \lambda_i)X \in \mathcal{W} \text{ for all } 1 \leq i \leq k \big\}.$$

This is a C^1–open neighborhood of $0 \in \mathfrak{X}_c(M)$, and $\mathcal{W}' \subseteq \mathcal{W}$. For $X \in \mathcal{W}'$ and $1 \leq i \leq k$ set $X_i := (\lambda_1 + \cdots + \lambda_i)X$, and note that $f_{X_i} \in \mathrm{Diff}^\infty_c(M)$. Clearly $\mathrm{supp}(f_{X_1}) \subseteq \mathrm{supp}(\lambda_1) \subseteq U_1$. Moreover, for $1 < i \leq k$ we have $X_{i-1} = X_i$ on $M \setminus \mathrm{supp}(\lambda_i)$, and hence $f_{X_{i-1}} = f_{X_i}$ on $M \setminus \mathrm{supp}(\lambda_i)$. We conclude that the diffeomorphism $(f_{X_{i-1}})^{-1} \circ f_{X_i}$ has compact support contained in $\mathrm{supp}(\lambda_i) \subseteq U_i$ for all $1 < i \leq k$.

Let $\mathcal{V} \subseteq \mathrm{Diff}^\infty_c(M)$ denote the open neighborhood of $e \in \mathrm{Diff}^\infty_c(M)$ corresponding to $\mathcal{W}'$ via (1). Define a map

$$\sigma : \mathcal{V} \to \mathrm{Diff}^\infty_{U_1}(M) \times \cdots \times \mathrm{Diff}^\infty_{U_k}(M)$$

$$\sigma(f_X) := \Big(f_{X_1}, (f_{X_1})^{-1} \circ f_{X_2}, (f_{X_2})^{-1} \circ f_{X_3}, \dots, (f_{X_{k-1}})^{-1} \circ f_{X_k}\Big)$$

and notice that it is smooth because X depends smoothly on f_X in view of our chart (1). Obviously we have $P(\sigma(f_X)) = f_{X_k} = f_X$ and thus $P \circ \sigma = \mathrm{id}_{\mathcal{V}}$. Moreover, it is immediate from the construction that $\sigma(e) = \sigma(f_0) = (e, \dots, e)$. □

3. Smooth Decomposition into foliation preserving diffeomorphisms

Let M be a smooth manifold without boundary and suppose $\mathcal{F}$ is a foliation on M. We will always assume that foliations are of constant rank. Let $\mathrm{Diff}^\infty_c(M, \mathcal{F})$ denote the group of compactly supported diffeomorphisms which preserve the leaves of $\mathcal{F}$. This is a Lie group modeled on the convenient vector space $\mathfrak{X}_c(M, \mathcal{F})$ of compactly supported vector fields tangential to $\mathcal{F}$, see [KM97].

A weaker version of the following proposition was proved in [HT03] using the Nash–Moser implicit function theorem. However, Proposition 2 applies to non-compact manifolds as well, and its proof is elementary.

Proposition 2. *Let M be a manifold without boundary and suppose $\mathcal{F}_1, \dots, \mathcal{F}_k$ are smooth foliations on M such that the corresponding distributions span the tangent bundle of M. Then the smooth map*

$$F : \mathrm{Diff}^\infty_c(M, \mathcal{F}_1) \times \cdots \times \mathrm{Diff}^\infty_c(M, \mathcal{F}_k) \to \mathrm{Diff}^\infty_c(M)$$
$$F(g_1, \dots, g_k) := g_1 \circ \cdots \circ g_k$$

has a local smooth right inverse at the identity. More precisely, there exist an open neighborhood $\mathcal{V}$ of $e \in \mathrm{Diff}^\infty_c(M)_o$ and a smooth map $\sigma : \mathcal{V} \to \mathrm{Diff}^\infty_c(M, \mathcal{F}_1)_o \times \cdots \times \mathrm{Diff}^\infty_c(M, \mathcal{F}_k)_o$ with $\sigma(e) = (e, \dots, e)$, such that $F \circ \sigma = \mathrm{id}_{\mathcal{V}}$.

Proof. Let us write $T\mathcal{F}_i$ for the subbundle of TM consisting of vectors tangent to $\mathcal{F}_i$. Let $\Sigma : T\mathcal{F}_1 \oplus \cdots \oplus T\mathcal{F}_k \to TM$ denote the vector bundle map given by addition. Choose a vector bundle map $s : TM \to T\mathcal{F}_1 \oplus \cdots \oplus T\mathcal{F}_k$ right inverse to Σ. This is possible since the distributions $\mathcal{F}_i$ span the tangent bundle.

Choose a Riemannian metric on M, and note that this provides Riemannian metrics on the leaves of $\mathcal{F}_i$. We obtain smooth exponential maps $\exp^{\mathcal{F}_i} : U_i \to M$, defined on an open neighborhood of the zero section $U_i \subseteq T\mathcal{F}_i$, which map the fibers of $T\mathcal{F}_i$ into the leaves of $\mathcal{F}_i$. Choose any linear connection on $T\mathcal{F}_1 \oplus \cdots \oplus T\mathcal{F}_k$. For $1 \leq i \leq k$ and $(\xi_1, \dots, \xi_k) \in T\mathcal{F}_1 \oplus \cdots \oplus T\mathcal{F}_k$ let $\tau_i(\xi_1, \dots, \xi_k)$ denote the point obtained by parallel transport in $T\mathcal{F}_1 \oplus \cdots \oplus T\mathcal{F}_k$ along the curve $t \mapsto \exp^{\mathcal{F}_i}(t\xi_i)$ up to time $t = 1$. It is evident that τ_i constitutes a smooth map:

$$\tau_i : T\mathcal{F}_1 \oplus \cdots \oplus U_i \oplus \cdots \oplus T\mathcal{F}_k \to T\mathcal{F}_1 \oplus \cdots \oplus T\mathcal{F}_k$$

We certainly find an open neighborhood of the zero section $W \subseteq TM$ such that the compositions

$$\alpha_i := \pi \circ \tau_i \circ \tau_{i+1} \circ \cdots \circ \tau_k \circ s : W \to M$$

are well defined for all $1 \leq i \leq k$. Here $\pi : T\mathcal{F}_1 \oplus \cdots \oplus T\mathcal{F}_k \to M$ denotes the vector bundle projection.

Set $\alpha := \alpha_1$, and consider $(\pi, \alpha) : W \to M \times M$. Its derivative at $0_x \in T_xM$ is invertible. Using the inverse function theorem and possibly shrinking W we may assume that $(\pi, \alpha) : W \to M \times M$ is a diffeomorphism onto an open neighborhood of the diagonal. Let $W' \subseteq T^*M \otimes TM$ be an open neighborhood of the zero section, and define

$$\mathcal{W} := \big\{X \in \mathfrak{X}_c(M) \,\big|\, \mathrm{img}(X) \subseteq W, \mathrm{img}(\nabla X) \subseteq W'\big\},$$

where ∇ denotes the Levi–Civita connection on M. This is a C^1–open neighborhood of $0 \in \mathfrak{X}_c(M)$. For $X \in \mathcal{W}$ define $f_X := \alpha \circ X \in C_c^\infty(M, M)$. We choose W and W' sufficiently small so that every f_X is a diffeomorphism. It is now clear that

$$\mathcal{W} \to \mathrm{Diff}_c^\infty(M), \quad X \mapsto f_X \tag{2}$$

constitutes a chart for $\mathrm{Diff}_c^\infty(M)$ centered at the identity, cf. the concept of local addition in [KM97, Section 42.4].

For $1 \le i \le k$ and $X \in \mathcal{W}$ define $f_{X,i} := \alpha_i \circ X$. Again, we choose W and W' sufficiently small so that $f_{X,i}$ is a diffeomorphism for every $X \in \mathcal{W}$ and all $1 \le i \le k$. Obviously we have $f_{X,1} = f_X$. Also note that $f_{X,k} \in \mathrm{Diff}_c^\infty(M, \mathcal{F}_k)$. Similarly, it follows from the construction that $f_{X,i} \circ (f_{X,i+1})^{-1} \in \mathrm{Diff}_c^\infty(M, \mathcal{F}_i)$ for all $1 \le i < k$.

Let $\mathcal{V}$ denote the open neighborhood of $e \in \mathrm{Diff}_c^\infty(M)$ corresponding to $\mathcal{W}$ via (2). Define a map

$$\sigma : \mathcal{V} \to \mathrm{Diff}_c^\infty(M, \mathcal{F}_1) \times \cdots \times \mathrm{Diff}_c^\infty(M, \mathcal{F}_k)$$
$$\sigma(f_X) := \Big(f_{X,1} \circ (f_{X,2})^{-1}, f_{X,2} \circ (f_{X,3})^{-1}, \ldots, f_{X,k-1} \circ (f_{X,k})^{-1}, f_{X,k}\Big)$$

and note that it is smooth because X depends smoothly on f_X in view of our chart (2). Moreover we have $F(\sigma(f_X)) = f_{X,1} = f_X$ and thus $F \circ \sigma = \mathrm{id}_{\mathcal{V}}$. Since the α_i restrict to the identity on the zero section we have $\sigma(e) = \sigma(f_0) = (e, \ldots, e)$. □

4. The exponential law and smooth parametrization of diffeomorphism

Suppose B is a smooth manifold without boundary and suppose G is a possibly infinite dimensional Lie group. Consider the space $C_c^\infty(B, G)$ of smooth maps from B to G which are constant equal to $e \in G$ outside a compact subset of B. This is a Lie group with point wise multiplication, see [KM97].

Lemma 1. *If G is locally smoothly perfect of order at most m, then so is $C_c^\infty(B, G)$.*

Proof. Let $\sigma : \mathcal{V} \to G^{2m}$ be a local right inverse as in Definition 1. Let $\mathcal{V}' \subseteq C_c^\infty(B, G)$ denote the subset of maps which take values in $\mathcal{V} \subseteq G$. This is an open neighborhood of $e \in C_c^\infty(B, G)$. Consider the smooth map

$$\sigma' := \sigma_* : \mathcal{V}' \to C_c^\infty(B, G^{2m}) = C_c^\infty(B, G)^{2m}$$

given by composition with σ. Its restriction to the connected component of $\mathcal{V}'$ clearly is a local smooth right inverse as required. □

Remark 1. Note that if G is only perfect then the argument in the proof of Lemma 1 breaks down, and one cannot conclude that $C_c^\infty(M, G)$ is perfect.

Let S and B be smooth manifolds without boundary. Set $M := B \times S$ and let $\mathcal{F}$ denote the foliation of M with leaves $\{\mathrm{pt}\} \times S$.

Proposition 3. *If $\mathrm{Diff}_c^\infty(S)$ is locally smoothly perfect of order at most m, then so is $\mathrm{Diff}_c^\infty(M, \mathcal{F})$.*

Proof. It follows from the exponential law in [KM97] that the homomorphism

$$C_c^\infty(B, \mathrm{Diff}_c^\infty(S)) \to \mathrm{Diff}_c^\infty(M, \mathcal{F})$$
$$\alpha \mapsto \big((b, s) \mapsto (b, \alpha(b)(s))\big)$$

is an isomorphism of Lie groups. The proposition now follows from Lemma 1 above. □

5. Herman's result for the torus

Let $T^n := \mathbb{R}^n/\mathbb{Z}^n$ denote the torus of dimension n. The group $\mathrm{Diff}^\infty(T^n)$ is a Fréchet–Lie group modeled on the Fréchet space $\mathfrak{X}(T^n)$ of vector fields on T^n. For $\beta \in T^n$ we let $R_\beta \in \mathrm{Diff}^\infty(T^n)$ denote the rotation by β. In this way we can consider T^n as subgroup of $\mathrm{Diff}^\infty(T^n)_o$.

Theorem 2 (Herman)**.** *There exist $\gamma \in T^n$ such that the map*

$$H : T^n \times \mathrm{Diff}^\infty(T^n)_o \to \mathrm{Diff}^\infty(T^n)_o,$$

$$H(\lambda, h) := R_\lambda \circ [R_\gamma, h]$$

has a local smooth right inverse at the identity. More precisely, there exist an open neighborhood $\mathcal{V}$ of $e \in \mathrm{Diff}^\infty(T^n)_o$ and a smooth map $\sigma : \mathcal{V} \to T^n \times \mathrm{Diff}^\infty(T^n)_o$ with $\sigma(e) = (0, e)$ such that $H \circ \sigma = \mathrm{id}_\mathcal{V}$.

Herman's result is an application of the Nash–Moser inverse function theorem. When inverting the derivative one is quickly led to solve the linear equation $Y = X - (R_\gamma)^*X$ for given $Y \in C^\infty(T^n, \mathbb{R}^n)$. This is accomplished using Fourier transformation. Here one has to choose γ sufficiently irrational so that tame estimates on the Sobolev norms of X in terms of the Sobolev norms of Y can be obtained. The corresponding small denominator problem can be solved due to a number theoretic result of to Khintchine. For a proof of Herman's theorem see [H73] and [H71].

Using the fact that the perfect group of Möbius transformations acts on T^1, Herman concluded that $\mathrm{Diff}^\infty(T^n)_o$ is a perfect group. In fact, one can estimate the number of commutators necessary, see [HT03, Section 4], and obtains

Corollary 1. $\mathrm{Diff}^\infty(T^n)$ *is locally smoothly perfect of order at most* 3*.*

6. Proof of Theorem 1

We start proving the following

Proposition 4. *Suppose $n \geq 2$. Then $\mathrm{Diff}_c^\infty(\mathbb{R} \times T^{n-1})$ is locally smoothly perfect of order at most* 9*.*

Proof. Consider the foliation $\mathcal{F}$ on $\mathbb{R} \times T^{n-1}$ with leaves $\{\mathrm{pt}\} \times T^{n-1}$. In view of Proposition 3 and Corollary 1 we know that $\mathrm{Diff}_c^\infty(\mathbb{R} \times T^{n-1}, \mathcal{F})$ is locally smoothly perfect of order at most 3.

One easily constructs diffeomorphisms φ_i of $\mathbb{R} \times T^{n-1}$ such that the foliations $\mathcal{F}_i := \varphi_i(\mathcal{F})$, $1 \leq i \leq 3$, span the tangent bundle of $\mathbb{R} \times T^{n-1}$. Indeed, choose functions $h_i : T^{n-1} \to \mathbb{R}$ such that $h_1 = 0$ and the critical sets of h_2 and h_3 are disjoint. Now define $\varphi_i(x, t) := (x + h_i(t), t)$. It is readily checked that these diffeomorphisms have the desired property. Clearly $\mathrm{Diff}^\infty(\mathbb{R} \times T^{n-1}, \mathcal{F}_i) \simeq \mathrm{Diff}^\infty(\mathbb{R} \times T^{n-1}, \mathcal{F})$ also is locally smoothly perfect of order at most 3. It follows from Proposition 2 that $\mathrm{Diff}_c^\infty(\mathbb{R} \times T^{n-1})$ is locally smoothly perfect of order at most $3 \cdot 3 = 9$. □

For a smooth manifold M of dimension n let $\mathrm{cov}(M)$ denote the minimal integer k such that there exists an open covering $\mathcal{U} = \{U_1, \dots, U_k\}$ of M for which every U_i is diffeomorphic to a disjoint union of copies of $\mathbb{R} \times T^{n-1}$. Using a triangulation and the fact that $\mathbb{R} \times T^{n-1}$ embeds into $\mathbb{R}^n$, one easily shows $\mathrm{cov}(M) \leq n + 1$. Therefore Theorem 3 below implies Theorem 1.

Theorem 3. *Let M be a smooth manifold of dimension $n \geq 2$. Then $\mathrm{Diff}^\infty_c(M)$ is locally smoothly perfect of order at most $9 \cdot \mathrm{cov}(M)$.*

Proof. Choose an open covering $\mathcal{U} = \{U_1, \dots, U_{\mathrm{cov}(M)}\}$ of M such that every U_i is diffeomorphic to a disjoint union of copies of $\mathbb{R} \times T^{n-1}$. Using Proposition 4 we see that $\mathrm{Diff}^\infty_{U_i}(M) = \bigoplus_{\pi_0(U_i)} \mathrm{Diff}^\infty_c(\mathbb{R} \times T^{n-1})$ is locally smoothly perfect of order at most 9. In view of Proposition 1 this implies that $\mathrm{Diff}^\infty_c(M)$ is locally smoothly perfect of order at most $9 \cdot \mathrm{cov}(M)$. □

It is easy to show that $\mathrm{cov}(\mathbb{R}^n) = \mathrm{cov}(S^n) = 2$. Hence we obtain

Corollary 2. *Let $n \geq 2$. Then $\mathrm{Diff}^\infty(S^n)$ and $\mathrm{Diff}^\infty_c(\mathbb{R}^n)$ are locally smoothly perfect of order at most 18.*

Note that the case $M = S^1$ is covered by Corollary 1. However, the case $M = \mathbb{R}$ is not.

References

[B78] A. Banyaga, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique.* Comment. Math. Helv. **53**(1978), 174–227.

[B97] A. Banyaga, *The structure of classical diffeomorphism groups.* Kluwer Academic Publishers Group, Dordrecht, 1979.

[E70] D.B.A. Epstein, *The simplicity of certain groups of homeomorphisms.* Compositio Math. **22**(1970), 165–173.

[E84] D.B.A. Epstein, *Commutators of C^∞–diffeomorphisms. Appendix to: 'A curious remark concerning the geometric transfer map" by John N. Mather.* Comment. Math. Helv. **59**(1984), 111–122.

[HR99] S. Haller and T. Rybicki, *On the group of diffeomorphisms preserving a locally conformal symplectic structure.* Ann. Global Anal. Geom. **17**(1999), 475–502.

[HT03] S. Haller and J. Teichmann, *Smooth perfectness through decomposition of diffeomorphisms into fiber preserving ones.* Ann. Global Anal. Geom. **23**(2003), 53–63.

[H71] M.R. Herman, *Simplicité du groupe des difféomorphismes de classe C^∞, isotopes à l'identité, du tore de dimension n.* C. R. Acad. Sci. Paris Sr. A **273**(1971), 232–234.

[H73] M.R. Herman, *Sur le groupe des diffomorphismes du tore.* Ann. Inst. Fourier (Grenoble) **23**(1973), 75–86.

[KM97] A. Kriegl and P.W. Michor, *The convenient setting of global analysis.* Mathematical Surveys and Monographs **53**, American Mathematical Society, 1997.

[M74] J.N. Mather, *Commutators of diffeomorphisms.* Comment. Math. Helv. **49**(1974), 512–528.

[M75] J.N. Mather, *Commutators of diffeomorphisms. II.* Comment. Math. Helv. **50**(1975), 33–40.

[M84] J.N. Mather, *A curious remark concerning the geometric transfer map.* Comment. Math. Helv. **59**(1984), 86–110.

[MD82] D. McDuff, *Local homology of groups of volume preserving diffeomorphisms. I.* Ann. Sci. École Norm. Sup. (4) **15**(1982), 609–648.

[MD83] D. McDuff, *Local homology of groups of volume-preserving diffeomorphisms. II.* Comment. Math. Helv. **58**(1983), 135–165.

[MD84] D. McDuff, *Local homology of groups of volume-preserving diffeomorphisms. III.* Ann. Sci. École Norm. Sup. (4) **16**(1984), 529–540.

[O97] H. Omori, *Infinite-dimensional Lie groups.* Translated from the 1979 Japanese original and revised by the author. Translations of Mathematical Monographs **158**. American Mathematical Society, Providence, RI, 1997.

[R95] T. Rybicki, *The identity component of the leaf preserving diffeomorphism group is perfect.* Monatsh. Math. **120**(1995), 289–305.

[T74] W. Thurston, *Foliations and groups of diffeomorphisms.* Bull. Amer. Math. Soc. **80**(1974), 304–307.

Stefan Haller, Department of Mathematics, University of Vienna, Nordbergstrasse 15, A-1090, Vienna, Austria.
E-mail address: stefan.haller@univie.ac.at

Josef Teichmann, Institute of Mathematical Methods in Economics, Technical University of Vienna, Wiedner Hauptstrasse 8–10, A-1040 Vienna, Austria.
E-mail address: josef.teichmann@fam.tuwien.ac.at