HOPF'S DECOMPOSITION AND RECURRENT SEMIGROUPS

Josef Teichmann

Abstract. Some results of ergodic theory are generalized in the setting of Banach lattices, namely Hopf's maximal ergodic inequality and the so called Hopf decomposition (see [6]). Then these results are applied to a recently analysed problem concerning recurrent semigroups on L^1-spaces (see [2],[3],[7]).

1. Introduction

The following stability result has been shown by W. Arendt, C. J. K. Batty, Ph. Bénilan [2] (see also C. J. K. Batty [3]). Consider the Laplacian $\triangle$ on $L^1(\mathbb{R}^n)$ and a positive potential $V \in L^\infty(\mathbb{R}^n)$: If $n = 1, 2$, then the semigroup generated by $\triangle - V$ converges strongly to 0 as $t \to \infty$ whenever $V \neq 0$; on the other hand, if $n \geq 3$, then there exist $V \neq 0$, such that the semigroup is not stable. In fact $\exp t(\triangle - V)$ leaves invariant a strictly positive function in $L^\infty(\mathbb{R}^n)$. I. McGillivray and M. Ouhabaz showed that the essential properties for this behaviour are recurrence (if $n = 1, 2$) and transience (if $n \geq 3$).

The purpose of this paper is to line out, that these phenomena can be observed in a much more general context. In fact it seems, that positive operators on Banach lattices form the right framework for the formulation of these results. A key role is played by the Hopf decomposition theorem, which we prove in an abstract context in section 3. This seems to be of independent interest. In section 4 we prove the stability results as immediate consequences. For convenience of the reader we put together the elementary properties of Banach lattices and positive operators needed in the sequel.

2. Vector- and Banach lattices

In this chapter we are going to outline the main principles and to prove a generalized version of Hopf's maximal ergodic inequality (see e.g. [1],[8] for some further informations).

Definition 1:
Let E be an ordered vector space. E is called <u>vector lattice</u>*, if for two elements $x, y \in E$ there is a supremum, i.e. an element $z \in E$ satisfying the following property:*

$$x \leq z \text{ and } y \leq z \text{ and } \forall z' \in E : x \leq z' \text{ and } y \leq z' \Rightarrow z \leq z'$$

We write $z = \sup(x, y)$.

The next definition treats basic subspace-structures of vector lattices:

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Definition 2: *Let E be a vector lattice:*

i.) *For $x \in E$ we define the following operations:*

$$x^+ := \sup(x,0) \qquad x^- := -\inf(x,0) \qquad |x| := \sup(x,-x)$$

ii.) *A subspace F of E is called* <u>*vector sublattice*</u>*, if the following condition is satisfied:*

$$\forall x,y \in F : \sup(x,y) \in F$$

iii.) *A subspace I of E is called* <u>*ideal*</u>*, if*

$$\forall x \in E \; \forall y \in I : |x| \le |y| \Rightarrow x \in I$$

iv.) *An ideal B of E is called a* <u>*band*</u>*, if*

$$\forall\, \emptyset \ne M \subset B : \quad M \text{ has a supremum in } E \Rightarrow \sup M \in B$$

v.) *Let $\emptyset \ne M \subset E$, then M^d denotes the following band:*

$$M^d := \{y \in E \mid \forall x \in M : \inf(|x|,|y|) = 0\}$$

Sometimes we shall denote $\inf(|x|,|y|) = 0$ by $x \perp y$.

Remarks 1:

i.) *The class of vector sublattices, ideals and bands, respectively, is closed under taking arbitrary intersections, so we can define a vector sublattice, an ideal and a band, respectively, generated by a nonempty set M.*

ii.) *An ideal is a vector sublattice for $|\sup(x,y)| \le |x| + |y|$ for all $x,y \in E$.*

iii.) *Two ideals I,J are called lattice disjoint, if their intersection is trivial. In this case we obtain:*

$$I \subset J^d \text{ and } J \subset I^d$$

If these two ideals are complemented to each other, i.e. $I + J = E$, then $I = J^d$ and $J = I^d$, then $I = I^{dd}$ and consequently I is a band. The relation $I = I^{dd}$ does not imply that I has a complement J. When I has a complement, I is called a <u>*projection band*</u>*, the projection belonging to I with kernel I^d is called* <u>*band projection*</u>*.*

iv.) *The following rules are valid in every vector lattice:*

$$\inf[(\sup_{i\in M} f_i, h)] = \sup_{i\in M}[\inf(f_i,h)] \qquad \sup[(\inf_{i\in M} f_i, h)] = \inf_{i\in M}[\sup(f_i,h)]$$

for any order-bounded family $\{f_i\}_{i\in M}$ and any $h \in E$. Furthermore we obtain for $x,y,z \in E$:

$$x = x^+ - x^- \qquad |x| = x^+ + x^- \qquad |x+y| \le |x| + |y|$$

$$x + y = \sup(x,y) + \inf(x,y) \qquad \sup(x+z, y+z) = \sup(x,y) + z$$

v.) *A vector lattice V is called* <u>*order-complete*</u>*, if every nonempty, order-bounded subset $M \subset V$ has a supremum.*

Crucial for our considerations is the following Riesz Decomposition Theorem.

Theorem 1: *(Riesz Decomposition)*
Let E be an order-complete vector lattice, then every band B is a projection band. We obtain in particular, for every subset $M \subset E$,

$$E = M^d \oplus M^{dd}$$

Proof: Let $B \subset E$ be a band, then $B \cap B^d$ is trivial, since $|x| = \inf(|x|, |x|) = 0$ for any $x \in B \cap B^d$. Let $z \in E_+ = \{x \in E \mid x \geq 0\}$ and define for all $x \in B_+$

$$f_x := \inf(x, z).$$

$0 \leq f_x \leq x$ implies that f_x is in B and $0 \leq f_x \leq z$ that the family $\{f_x\}_{x \in B_+}$ is order-bounded. E is order-complete, so $\{f_x\}_{x \in B_+}$ has a supremum, $z_1 := \sup_{x \in B_+} f_x$.

B is a band, so we find that z_1 is in B. For all $x \in B$ we have the following inequality after Rem.1.iv.):

$$0 \leq \inf(z - z_1, |x|) = \inf(z, |x| + z_1) - z_1 \leq 0$$

Consequently $z - z_1 \in B^d$, i.e. every positive element of E can be decomposed $z = z_1 + z_2$ with $z_1 \in B$ and $z_2 \in B^d$. So we can conclude. □

Norm and order have to be compatible in normed vector lattices, therefore the following definitions:

Definition 3: *Let E, F be vector lattices:*

i.) *A seminorm p on E is called lattice seminorm, if*

$$\forall x, y \in E : |x| \leq |y| \Rightarrow p(x) \leq p(y) \ .$$

A vector lattice with a lattice norm is called normed vector lattice, a complete normed vector lattice is called Banach lattice.

ii.) *A linear map $T : E \to F$ is called positive, if $T(E_+) \subset F_+$. A positive linear functional $\phi : E \to \mathbb{R}$ is called strictly positive, if*

$$\forall x \in E_+ : x \neq 0 \Rightarrow \phi(x) > 0 \ .$$

iii.) *Let p be a lattice seminorm on E. Then p is called (σ)-order-continuous, if for any monotone decreasing net (sequence) $\{x_i\}_{i \in I}$ the following condition holds:*

$$\inf_{i \in I} x_i = 0 \Rightarrow \inf_{i \in I} p(x_i) = 0$$

Remarks 2:

i.) *The usual factorization and extension procedures of functional analysis can be applied in the case of normed vector lattices. One striking property of linear operators on Banach lattices is that positive operators are automatically continuous.*

ii.) *A positive functional ϕ on a vector lattice induces a lattice seminorm p by $p(x) := \phi(|x|)$ for $x \in E$. Every strictly positive functional induces a norm.*

iii.) *Let E be a normed vector lattice, then every band $B \subset E$ is closed. For every nonempty subset $M \subset E$ we obtain $\langle M \rangle = M^{dd}$ Here $\langle M \rangle$ denotes the band generated by M.*

Now we are going to prove a lattice-version of Hopf's maximal ergodic inequality:

Theorem 2: *(maximal ergodic inequality of E. Hopf)*
Let E be an order-complete vector lattice, $T : E \to E$ a positive operator and $\phi : E \to \mathbb{R}$ a positive linear functional satisfying $T'\phi \leq \phi$. For $x \in E$ we define bands E_n with associated band projection $p_n : E \to E$ for $n \in \mathbb{N}_+$ by

$$E_n := [M_n(x)^-]^d \text{ for } n \in \mathbb{N}_+ \quad .$$

$$M_n(x) := \sup_{1 \leq m \leq n} S_m(x) \text{ and } S_n(x) := \sum_{i=0}^{n-1} T^i x \text{ for } n \in \mathbb{N}_+$$

We obtain: $\forall x \in E \quad \forall n \in \mathbb{N}_+ : \phi(p_n(x)) \geq 0 \quad .$

Proof: We fix $n \in \mathbb{N}_+$, $x \in E$, so we have $M_n(x)^+ \geq S_m(x)$ for $1 \leq m \leq n$ and consequently because of the positivity of T

$$T(M_n(x)^+) + x \geq S_{m+1}(x) \text{ for } 1 \leq m \leq n \quad .$$

So we obtain $x \geq M_n(x) - T(M_n(x)^+)$ for all $n \in \mathbb{N}$. If we apply $\phi \circ p_n$ to this equation, we can conclude

$$\phi(p_n(x)) \geq \phi\{p_n[M_n(x) - T(M_n(x)^+)]\} \geq 0 \text{ for all } n \in \mathbb{N},$$

because $p_n(M_n(x)) = M_n(x)^+$ ($\inf(x^+, x^-) = 0$ for all $x \in E$) and $T'\phi \leq \phi$. $\square$

Example 1: *Let (Ω, F, μ) be a measured space, $E = L_p(\Omega)$ for $1 \leq p < \infty$ and the linear functional ϕ be given by*

$$\phi(f) := \int_\Omega fg d\mu \text{ for all } f \in E \quad ,$$

where $g \in L_q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$ was chosen positive. Then for every positive operator T satisfying $T'g \leq g$ the above inequalities are valid. In the case $p = 1$ and $g = 1_\Omega$ we obtain the classical version of Hopf's inequality(see [6]).

Definition 4: *Let E be a vector lattice:*

i.) $x \in E_+$ is called a (weak) order-unit of E, if the ideal (band) generated by x is the whole space E.

ii.) Let E be a normed vector lattice and $x \in E_+$, then x is called a quasi-interior point of E, if the closed ideal generated by x is the whole space E.

Remarks 3: *Let E be a Banach lattice:*

i.) $x \in E$ is an inner point of E_+ if $x \in E_+$ is an order-unit.

ii.) Let E be separable, then quasi-interior points exist.

Definition 5: *Let E be a vector lattice:*

i.) A functional $\phi : E \to \mathbb{R}$ is called order-bounded, if the following condition is satisfied:

$$\forall x, y \in E : x \leq y \Rightarrow \phi([x, y]) \text{ is bounded}$$

ii.) The vector space of all order-bounded functionals is denoted by $E^\sharp$.

Remarks 4:

i.) Let E be a vector lattice, then all positive functionals are order-bounded. The vector lattice $E^\sharp$ ($\phi \le \psi \iff \psi - \phi$ is positive functional) is order-complete. The following formula will be useful: For $\phi, \psi \in E^\sharp_+$

$$\forall x \in E_+ : [\sup(\phi, \psi)](x) = \sup\{\phi(y) + \psi(x-y) \mid 0 \le y \le x\} \quad .$$

ii.) Let E be a Banach lattice, then the topological dual space E' and the space of all order-bounded linear functionals $E^\sharp$ coincide, the norm on E' is a lattice norm, E' is an order-complete Banach lattice.

In order to be able to handle a bigger class of linear functionals we introduce ϕ-reachable functionals on a normed vector lattice, where the norm is induced by a strictly positive linear functional ϕ:

Definition 6: *Let E be a vector lattice:*
Let $\phi : E \to \mathbb{R}$ be a strictly positive functional and $\psi \in E^\sharp_+$, then ψ is called ϕ-reachable, if the following condition is satisfied:

$$\sup_{\alpha > 0}[\inf(\alpha\phi, \psi)] = \psi$$

In the second chapter we shall apply these notions in order to prove a generalized version of Hopf's decomposition-theorem.

3. Hopf's decomposition of a positive contraction semigroup

Given a positive contraction semigroup on a vector lattice. Under certain assumptions one can decompose the vector lattice into two bands, where some regularity conditions are satisfied. One is lead in a natural way to the ideas of recurrence and transience, to so called Dirichlet spaces and to some minimality properties. At first we are going to treat the case of discrete semigroups, the continuous case will be a corollary.

Theorem 3: *(Hopf's decomposition I)*
Let E be an order-complete vector lattice containing a weak order-unit, $T : E \to E$ a positive operator and $\phi : E \to \mathbb{R}$ a strictly positive, order-continuous linear functional so that $T'\phi \le \phi$ (we say, T is contractive with respect to ϕ or ϕ is a subinvariant linear form of T).

Then there exists a unique decomposition of E into two bands B_p, B_q with associated band projections p, q given by the following defining property:

$$\forall \psi \in E^\sharp_+ : T'\psi \le \psi \text{ and } \psi \ \phi\text{-reachable} \Rightarrow q'(T'\psi) = q'(\psi)$$

$$\exists\, \eta \in E^\sharp_+ : T'\eta \le \eta \text{ and } \eta \ \phi\text{-reachable}, \quad q'\eta = 0, \quad (\eta - T'\eta) \text{ is strictly positive on } B_p \text{ and for all } x \in B_{p+} : \lim_{n\to\infty} \langle T^n x, \eta\rangle = 0$$

In addition, one can choose η smaller than ϕ.

Proof: We denote by $\mathfrak{P}$ the set of band projections $p : E \to E$. Then we can define the following map $\lambda : \mathfrak{P} \to \mathbb{R}_{\ge 0}$, which reminds a measure:

$$\lambda(p) := \phi(p(x)) \text{ for all } p \in \mathfrak{P}$$

This map is bounded by $\phi(x)$ and monotone with respect to the natural ordering of $\mathfrak{P}$. Moreover, for all $p \in \mathfrak{P}$,

$$\lambda(p) = 0 \iff p = 0 \quad ,$$

since $p(x) > 0$ for all $0 \neq p \in \mathfrak{P}$. Now we are going to apply a variant of a principle usually used in measure theory.

We denote by P the following property on $\mathfrak{P}$, i.e. a map from $\mathfrak{P}$ to $\{0,1\}$.

$$\forall 0 \neq p \in \mathfrak{P} : P(p) = 1 \iff \exists \psi \in E^\sharp_+ : \psi \leq \phi, \quad T'\psi \leq \psi \text{ and}$$

$$\psi - T'\psi \text{ is strictly positive on } B_p.$$

Let B_p be the band associated to a band projection p in E. We define $P(0) = 1$. The so defined property P satisfies the following condition:

$$\forall p, q \in \mathfrak{P} : P(p) = 1 \text{ and } q \leq p \Rightarrow P(q) = 1 \qquad (B)$$

If a property on $\mathfrak{P}$ satisfies condition (B), there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ in $\mathfrak{P}$, so that

$$\forall n, m \in \mathbb{N} : n \neq m \Rightarrow p_n p_m = 0 \text{ and}$$

$$\forall n \in \mathbb{N} : P(p_n) = 1 \text{ and } \lambda(p_{n+1}) \geq \frac{\alpha_n}{2} \quad ,$$

where $\alpha_n := \sup\{\lambda(q) \mid P(q) = 1 \quad , 0 \leq q \leq 1 - p_0 - \ldots - p_n\}$ for $n \in \mathbb{N}$. We can define a sequence $\{p_n\}_{n\in\mathbb{N}}$ recursively by choosing a band projection p_0 with $P(p_0) = 1$ and applying the axiom of choice. Furthermore we have

$$\sum_{n=0}^{\infty} \alpha_n \leq \lambda(1) < \infty \quad ,$$

consequently the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ converges to 0. Let p be the unique band projection with

$$\ker p = \bigcap_{n\in\mathbb{N}} \ker(p_n) \quad .$$

Then we state for $q := 1 - p$:

$$\forall 0 \leq r \leq q : P(r) = 1 \Rightarrow r = 0 \qquad (*)$$

In fact $\lambda(r) \leq \alpha_n$ is valid for all $n \in \mathbb{N}$. In the next step we shall show that $P(p) = 1$:

We denote by ψ_n the subinvariant linear form associated to T, which exists by assumption on p_n for $n \in \mathbb{N}$:

$$\xi_N := \sum_{n=0}^{N} \frac{1}{2^{n+1}} \psi_n \leq \phi \text{ for all } N \in \mathbb{N}$$

Then $\{\xi_n\}_{N\in\mathbb{N}}$ is a sequence of subinvariant linear forms of T bounded by ϕ. $E^\sharp$ is order-complete, so we obtain:

$$\xi := \sup_{N\in\mathbb{N}} \xi_N \text{ exists and } \forall x \in E_+ : \lim_{N\to\infty} \langle x, \xi_N \rangle = \langle x, \xi \rangle$$

Consequently we have the following inequality for $x \in E_+$:

$$\langle x, T'\xi\rangle = \lim_{N\to\infty} \langle Tx, \xi_N\rangle \le \lim_{N\to\infty} \langle x, \xi_N\rangle = \langle x, \xi\rangle \quad ,$$

so ξ is a subinvariant linear form. Let $0 \ne x \in B_{p+}$, now we are going to verify that:

$$\sup_{N\in\mathbb{N}} (\sum_{n=0}^{N} p_n(x)) = x \quad .$$

This sequence is clearly bounded by x, so we have a supremum in the order-complete vector lattice E contained in the band B_p, since all elements of the sequence are contained in B_p. On the other hand we have

$$\forall N \in \mathbb{N} : 0 \le x - \sup_{N\in\mathbb{N}} (\sum_{n=0}^{N} p_n(x)) \le x - \sum_{n=0}^{N} p_n(x) \in \ker p_N \quad .$$

So the claim is proved and there exists a number $n \in \mathbb{N}$, so that $p_n(x) > 0$. Now we can estimate:

$$\langle x, \xi - T'\xi\rangle = \lim_{N\to\infty} \langle x - Tx, \xi_N\rangle \ge \langle x, \frac{1}{2^{n+1}}(\psi_n - T'\psi_n)\rangle \ge$$

$$\ge \langle p_n(x), \frac{1}{2^{n+1}}(\psi_n - T'\psi_n)\rangle > 0 \quad ,$$

consequently $\xi - T'\xi$ is strictly positive on B_p and therefore $P(p) = 1$.
Since ξ is a subinvariant linear form of T, the following monotone limit exists for $x \in E_+$:

$$\langle x, \xi_0\rangle := \lim_{n\to\infty} \langle T^n x, \xi\rangle \quad .$$

ξ_0 is a positive functional on E and

$$\langle x, T'\xi_0\rangle \le \langle x, {T'}^{n+1}\xi\rangle \text{ and } \langle x, \xi_0\rangle \le \langle Tx, {T'}^{n-1}\xi\rangle$$

is valid for all $n \in \mathbb{N}_+$. Passing to the limit we obtain:

$$T'\xi_0 = \xi_0 \text{ and } \xi_0 \le \xi$$

Now define $\eta := \xi - \xi_0$, then the condition $(*)$:

$$q'(\eta) = 0 \text{ and } \eta - T'\eta \text{ strictly positive on } B_p$$

In particular we have by the above construction:

$$\forall x \in B_{p+} : \lim_{n\to\infty} \langle T^n x, \eta\rangle = 0$$

The rest of the proof is to show the following claim:

$$\forall \psi \in E_+^\sharp : T'\psi \le \psi \text{ and } \psi \ \phi\text{-reachable} \Rightarrow q'(T'\psi) = q'(\psi)$$

Let ψ be a ϕ-reachable subinvariant linear form of T. By Rem.2.i.) and condition $(*)$ we can conclude, since for all $\alpha \in \mathbb{R}_+$ the subinvariant linear form $\inf(\alpha\phi, \psi)$ is bounded by the order-continuous linear functional $\alpha\phi$:

$$\forall \alpha \in \mathbb{R}_+ : q'(\inf(\alpha\phi, \psi)) = q'T'(\inf(\alpha\phi, \psi)) \le q'T'(\psi)$$

ψ is a ϕ-reachable subinvariant linear form, so

$$q'(\psi) = q'[\sup_{\alpha>0} \inf(\alpha\psi, \phi)] \le q'T'(\psi) \quad .$$

This is the existence of the claimed decomposition, uniqueness is obvious. $\square$

Remarks 5:

i.) *The existence of a strictly positive mapping λ on $\mathfrak{P}$ is guaranteed by the existence of a weak order-unit. Weak order-units exist in the case of not "too big" vector lattices.*

ii.) *On L^1-spaces bands and equivalence classes (up to sets of measure zero) of measurable sets are in one-to-one correspondence. This observation was the starting point of our investigations.*

For further applications we shall need another equivalent characterization of Hopf's decomposition, which will be useful in the case of continuous semigroups.

Theorem 4: *(Hopf's decomposition II)*
Let E be an order-complete vector lattice containing a weak order-unit, $T : E \to E$ a positive operator and $\phi : E \to \mathbb{R}$ a strictly positive, order-continuous linear functional so that $T'\phi \leq \phi$.

Then there exists a unique decomposition of E into two bands B_p, B_q with associated band projections p, q given by the following defining property:

$$\forall \psi \in E_+^\sharp : \psi \ \phi\text{-reachable} \ \Rightarrow \sum_{i=0}^{\infty} \langle T^i x, \psi \rangle \in \{0, \infty\} \ \text{for all } x \in B_{q+}$$

$$\exists\, \xi \in E_+^\sharp : \xi \ \phi\text{-reachable}, \quad q'\xi = 0, \quad (\xi - T'\xi) \ \text{strictly positive on } B_p$$

$$\text{and for all } x \in B_p : \sum_{i=0}^{\infty} \langle T^i x, \xi \rangle < \infty$$

In particular, one can choose ξ, so that for all $x \in E_+$

$$\sum_{i=0}^{\infty} \langle T^i x, \xi \rangle \leq \langle x, \phi \rangle \quad .$$

This decomposition coincides with the one given in Theorem 3.

Proof: The above property defines a unique decomposition. We are going to show existence by means of the results of Theorem 3:

Let $\eta \leq \phi$ be a subinvariant linear form of T vanishing on B_q and being strictly positive on B_p so that

$$\forall x \in B_{p+} : lim_{n\to\infty} \langle T^n x, \eta \rangle = 0 \quad .$$

The existence is given by Theorem 3. We define $\xi := \eta - T'\eta$. Let $x \in B_{p+}$, then we obtain for $n \in \mathbb{N}$:

$$\sum_{i=0}^{n} \langle T^i x, \xi \rangle = \langle x, \eta \rangle - \langle x, {T'}^{n+1} \eta \rangle \quad .$$

The linear functional $\xi \leq \eta \leq \phi$ satisfies the claimed property of Theorem 4. The next step is devoted to the second part of the claim. Let ψ be a ϕ-reachable linear functional on E and $x \in E_+$ be fixed so that

$$0 \leq \sum_{i=0}^{\infty} \langle T^i x, \psi \rangle < \infty \quad .$$

For every fixed $\alpha > 0$ we define a sequence $\{\mu_N\}_{N\in\mathbb{N}}$ by

$$\mu_N = \inf(\alpha\phi, \sum_{i=0}^{N} {T'}^i \psi) \text{ for } N \in \mathbb{N}$$

The sequence is increasing and we have:

$$T'\mu_N \leq \alpha\phi \text{ and } T'\mu_N \leq \sum_{i=0}^{N+1} {T'}^i \psi$$

Consequently we obtain $T'\mu_N \leq \mu_{N+1}$ for $N \in \mathbb{N}$ and therefore

$$\mu := \sup_{N\in\mathbb{N}} \mu_N$$

is a subinvariant linear form of T, since

$$\langle Ty, \mu \rangle = \lim_{N\to\infty} \langle Ty, \mu_N \rangle \leq \lim_{N\to\infty} \langle y, \mu_{N+1} \rangle = \langle y, \mu \rangle \text{ for } y \in E_+ \quad .$$

By Theorem 3 we conclude that for all $n \in \mathbb{N}$

$$\forall x \in B_{q+} : \langle x, \mu \rangle = \langle x, {T'}^n \mu \rangle \quad .$$

Let $n \in \mathbb{N}$ be fixed, so

$$\langle x, {T'}^n \mu \rangle = \lim_{N\to\infty} \langle T^n x, \mu_N \rangle \leq \lim_{N\to\infty} [\sum_{i=n}^{N+n} \langle T^i x, \psi \rangle] \quad ,$$

which allows us to draw the beautiful conclusion: $\langle x, \mu \rangle = 0$ Consequently for all $N \in \mathbb{N}$ we obtain: $\langle x, \mu_N \rangle = 0$. By ϕ-reachability we finish the proof, since

$$\langle x, \mu_0 \rangle \text{ and } \psi \ \phi\text{-reachable } \Rightarrow \psi(x) = 0 \quad .$$

One obtains by the same argument $\langle T^n x, \psi \rangle = 0$ for all $n \in \mathbb{N}$, so

$$\sum_{i=0}^{\infty} \langle T^n x, \psi \rangle = 0$$

This is the claim. $\square$

We reformulate the theorems above for continuous semigroups, the proofs are slight modifications.

Definition 7: *Let E be a vector lattice and $T := \{T_t\}_{t>0}$ a semigroup of positive operators. Let ϕ be a strictly positive functional on E, then the semigroup is called weakly measurable with respect to ϕ, if the maps*

$$\left\{ \begin{array}{r} \mathbb{R}_+ \to \mathbb{R}_{\geq 0} \\ t \mapsto \langle T_t x, \psi \rangle \end{array} \right\}$$

are measurable for $x \in E_+$ and $0 \leq \psi \leq \phi$ in $E^\sharp$.

The reformulations are the following:

Theorem 5: *Let E be an order-complete vector lattice containing a weak order-unit, T a semigroup of positive operators and $\phi : E \to \mathbb{R}$ a strictly positive, order-continuous linear functional so that ${T_t}'\phi \leq \phi$ for all $t > 0$ (we say, T is contractive with respect to ϕ or ϕ is a subinvariant linear form of T). Furthermore let T be weakly measurable with respect to ϕ.*

Then there exists a unique decomposition of E into two bands B_p, B_q with associated band projections p, q given by the following defining property:

$$\forall \psi \in E_+^\sharp : \psi \text{ subinvariant linear form of } T \text{ and } \psi\ \phi\text{-reachable, then}$$

$$q'({T_t}'\psi) = q'(\psi) \text{ for all } t > 0$$

$$\exists\, \eta \in E_+^\sharp : \eta \text{ subinvariant linear form of } T \text{ and } \eta\ \phi\text{-reachable}, \quad q'\eta = 0,$$

$$(\eta - {T_t}'\eta) \text{ is strictly positive on } B_p \text{ for } t > 0$$

$$\text{and for all } x \in B_{p+} : \lim_{t\to\infty} \langle T_t x, \eta \rangle = 0$$

In addition, η can be chosen smaller than ϕ.

In the next theorem we shall use explicitly the rather weak measureability conditions on T:

Theorem 6: *Let E be an order-complete vector lattice containing a weak order-unit, T a semigroup of positive operators and $\phi : E \to \mathbb{R}$ a strictly positive, order-continuous linear functional so that ${T_t}'\phi \leq \phi$ for all $t > 0$. Furthermore let T be weakly measurable with respect to ϕ.*

Then there exists a unique decomposition of E into two bands B_p, B_q with associated band projections p, q given by the following defining property:

$$\forall \psi \in E_+^\sharp : \psi\ \phi\text{-reachable} \Rightarrow \int_0^\infty \langle T_t x, \psi \rangle dt \in \{0, \infty\} \text{ for all } x \in B_{q+}$$

$$\exists\, \xi \in E_+^\sharp : \xi\ \phi\text{-reachable}, \quad q'\xi = 0, \quad (\xi - T_t'\xi) \text{ strictly positive on } B_p \text{ for } t > 0$$

$$\text{and for all } x \in B_p : \int_0^\infty \langle T_t x, \xi \rangle dt < \infty$$

In particular, one can choose ξ so that for all $x \in E_+$

$$\int_0^\infty \langle T_t x, \xi \rangle dt \leq \langle x, \phi \rangle \quad .$$

This decomposition coincides with the one given in Theorem 5.

Remarks 6:

i.) One should observe that in the proof of Theorem 4 only the conclusions of Theorem 3 are used, but not order-continuity of ϕ.

ii.) In fact the theorems on continuous semigroups can be shown directly from the results of Theorem 3 and 4 without reformulating the given demonstration. Working out this idea one observes that under our assumptions on the semigroup, Hopf's decomposition is already given by one single operator of the semigroup. Consequently they all coincide, all information up to this level is contained in one operator.

4. Recurrent semigroups of positive operators

The theorems of chapter 2 will be used in the sequel to justify the definitions of transience and recurrence (see [4] or any book on probability theory, i.g. [5]). From now on we shall make use of a calculus, which enables us to formulate discrete and continuous versions together, because there is no structural difference in theory. Let G be either $\mathbb{R}_+$ or $\mathbb{N}_+$ and μ the restriction of the associated canonical Haar-measure.

Definition 8: *Let E be a vector lattice and ϕ a strictly positive functional.*

i.) A semigroup $T = \{T_g\}_{g \in G}$, weakly measurable with respect to ϕ and having ϕ as subinvariant linear form, is called <u>recurrent</u>, if

$$\forall x \in E_+ \forall \psi \in E_+^\sharp : \psi \ \phi\text{-reachable} \Rightarrow \int_G \langle T_g x, \psi \rangle d\mu(g) \in \{0, \infty\}$$

ii.) A semigroup $T = \{T_g\}_{g \in G}$, weakly measurable with respect to ϕ and having ϕ as subinvariant linear form, is called <u>transient</u>, if

$$\exists \, \eta \in E_+^\sharp : \eta \text{ strictly positive and } \eta \ \phi\text{-reachable} :$$

$$\int_G \langle T_g x, \eta \rangle d\mu(g) < \infty \text{ for } x \in E_+$$

iii.) Let T be a semigroup and $I \subset E$ an ideal, I is called <u>T-invariant</u>, if for all $g \in G$

$$T_g(I) \subset I \quad .$$

iv.) A semigroup T is called <u>irreducible</u>, if for every $x > 0$ and for every $\psi > 0$ there exists a $g \in G$ so that $\langle T_g x, \psi \rangle > 0$. In the case of C_0-semigroups of positive contractions on a Banach lattice, this is equivalent to saying that there is no non-trivial closed T-invariant ideal.

Theorem 7: *Let E be an order-complete vector-lattice with a weak order-unit, ϕ a strictly positive, order-continuous linear functional and T a semigroup of positive operators, for which ϕ is subinvariant linear form and which is measurable with respect to ϕ. Then we obtain:*

$$T \text{ irreducible} \Rightarrow T \text{ transient or recurrent}$$

Proof: By Theorem 3 and Theorem 5, respectively, there is a band B_p and a subinvariant linear form η of T, which is strictly positive on B_p. For $x \in B_q$ we obtain:

$$\langle T_g x, \eta \rangle \leq \langle x, \eta \rangle = 0$$

Consequently one can conclude, that either $p = 0$ or $q = 0$. □

In general, under the above assumptions B_q is T-invariant. The following definition leads to the notion of dominance of semigroups which can be investigated together with recurrence.

Definition 9: *Let E be a vector lattice:*
We shall say that a semigroup S of positive operators is <u>*dominated*</u> *by a semigroup T if*

$$\forall g \in G,\ x \in E_+ : S_g x \leq T_g x$$

Theorem 8: *Let E be an order-complete vector lattice with a weak order-unit, ϕ a strictly positive linear functional and T an irreducible semigroup of positive operators, for which ϕ is subinvariant linear form and which is measurable with respect to ϕ. If T is recurrent, then for any semigroup $S \neq T$ of positive operators dominated by T we obtain:*

$$\lim_{g\to\infty} \langle S_g x, \phi\rangle = 0 \text{ for all } x \in E_+ \text{ and } T'_g\phi = \phi \text{ for all } g \in G.$$

Proof: By Rem. 6.i.) we arrive immediately at the following result:

$$\forall \psi \in E_+^\sharp : {T'}_g\psi \leq \psi \text{ for all } g \in G \text{ and } \psi\ \phi\text{-reachable} \Rightarrow {T'}_g\psi = \psi \text{ for all } g \in G$$

So the second part of the statement is already proved. Now let $S \neq T$ be a semigroup of positive operators dominated by T. The positive linear functional h is defined in the following way:

$$h(x) := \lim_{g\to\infty} \langle S_g x, \phi\rangle \text{ for } x \in E_+$$

The limit is monotone, so there are no existence-problems, furthermore $S'_g h = h$ for all $g \in G$. Consequently

$$\forall g \in G : h \leq T'_g h$$

Now we apply the integral criterion for recurrence. Let $0 < \tilde{g} \leq g$ be elements in G:

$$\begin{aligned} 0 \leq \int_0^g \langle T_s x, T'_{\tilde{g}}h - h\rangle d\mu(s) &= \int_{\tilde{g}}^{g+\tilde{g}} \langle T_s x, h\rangle d\mu(s) - \int_0^g \langle T_s x, h\rangle d\mu(s) = \\ &= \int_g^{g+\tilde{g}} \langle T_s x, h\rangle d\mu(s) - \int_0^{\tilde{g}} \langle T_s x, h\rangle d\mu(s) \leq \tilde{g}\phi(x) \end{aligned}$$

for $x \in E_+$. Recurrence and irreducibility allow us to conclude that ${T_g}'h = h$ for all $g \in G$. Again by irreducibility we obtain that either h is strictly positive or $h = 0$. If h was strictly positive, we would obtain

$$\forall g \in G\ \forall x \in E_+ : h(T_g x - S_g x) = 0 \quad ,$$

being a contradiction to $S \neq T$. Consequently $h = 0$, as claimed. □

Remarks 7:

i.) *In the proof of Theorem 8 we have also shown that a recurrent, irreducible semigroup possesses neither subinvariant nor superinvariant non-zero linear forms:*

$$\forall \psi \in E_+^\sharp : \psi \leq \phi \text{ and } T_g\psi \geq \psi \text{ for a } g \in G \Rightarrow T_g\psi = \psi$$

$$\forall \psi \in E_+^\sharp : \psi \leq \phi \text{ and } \psi \geq T_g\psi \text{ for a } g \in G \Rightarrow T_g\psi = \psi$$

Theorem 8 can be applied to find simple proofs of Frobenius-Perron-like theorems.

ii.) *One could name the above property "minimal non-stability" of semigroups. In [7] the authors proved that submarkovian, irreducible, recurrent semigroups (e.g. the Gaussian semigroup in dimension $n = 1, 2$) are minimally non-stable. The proof given in our paper generalizes and simplifies the setting.*

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Josef Teichmann, University of Vienna, Institute of mathematics, Strudlhofgasse 4, A-1090 Vienna, Austria, E-Mail: jteichma@radon.mat.univie.ac.at