Short-Maturity Asymptotics for Fast Mean-Reverting Stochastic Volatility Models

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Heston Model

In this talk, we will mainly deal with the Heston model.

Under the **risk-neutral measure** it is:

$$\begin{aligned} dS_t &= rS_t dt + \sigma_t S_t \, dW_t^1 \,, \\ \sigma_t &= \sqrt{Y_t} \,, \\ dY_t &= \kappa(\theta - Y_t)dt + \nu\sqrt{Y_t} \, dW_t^2 \,, \end{aligned}$$

where W^1, W^2 are two standard Brownian motions with covariation $d\langle W^1, W^2\rangle_t = \rho dt$, where $|\rho| < 1$.

We assume that $\boldsymbol{2\kappa\theta > \nu^2}$, $\nu, \kappa, \theta, Y_0 = y > 0$, so that the square-root (or CIR) process (Y_t) stays positive at all times.

Implied Volatility

Black-Scholes implied volatility:

$$\text{Call}_{Heston}(S, y; K, T) = \text{Call}_{BS}(S; K, T; \sigma_{imp}(S, y; K, T))$$

In what follows we use *log-moneyness*:

$$x = \log(K/S_0)$$

and the notation

$$\sigma_{imp}(x, y, T)$$

where T is *time-to-maturity*.

Implied Volatility at Short Maturity

Large deviation result.

Geodesic distance:

$$y d_x^2 + 2\rho\nu y d_x d_y + \nu^2 y d_y^2 = 1$$

$$d(x = 0, y) = 0$$

$$d(x, y) > 0 \quad \text{for} \quad x > 0$$

At short maturity:

$$\sigma_{imp}(x, y, T = 0) = \frac{x}{d(x, y)}$$

where d (which does not depend on κ) is computed numerically.

Avellaneda-BoyerOlson-Busca-Friz (2003)

Berestycki-Busca-Florent (2004)

Fast Mean-Reverting Heston Model

By fast mean-reverting stochastic volatility, we mean that the rate of mean reversion $\boldsymbol{\kappa}$ **is large**. In order to ensure that volatility is not “dying” or “exploding” we also impose that the *vol-vol* parameter ν is large of the order of $\sqrt{\kappa}$. In order to achieve this scaling, we introduce a small parameter $0 < \epsilon \ll 1$, and we replace (κ, ν) by $(\kappa/\varepsilon^2, \nu/\varepsilon)$ so that the model becomes:

$$
\begin{aligned}
dS_t &= rS_t dt + S_t \sqrt{Y_t} dW_t^1 \,, \\
dY_t &= \frac{\kappa}{\varepsilon^2}(\theta - Y_t)dt + \frac{\nu}{\varepsilon}\sqrt{Y_t} dW_t^2 \,.
\end{aligned}
$$

The small quantity ε^2 represents the **intrinsic time scale of the volatility** process (Y_t), or, in other words, its de-correlation time. Observe that the condition $2\left(\frac{\kappa}{\varepsilon^2}\right)\theta > \left(\frac{\nu}{\varepsilon}\right)^2$ is equivalent to $2\kappa\theta > \nu^2$ and therefore independent of ε.

Asymptotics at Fixed Maturities

Fouque-Papanicolaou-Sircar (2000) for general FMR SV models:

"implied vol is affine in LMMR"

$$\sigma_{imp}(x, y, T) = \bar{\sigma} + \varepsilon\rho\, C\left(\frac{x}{T}\right) + O(\varepsilon^2)\,,$$

where $\overline{\sigma}$ is the **effective volatility**

$$\overline{\sigma}^2 = \int \sigma(y)^2 \Phi_Y(dy)\,,$$

Φ_Y being the invariant distribution of the process Y.
In the case of **Heston**, $\sigma(y) = \sqrt{y}$, $\Phi_Y = \Gamma(\theta, \frac{\theta\nu^2}{2\kappa})$, and $\boldsymbol{\overline{\sigma}^2 = \theta}$.
C is a constant which depends on the model parameters.

Prices and Pricing PDE's

$$P^\varepsilon(t,x,y) = I\!E^\star\left\{e^{-r(T-t)}h(S^\varepsilon_T)|S^\varepsilon_t = x, Y^\varepsilon_t = y\right\}$$

$$\begin{aligned}\frac{\partial P^\varepsilon}{\partial t} + \frac{1}{2}yx^2\frac{\partial^2 P^\varepsilon}{\partial x^2} + \frac{\rho\nu}{\varepsilon}xy\frac{\partial^2 P^\varepsilon}{\partial x \partial y} + \frac{\nu^2}{2\varepsilon^2}y\frac{\partial^2 P^\varepsilon}{\partial y^2} & \\ +r\left(x\frac{\partial P^\varepsilon}{\partial x} - P^\varepsilon\right) + \frac{1}{\varepsilon^2}(m-y)\frac{\partial P^\varepsilon}{\partial y} &= 0\end{aligned}$$

to be solved for $t < T$ with the **terminal condition**

$$P^\varepsilon(T,x,y) = h(x)$$

Operator Notation

$$\left(\frac{1}{\varepsilon^2}\mathcal{L}_0 + \frac{1}{\varepsilon}\mathcal{L}_1 + \mathcal{L}_2\right) P^\varepsilon = 0$$

with

$$\begin{aligned}
\mathcal{L}_0 &= \frac{1}{2}\nu^2 y \frac{\partial^2}{\partial y^2} + (m - y)\frac{\partial}{\partial y} \qquad = \mathcal{L}_{CIR} \\
\mathcal{L}_1 &= \rho\nu x y \frac{\partial^2}{\partial x \partial y} \\
\mathcal{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2} y x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right) \qquad = \mathcal{L}_{BS}(\sqrt{y})
\end{aligned}$$

Formal Expansion

Expand:

$$P^\varepsilon = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \varepsilon^3 P_3 + \cdots$$

Compute:

$$\left(\frac{1}{\varepsilon^2}\mathcal{L}_0 + \frac{1}{\varepsilon}\mathcal{L}_1 + \mathcal{L}_2\right)\left(P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \varepsilon^3 P_3 + \cdots\right) = 0$$

Group the terms by powers of ε:

$$\begin{aligned}
\frac{1}{\varepsilon^2}\mathcal{L}_0 P_0 \quad &+ \quad \frac{1}{\varepsilon}\left(\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0\right) \\
&+ \quad \left(\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0\right) \\
&+ \quad \varepsilon\left(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1\right) \\
&+ \quad \cdots = 0
\end{aligned}$$

Diverging terms

- Order $1/\varepsilon^2$:

$$\mathcal{L}_0 P_0 = 0$$

$$\mathcal{L}_0 = \mathcal{L}_{CIR}, \text{ acting on } y \quad \Longrightarrow \quad \underline{P_0 = P_0(t,x)}$$

$$\text{with} \quad P_0(T,x) = h(x)$$

- Order $1/\varepsilon$:

$$\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0$$

$$\mathcal{L}_1 \text{ takes derivatives w.r.t. } y \quad \Longrightarrow \quad \mathcal{L}_1 P_0 = 0$$

$$\Longrightarrow \quad \mathcal{L}_0 P_1 = 0$$

$$\text{As for } P_0 : \qquad \underline{P_1 = P_1(t,x)} \qquad \text{with} \qquad P_1(T,x) = 0$$

- **Important observation:**

$$P_0 + \varepsilon P_1 \textbf{ does not depend on } y$$

Zero Order Term

$$\mathcal{L}_0 P_2 + (\mathcal{L}_1 P_1 = 0) + \mathcal{L}_2 P_0 = 0$$

Poisson equation in P_2 with respect to $\mathcal{L}_0$ and the variable y.
Solution:

$$P_2 = (-\mathcal{L}_0)^{-1}(\mathcal{L}_2 P_0)$$

<u>Only if</u> $\mathcal{L}_2 P_0$ is **centered**
with respect to the
invariant distribution of Y.

Poisson Equations

$$\mathcal{L}_0 \chi + g = 0$$

Expectations w.r.t. the invariant distribution of the CIR process:

$$\langle g \rangle = -\langle \mathcal{L}_0 \chi \rangle = -\int (\mathcal{L}_0 \chi(y)) \Phi(y) dy = \int \chi(y) (\mathcal{L}_0^\star \Phi(y)) dy = 0$$

$$\lim_{t \to +\infty} I\!E \left\{ g(Y_t) | Y_0 = y \right\} = \langle g \rangle = 0 \quad \text{(exponentially fast)}$$

$$\chi(y) = \int_0^{+\infty} I\!E \left\{ g(Y_t) | Y_0 = y \right\} dt$$

checked by applying $\mathcal{L}_0$

Leading Order Term

Centering:

$$\langle \mathcal{L}_2 P_0 \rangle = \langle \mathcal{L}_2 \rangle P_0 = 0$$

$$\begin{aligned}
\langle \mathcal{L}_2 \rangle &= \left\langle \frac{\partial}{\partial t} + \frac{1}{2} y x^2 \frac{\partial^2}{\partial x^2} + r \left(x \frac{\partial}{\partial x} - \cdot \right) \right\rangle \\
&= \frac{\partial}{\partial t} + \frac{1}{2} \langle y \rangle x^2 \frac{\partial^2}{\partial x^2} + r \left(x \frac{\partial}{\partial x} - \cdot \right)
\end{aligned}$$

Effective volatility: $\bar{\sigma}^2 = \langle y \rangle = \theta$

The zero order term $P_0(t,x)$ is the solution of the Black-Scholes equation

$$\mathcal{L}_{BS}(\bar{\sigma}) P_0 = 0$$

with the terminal condition $P_0(T,x) = h(x)$

Back to $P_2(t,x,y)$

The **centering condition** $\langle \mathcal{L}_2 P_0 \rangle = 0$ being satisfied:

$$\mathcal{L}_2 P_0 = \mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0 \rangle = \frac{1}{2}\left(y - \bar{\sigma}^2\right) x^2 \frac{\partial^2 P_0}{\partial x^2}$$

$$= \frac{1}{2} \mathcal{L}_0 \phi(y) x^2 \frac{\partial^2 P_0}{\partial x^2}$$

for ϕ a solution of the Poisson equation:

$$\mathcal{L}_0 \phi = y - \langle y \rangle$$

Then

$$P_2(t,x,y) = -\mathcal{L}_0^{-1}\left(\mathcal{L}_2 P_0\right) = -\frac{1}{2}\left(\phi(y) + c(t,x)\right) x^2 \frac{\partial^2 P_0}{\partial x^2}$$

Terms of order ε

Poisson equation in P_3:

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0$$

Centering condition:

$$\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle = 0$$

Equation for P_1:

$$\langle \mathcal{L}_2 P_1 \rangle = -\langle \mathcal{L}_1 P_2 \rangle = \frac{1}{2} \left\langle \mathcal{L}_1 \left[\left(\phi(y) + c(t,x)\right) x^2 \frac{\partial^2 P_0}{\partial x^2} \right] \right\rangle$$

P_1 independent of y and $\mathcal{L}_1$ takes derivatives w.r.t. y

$$\Longrightarrow \qquad \mathcal{L}_{BS}(\bar{\sigma}) P_1 = \frac{1}{2} \langle \mathcal{L}_1 \phi(y) \rangle \left[x^2 \frac{\partial^2 P_0}{\partial x^2} \right]$$

with $P_1(T,x) = 0$

The correction $P_1^\varepsilon(t,x) = \varepsilon P_1(t,x)$

$$
\begin{aligned}
\mathcal{L}_{BS}(\bar\sigma)P_1^\varepsilon - \frac{\varepsilon\nu}{2}\left\langle \left(\rho x y \frac{\partial^2}{\partial x \partial y}\right)\phi(y)\right\rangle \left[x^2 \frac{\partial^2 P_0}{\partial x^2}\right] &= 0 \\
\mathcal{L}_{BS}(\bar\sigma)P_1^\varepsilon + V_3^\varepsilon x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2}\right) &= 0
\end{aligned}
$$

BS equation with source and zero terminal condition

with the **small parameter** V_3^ε given by:

$$
V_3^\varepsilon = \frac{-\varepsilon\rho\nu}{2}\langle y\phi'(y)\rangle
$$

Explicit Formula for the Corrected Price

$$P_1^{\varepsilon} = (T-t)V_3^{\varepsilon} x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2}\right)$$

where V_3^{ε} is a small number of order ε.

The corrected price is given explicitly by

$$P_0 + (T-t)V_3^{\varepsilon} x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2}\right)$$

where P_0 is the Black-Scholes price with constant volatility $\bar{\sigma} = \sqrt{\theta}$

Comments

- The small constant V_3^ε is a complex functions of the original model parameters $(\theta, \nu, \rho, \varepsilon)$

- Only $(\bar{\sigma}, V_3^\varepsilon)$ are needed to compute the corrected price

- **Probabilistic representation** of $(P_0 + P_1^\varepsilon)(t,x)$:

$$\bar{\mathbb{E}} \left\{ e^{-r(T-t)} h(\bar{X}_T) + \int_t^T e^{-r(s-t)} H(s, \bar{X}_s) ds | \bar{X}_t = x \right\}$$

- **Put-Call Parity** is preserved at the order $\mathcal{O}(\varepsilon)$

- The V_3^ε term is the **skew effect**

$$\rho = 0 \Longrightarrow V_3^\varepsilon = 0$$

Corrected Call Option Prices

$$h(x) = (x - K)^+$$

and

$$P_0(t,x) = C_{BS}(t,x;K,T;\bar{\sigma})$$

Compute the **Delta**, the **Gamma** and the **Delta-Gamma**

Deduce the **source**

$$H = V_3^\varepsilon x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2}\right)$$

and the **correction**

$$P_1^\varepsilon(t,x) = (T-t)H(t,x) = \frac{x e^{-d_1^2/2}}{\overline{\sigma}\sqrt{2\pi}}\left(-V_3 \frac{d_1}{\overline{\sigma}}\right)$$

Expansion of Implied Volatilities

Recall

$$C_{BS}(t, x; K, T; \mathbf{I}) = C^{\text{observed}}$$

Expand

$$\mathbf{I} = \bar{\sigma} + \varepsilon I_1 + \cdots$$

Deduce for given (K, T):

$$C_{BS}(t, x; \bar{\sigma}) + \varepsilon I_1 \frac{\partial C_{BS}}{\partial \sigma}(t, x; \bar{\sigma}) + \cdots = P_0(t, x) + P_1^{\varepsilon}(t, x) + \cdots$$

$$\Longrightarrow \qquad \varepsilon I_1 = P_1^{\varepsilon}(t, x) \left[\mathbf{Vega}(\bar{\sigma})\right]^{-1}$$

Compute the $\mathbf{Vega} = \partial C_{BS} / \partial \sigma = x e^{-d_1^2/2} \sqrt{T - t} / \sqrt{2\pi}$

and deduce

Calibration Formulas

The **implied volatility** is an **affine function** of the **LMMR**:

log-moneyness-to-maturity-ratio $= \log(K/x)/(T-t)$

$$\mathbf{I} = a\,[\mathbf{LMMR}] + b + \mathcal{O}(\varepsilon^2)$$

with

$$\begin{aligned} a &= \frac{V_3^\varepsilon}{\bar\sigma^3} \\ b &= \overline{\sigma} - \frac{V_3^\varepsilon}{\overline{\sigma}^3}\left(r - \frac{1}{2}\overline{\sigma}^2\right) \end{aligned}$$

or for **calibration purpose**:

$$\begin{aligned} \overline{\sigma} &= b + a(r - \frac{b^2}{2}) \\ V_3^\varepsilon &= ab^3 \end{aligned}$$

Fast Mean-Reverting Short-Maturity Scaling

We combine fast mean reversion and short maturity by considering **maturities of order ε**, that is short maturities but long compared to the intrinsic volatility time scale ε^2. We set $\mathbf{T = \varepsilon t}$ where $t > 0$.

A typical situation: $T \sim 15$ days and $\varepsilon^2 \sim 2$ days.

$t \mapsto \epsilon t$ gives (in distribution) the **rescaled process**:

$$\begin{aligned} dS_{\epsilon,t} &= \epsilon r S_{\epsilon,t} dt + S_{\epsilon,t}\sqrt{\epsilon Y_{\epsilon,t}}\, dW_t^1 \,, \\ dY_{\epsilon,t} &= \frac{\kappa}{\epsilon}(\theta - Y_{\epsilon,t})dt + \frac{\nu}{\sqrt{\epsilon}}\sqrt{Y_{\epsilon,t}}\, dW_t^2, \end{aligned}$$

preserving the constant correlation ρ.

The **log-price** $X_{\epsilon,t} = \log S_{\epsilon,t}$ is given by

$$X_{\epsilon,t} = x_0 + \epsilon r t - \frac{\epsilon}{2}\int_0^t Y_{\epsilon,s}\, ds + \sqrt{\varepsilon}\int_0^t \sqrt{Y_{\epsilon,s}}\, dW_s^1 \,.$$

Asymptotic Results

- **Large deviation principle**
- **Option pricing**
- **Implied volatilities**

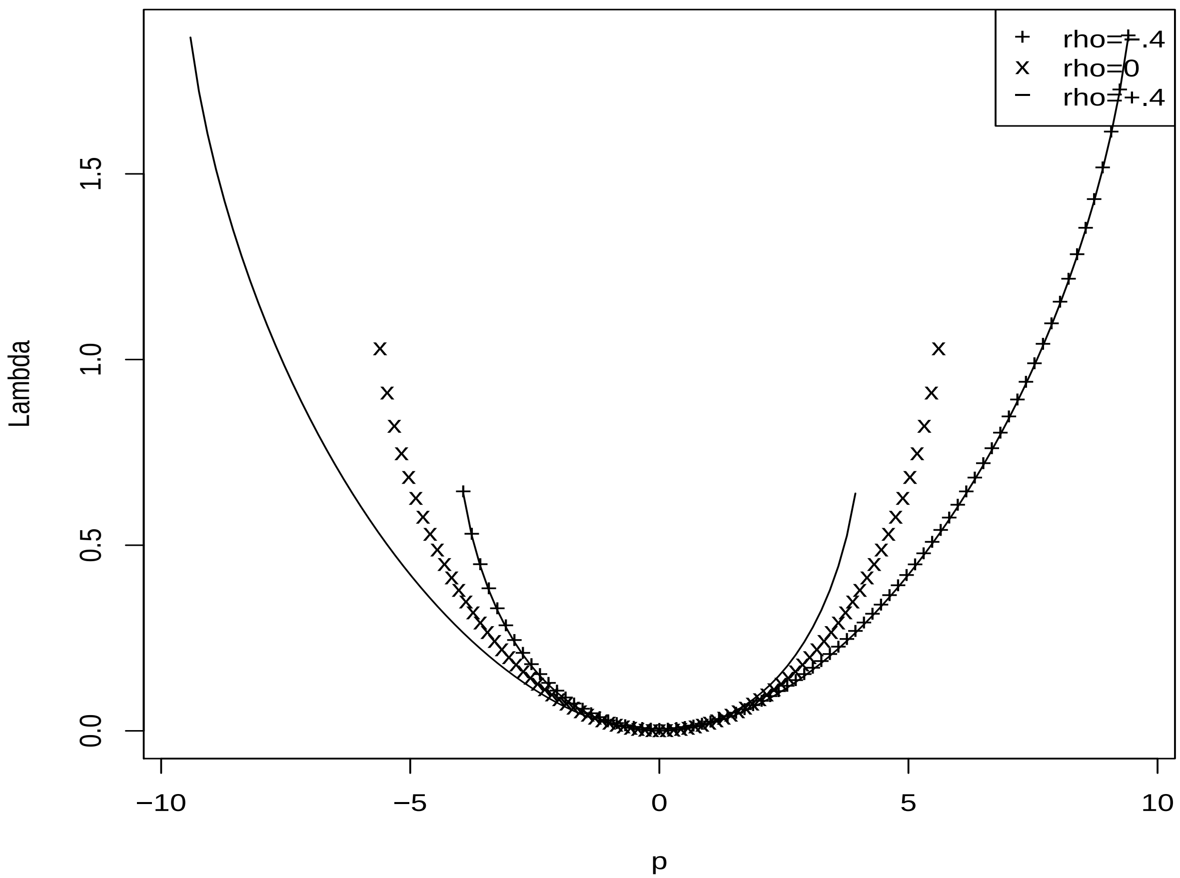
Proofs based on explicit computation of **moment generating functions**.

Large Deviation Principle

For each $t > 0$, $\{X_{\epsilon,t} : \epsilon > 0\}$ satisfies the large deviation principle with **rate function** $I(q; x_0, t) = \Lambda^*(q - x_0; 0, t)$,
where $\Lambda^*(q; x, t) \equiv \sup_{p \in R}\{qp - \Lambda(p; x, t)\}$ is the Legendre transform of
$\Lambda(p; x, t) : R \times R \times R_+ \mapsto R \cup \{+\infty\}$ given explicitly by:

$$\begin{aligned}
\Lambda(p; x, t) &= xp + \frac{\kappa\theta t}{\nu^2}\left((\kappa - \nu\rho p) - \sqrt{(\kappa - \rho\nu p)^2 - \nu^2 p^2}\right), \\
&\qquad \text{for } -\frac{\kappa}{\nu(1-\rho)} \leq p \leq \frac{\kappa}{\nu(1+\rho)}, \\
&= +\infty \qquad \text{otherwise.}
\end{aligned}$$

Note that $\Lambda^*(q; x, t) = \Lambda^*(q - x; 0, t)$ and $\Lambda(p; x, t) = t\,\Lambda(p; \frac{x}{t}, 1)$.
Λ^* will be given explicitly.



The parameters are $t = 1$, $\kappa = 1.15$, $\theta = .04$, $\nu = .2$ *and* $\rho = -.4, 0, +.4$

Pricing Short-Maturity Out-of-The-Money Options

Suppose $\log(\frac{K}{S_0}) > 0$, and $t > 0$ fixed. Then

$$\lim_{\epsilon \longrightarrow 0} \epsilon \log E[e^{-r\varepsilon t}(S_{\epsilon,t} - K)^+ | S_{\epsilon,0} = S_0, Y_{\epsilon,0} = y_0] = \\ -\Lambda^* \left(\log(\frac{K}{S_0}); 0, t \right),$$

independently of the initial square-volatility level y_0.

Note that the maturity of the option is $T = \varepsilon t$ which goes to zero in the limit.

The discounting factor $e^{-r\varepsilon t}$ plays no role in this asymptotic result.

Implied Volatilities

The asymptotic implied volatilities can be computed.

Let $\sigma^{\varepsilon}_{imp}(x, y, t)$ denote the Black-Scholes implied volatility for the **European call option** with strike price K, **out-of-the-money** so that $x = \log(K/S_0) > 0$, with short maturity $\boldsymbol{T = \varepsilon t}$ for $t > 0$ fixed, and computed under the fast mean-reverting dynamics. Then

$$\lim_{\epsilon \to 0} \sigma^{\epsilon}_{imp}(t, x)^2 = \frac{x^2}{2\Lambda^*(x; 0, t)t} .$$

Similarly, by considering Out-of-The-Money put options, one obtains the same formula for $x < 0$.

The At-The-Money volatility is obtained by taking the limit $x \to 0$ and coincides with the **effective volatility** $\overline{\sigma} = \sqrt{\theta}$.

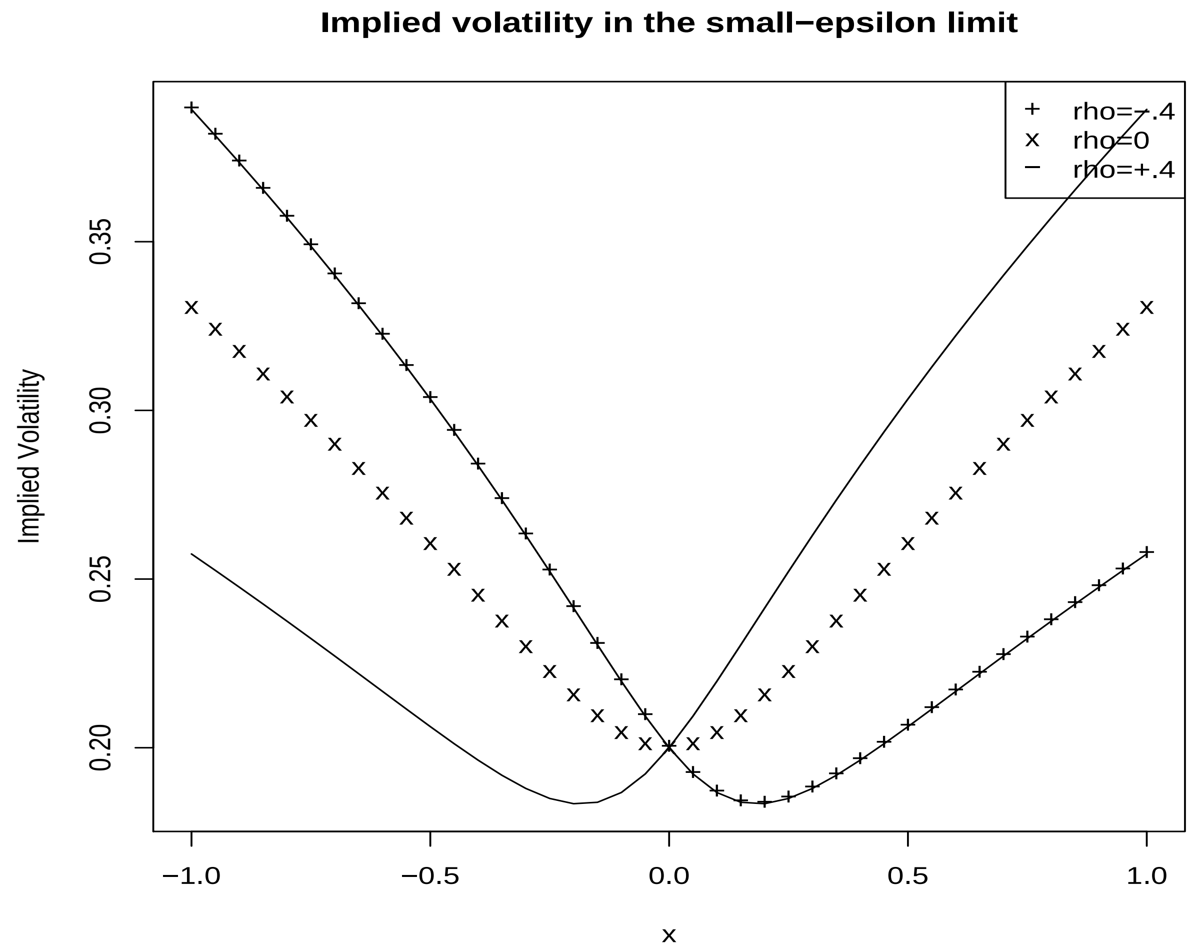
Explicit Formula for Λ^*

$$\Lambda^*(q;0,t) = qp(q;t) - \Lambda(p(q;t);0,t),$$

where $p(q;t)$ is given by

$$\begin{aligned} p(q;t) \quad &= \quad \frac{\kappa}{\nu(1-\rho^2)}\left(-\rho + \frac{q\nu + \kappa\theta t\rho}{\sqrt{(q\nu + \kappa\theta t\rho)^2 + (1-\rho^2)\kappa^2\theta^2 t^2}}\right) \\ &\in \mathrm{int}(\mathrm{Dom}(\Lambda)) = \left(-\frac{\kappa}{\nu(1-\rho)}, \frac{\kappa}{\nu(1+\rho)}\right). \end{aligned}$$

$\Lambda^*(q;0,t)$ is finite for all $q \in R$.
It is strictly increasing for $q > 0$ and strictly decreasing for $q < 0$.
$\Lambda^*(0;0,t) = 0$.
$\Lambda^*(q;0,t)$ is continuous in $(q,t) \in R \times R_+$.



The parameters are $t = 1$, $\kappa = 1.15$, $\theta = .04$, $\nu = .2$ *and* $\rho = -.4, 0, +.4$

Moment Generating Function

$$\begin{aligned}\Lambda_\epsilon(p) &= \Lambda_\epsilon(p;x,y,t) = \epsilon \log E[e^{\frac{p}{\epsilon}X_{\epsilon,t}}|X_{\epsilon,0}=x, Y_{\epsilon,0}=y] \\ &= \epsilon \log E[S_{\epsilon,t}^{\frac{p}{\epsilon}}|S_{\epsilon,0}=e^x, Y_{\epsilon,0}=y] \\ &= \epsilon r p t + \epsilon \log E[\tilde{S}_{\epsilon,t}^{\frac{p}{\epsilon}}|\tilde{S}_{\epsilon,0}=e^x, Y_{\epsilon,0}=y],\end{aligned}$$

where $\tilde{S}_{\varepsilon,t}$ is the **discounted stock price**.

$$\begin{aligned}&E[\tilde{S}_{\epsilon,t}^{\frac{p}{\epsilon}}|\tilde{S}_{\epsilon,0}=e^x, Y_{\epsilon,0}=y] \\ &= e^{\frac{xp}{\varepsilon}} E[e^{-\frac{p}{2}\int_0^t Y_{\epsilon,s}ds + \frac{p}{\sqrt{\varepsilon}}\int_0^t \sqrt{Y_{\epsilon,s}}dW_s^1}|Y_{\epsilon,0}=y] \\ &= e^{\frac{xp}{\varepsilon}} E[e^{-\frac{p}{2}\int_0^t Y_{\epsilon,s}ds + \frac{p\rho}{\sqrt{\varepsilon}}\int_0^t \sqrt{Y_{\epsilon,s}}dW_s^2 + \frac{p\sqrt{1-\rho^2}}{\sqrt{\varepsilon}}\int_0^t \sqrt{Y_{\epsilon,s}}dW_s^3}|Y_{\epsilon,0}=y] \\ &= e^{\frac{xp}{\varepsilon}} E[e^{-\frac{p}{2}\int_0^t Y_{\epsilon,s}ds + \frac{p\rho}{\sqrt{\varepsilon}}\int_0^t \sqrt{Y_{\epsilon,s}}dW_s^2 + \frac{p^2(1-\rho^2)}{2\varepsilon}\int_0^t Y_{\epsilon,s}ds}|Y_{\epsilon,0}=y] \\ &= e^{\frac{xp}{\varepsilon}} E[e^{\frac{p\rho}{\sqrt{\varepsilon}}\int_0^t \sqrt{Y_{\epsilon,s}}dW_s^2 - \frac{p^2\rho^2}{2\varepsilon}\int_0^t Y_{\epsilon,s}ds} e^{\frac{p(p-\varepsilon)}{2\varepsilon}\int_0^t Y_{\epsilon,s}ds}|Y_{\epsilon,0}=y],\end{aligned}$$

Moment Generating Function (continued)

Using Girsanov tranform, one obtains that

$$E[\tilde{S}_{\epsilon,t}^{\frac{p}{\epsilon}} | \tilde{S}_{\epsilon,0} = e^x, Y_{\epsilon,0} = y] \quad = \quad e^{\frac{xp}{\varepsilon}} E^Q[e^{\frac{p(p-\varepsilon)}{2\varepsilon} \int_0^t Z_{\epsilon,s} ds} | Z_{\epsilon,0} = y],$$

where, under the probability Q, the process $Z_{\epsilon,t}$ satisfies

$$dZ_{\epsilon,t} \quad = \quad \frac{1}{\epsilon} \left(\kappa\theta - (\kappa - \nu\rho p) Z_{\epsilon,t} \right) dt + \frac{\nu}{\sqrt{\epsilon}} \sqrt{Z_{\epsilon,t}} dW_t^Q,$$

driven by a Brownian motion W^Q.
This result is derived in Andersen and Piterbarg (2007). Note that the proof allows the possibility of "$+\infty = +\infty$".
Their statement is limited to the case of $p(p-\epsilon) > 0$, but the proof is not limited to that case, allowing $p \in R$.

Explicit evaluation of Λ_ϵ

The following **two inequalities** play important roles:

$$(\rho\nu p-\kappa)^2 \geq p(p-\epsilon)\nu^2\,, \qquad \rho\nu p<\kappa.$$

When they are both satisfied, then by results on **exponential functionals of CIR processes** (Albanese and Lawi (2005), or Hurd and Kuznetsov (2008)), we have

$$E^Q[e^{\frac{p(p-\varepsilon)}{2\varepsilon}\int_0^t Z_{\epsilon,s}ds}|Z_{\epsilon,0}=y]=e^{m_\varepsilon(t)-n_\varepsilon(t)y}, \qquad \text{with}$$

$$\begin{aligned}
m_\epsilon(t) &= \frac{\kappa\theta t}{\sigma^2}(b-\bar{b})+\frac{2\kappa\theta}{\sigma^2}\log\left(\frac{\bar{b}e^{\bar{b}t/2}}{\bar{b}\cosh(\frac{\bar{b}t}{2})+b\sinh(\frac{\bar{b}t}{2})}\right),\\
n_\epsilon(t) &= \frac{-p(p-\epsilon)}{\varepsilon}\left(\frac{\sinh(\frac{\bar{b}t}{2})}{\bar{b}\cosh(\frac{\bar{b}t}{2})+b\sinh(\frac{\bar{b}t}{2})}\right),\\
\bar{b} &= \frac{1}{\epsilon}\sqrt{(\kappa-\nu\rho p)^2-\nu^2 p(p-\epsilon)}\,, \qquad b=\frac{\kappa-\nu\rho p}{\epsilon}\,.
\end{aligned}$$

Explicit evaluation of Λ_ϵ (continued)

Note that when the limit exists as $\varepsilon \to 0$, the only contribution from $\varepsilon\,(m_\epsilon(t) - n_\epsilon(t)y)$ comes from the first term of $m_\varepsilon(t)$ which leads to formula for $\Lambda(p;x,t)$.

If one of the two inequalities $(\rho\nu p - \kappa)^2 \geq p(p-\epsilon)\nu^2\,, \quad \rho\nu p < \kappa$ is violated then $\Lambda_\varepsilon = +\infty$. A careful analysis shows that:

$\Lambda_\epsilon(p)$ is lower semicontinuous and convex in p.

For $\epsilon > 0$ small enough,

$$\Lambda_\epsilon(p;x,y,t) = \epsilon rpt + xp + \epsilon\,(m_\epsilon(t) - n_\epsilon(t)y)$$

holds when $c_{1,\epsilon} \leq p \leq c_{2,\epsilon}$ with

$$\begin{aligned} c_{1,\epsilon} &= \frac{(\epsilon\nu - 2\kappa\rho) - \sqrt{(\epsilon\nu - 2\kappa\rho)^2 + 4\kappa^2(1-\rho^2)}}{2\nu(1-\rho^2)} \leq 0, \\ c_{2,\epsilon} &= \frac{(\epsilon\nu - 2\kappa\rho) + \sqrt{(\epsilon\nu - 2\kappa\rho)^2 + 4\kappa^2(1-\rho^2)}}{2\nu(1-\rho^2)} \geq 0. \end{aligned}$$

Convergence of Λ_ϵ to Λ, and LDP

The function Λ is lower semicontinuous and essentially smooth in p. Moreover, $\Lambda_\epsilon(\cdot; x, y, t)$ **Γ-converges** to $\Lambda(\cdot; x, t)$: for each $x \in R$, $y > 0$, $t > 0$

1. For every $p \in R$, there exists $\{p_\epsilon\}$ with $p_\epsilon \longrightarrow p$ such that
$$\lim_{\epsilon \longrightarrow 0^+} \Lambda_\epsilon(p_\epsilon; x, y, t) = \Lambda(p; x, t).$$
2. For every $p \in R$ and every $p_\epsilon \longrightarrow p$,
$$\liminf_{\epsilon \longrightarrow 0^+} \Lambda_\epsilon(p_\epsilon; x, y, t) \geq \Lambda(p; x, t).$$

A close inspection of the proof for the usual form of the **Gärtner-Ellis theorem** shows that the theorem generalizes under Γ-convergence. Note that without essential smoothness of the function Λ in the example above, one cannot conclude that the large deviation lower bound holds. **Our LDP result follows**.

OTM Option Pricing (lower bound)

For $\delta > 0$ we have

$$\begin{aligned} E[(S_{\epsilon,t} - K)^+] &\geq E[\mathbf{1}_{\{S_{\epsilon,t} - K > \delta\}}(S_{\epsilon,t} - K)^+] \\ &\geq \delta P(S_{\epsilon,t} > K + \delta). \end{aligned}$$

By LDP, it follows that

$$\begin{aligned} \liminf_{\epsilon \to 0^+} \epsilon \log E[(S_{\epsilon,t} - K)^+] &\geq \liminf_{\epsilon \to 0^+} \epsilon \log P(X_{\epsilon,t} > \log(K + \delta)) \\ &\geq -\inf_{q > \log(K+\delta)} \Lambda^*(q - \log S_0; 0, t) \\ &= -\Lambda^* \left(\log \left(\frac{K + \delta}{S_0} \right); 0, t \right). \end{aligned}$$

The last equality follows from the fact that $\log(\frac{K}{S_0}) > 0$, and that $\Lambda^*(q; 0, t)$ is non-decreasing for q in the region $q \geq 0$.

Taking $\delta \to 0^+$, by continuity of Λ^*, we obtain the desired lower bound.

OTM Option Pricing (upper bound)

For $p, q > 1$ such that $p^{-1} + q^{-1} = 1$, we have:

$$E[(S_{\epsilon,t} - K)^+] \le E^{1/p}[|(S_{\epsilon,t} - K)^+|^p] E^{1/q}[\mathbf{1}_{\{S_{\epsilon,t} - K \ge 0\}}] \,.$$

Therefore

$$\begin{aligned} \epsilon \log E[(S_{\epsilon,t} - K)^+] &\le \frac{\epsilon}{p} \log E[(S_{\epsilon,t})^p] + \epsilon(1 - \frac{1}{p}) \log P(S_{\epsilon,t} \ge K) \\ &\le \frac{1}{p} \Lambda_\epsilon(\epsilon p) + (1 - \frac{1}{p}) \epsilon \log P(S_{\epsilon,t} \ge K) \,. \end{aligned}$$

Taking $\lim_{p \to +\infty} \limsup_{\epsilon \to 0}$ on both sides, and noting that $\lim_{\epsilon \to 0} \Lambda_\epsilon(\epsilon p) = 0$, we deduce (by LDP) the desired upper bound

$$\limsup_{\epsilon \longrightarrow 0} \epsilon \log E[(S_{\epsilon,t} - K)^+] \le -\Lambda^* \left(\log(\frac{K}{S_0}); 0, t \right) .$$

Asymptotic Implied Volatility

We denote the log-moneyness by $x = \log(K/S_0) > 0$, and for simplicity $\sigma^{\epsilon}_{imp}(t,x) = \sigma_{\varepsilon}$, t and x being fixed here.

First, we show that

$$\lim_{\epsilon \to 0} \sigma_{\epsilon}\sqrt{\epsilon t} = 0.$$

We know that $\Lambda^*(x;0,t) > 0$. Let $0 < \delta < \Lambda^*(x;0,t)$. By the LDP upper bound, for $\epsilon > 0$ small enough

$$\begin{aligned} & e^{-(\Lambda^*(x;0,t)-\delta)/\epsilon} \geq E[(S_{\epsilon,t} - K)^+] \\ & = e^{r\varepsilon t} S_0 \Phi\left(\frac{-x + r\varepsilon t + \frac{1}{2}\sigma_{\epsilon}^2 \epsilon t}{\sigma_{\epsilon}\sqrt{\epsilon t}}\right) - K\Phi\left(\frac{-x + r\varepsilon t - \frac{1}{2}\sigma_{\epsilon}^2 \epsilon t}{\sigma_{\epsilon}\sqrt{\epsilon t}}\right), \end{aligned}$$

Since $E[(S_{\epsilon,t} - K)^+] \geq 0$, the right-hand side must converge to zero as $\varepsilon \to 0$, which implies $\lim_{\epsilon \to 0} \sigma_{\epsilon}\sqrt{\epsilon t} = 0$

Asymptotic Implied Volatility (lower bound)

We use the classical notation

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + r\varepsilon t + \frac{\sigma_\epsilon^2}{2}\epsilon t}{\sigma_\varepsilon \sqrt{\epsilon t}}.$$

Let $\delta > 0$, by the lower bound LDP, for $\epsilon > 0$ small enough, we have

$$\begin{aligned} e^{-(\Lambda^*(x;0,t)+\delta)/\epsilon} &\leq E[(S_{\epsilon,t} - K)^+] \\ &\leq e^{r\varepsilon t} S_0 \Phi\left(d_1\right) = e^{r\varepsilon t} S_0 \left(1 - \Phi\left(-d_1\right)\right) \\ &\leq e^{r\varepsilon t} S_0 \left(\frac{1}{-d_1}\right) \Phi'\left(-d_1\right), \end{aligned}$$

from classical estimate on Φ. Using $\lim_{\epsilon\to 0} \sigma_\epsilon \sqrt{\epsilon t} = 0$ and $S_0 < K$, we know that $\lim_{\varepsilon\to 0} d_1 = -\infty$. Taking $(\epsilon \log)$ on both sides, one sees that the leading order term on the right-hand side is given by

$$-\varepsilon \frac{\left(\log(\frac{S_0}{K})\right)^2}{2(\sigma_\varepsilon\sqrt{\varepsilon t})^2} = -\frac{x^2}{2\sigma_\varepsilon^2 t} \Longrightarrow -(\Lambda^*(x,;0,t)+\delta) \leq -\frac{x^2}{2\lim_{\epsilon_n\to 0}\sigma_{\epsilon_n}^2 t}.$$

Asymptotic Implied Volatility (upper bound)

Denoteby $P_{BS} = P_{BS}(\sigma_\epsilon)$ the measure under which S_ϵ follows the **Black-Scholes model with constant volatility** $\sigma_\varepsilon = \sigma_\varepsilon(t,x)$:

$$dS_{\varepsilon,s} = S_{\varepsilon,s}\left(rds + \sigma_\varepsilon dW_s\right) \, ,$$

where W is a Brownian motion under P_{BS} (note that here t is fixed and the maturity of the call option is εt). Using the notation

$$d_2 = \frac{\log\left(\frac{S_0}{K+\delta}\right) + r\varepsilon t - \frac{\sigma_\epsilon^2}{2}\epsilon t}{\sigma_\varepsilon\sqrt{\epsilon t}}, \quad \text{one obtains:}$$

$$\begin{aligned} e^{-(\Lambda^*(x;0,t)-\delta)/\epsilon} &\geq E^P[(S_{\epsilon,t}-K)^+] = E^{P_{BS}}[(S_{\epsilon,t}-K)^+] \\ &\geq \delta P_{BS}(S_{\epsilon,t} > K+\delta) \\ &= \delta\left(1-\Phi\left(-d_2\right)\right) \\ &\geq \delta\left(\frac{-d_2}{1+d_2^2}\right)\phi(-d_2), \end{aligned}$$

Asymptotic Implied Volatility (upper bound continued)

Arguing as in the case of the lower bound, we know that $\lim_{\varepsilon\to 0} d_2 = -\infty$. Taking $(\varepsilon \log)$ on both sides, the leading order term on the right-hand side is given by

$$-\frac{\left(\log(\frac{S_0}{K+\delta})\right)^2}{2\sigma_\varepsilon^2 t} \Longrightarrow -(\Lambda^*(x;0,t)-\delta) \geq -\frac{\left(\log(\frac{K+\delta}{S_0})\right)^2}{2\lim_{\epsilon_n\to 0^+}\sigma_{\epsilon_n}^2 t}\,.$$

Sending $\delta \to 0^+$ gives the desired upper bound, which concludes the proof of

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon^2 = \sigma(t,x)^2 = \frac{x^2}{2\Lambda^*(x;0,t)t}\,,$$

in this regime "fast mean reverting volatility and short maturity", and for an OTM call option ($x > 0$).

The same formula for $x < 0$ is derived similarly by considering OTM put options.

The ATM Limit

Using the explicit formula for $\Lambda^*(x;0,t)$, one can derive the At-The-Money limit:

$$\lim_{x\to 0} \sigma(t,x)^2 = \theta\,,$$

by checking that near zero

$$\begin{aligned} p(q;t) &= \frac{q}{\theta t} + O(q^2), \\ \Lambda(p;0,t) &= \frac{\theta t}{2} p^2 + O(p^3), \end{aligned}$$

and consequently

$$\Lambda^*(q;0,t) = q\left(\frac{q}{\theta t}\right) - \frac{\theta t}{2}\left(\frac{q}{\theta t}\right)^2 + O(q^3) = \frac{q^2}{2\theta t} + O(q^3).$$

Asymptotic ATM Implied Volatility

In fact, we can also derive the limit as $\varepsilon \to 0$ of the At-The-Money implied volatility $\sigma_\varepsilon(t,0)$.

The asymptotic At-The-Money volatility is given by

$$\lim_{\varepsilon \longrightarrow 0} \sigma_\varepsilon(t,0)^2 = \lim_{x \longrightarrow 0} \sigma(t,x)^2 = \theta\,.$$

This is not a large deviation result but rather an **averaging result** of the type studied by FPS. Since it involves convergence in distribution, it is more convenient to work with put options whose payoffs are continuous and bounded. The ATM volatility is defined by the unique positive number $\sigma_\varepsilon(0,t) = \sigma_\varepsilon(0)$ satisfying

$$E[(S_0 - S_{\varepsilon,t})^+] = S_0\Phi(-d_2) - e^{r\varepsilon t} S_0 \Phi(-d_1)\,, \quad d_{1,2} = \frac{(r \pm \frac{1}{2}\sigma_\varepsilon(0)^2)\sqrt{\varepsilon t}}{\sigma_\varepsilon(0)}\,.$$

Asymptotic ATM Implied Volatility (continued)

Dividing on both sides by $\sqrt{\varepsilon}\, S_0$, on gets Eq *

$$E\left[\left(-\sqrt{\varepsilon}\int_0^t r\,\frac{S_{\varepsilon,s}}{S_0}ds - \int_0^t \frac{S_{\varepsilon,s}}{S_0}\sqrt{Y_{\varepsilon,s}}dW_s^1\right)^+\right] = \frac{1}{\sqrt{\varepsilon}}\left(\Phi(-d_2) - e^{r\varepsilon t}\Phi(-d_1)\right)$$

The following integrals convergence to zero in probability

$$\sqrt{\varepsilon}\int_0^t r\,\frac{S_{\varepsilon,s}}{S_0}ds \quad \text{and} \quad \int_0^t \left(\frac{S_{\varepsilon,s}}{S_0} - 1\right)\sqrt{Y_{\varepsilon,s}}dW_s^1\,.$$

The convergence of the quadratic variation of the martingale term, $\int_0^t Y_{\varepsilon,s}\,ds \to \bar{\sigma}^2 t$, implies the convergence in distribution

$$\left(-\sqrt{\varepsilon}\int_0^t r\,\frac{S_{\varepsilon,s}}{S_0}ds - \int_0^t \frac{S_{\varepsilon,s}}{S_0}\sqrt{Y_{\varepsilon,s}}dW_s^1\right) \to \int_0^t \bar{\sigma}\,dW_s^1 = \bar{\sigma}W_t^1\,,$$

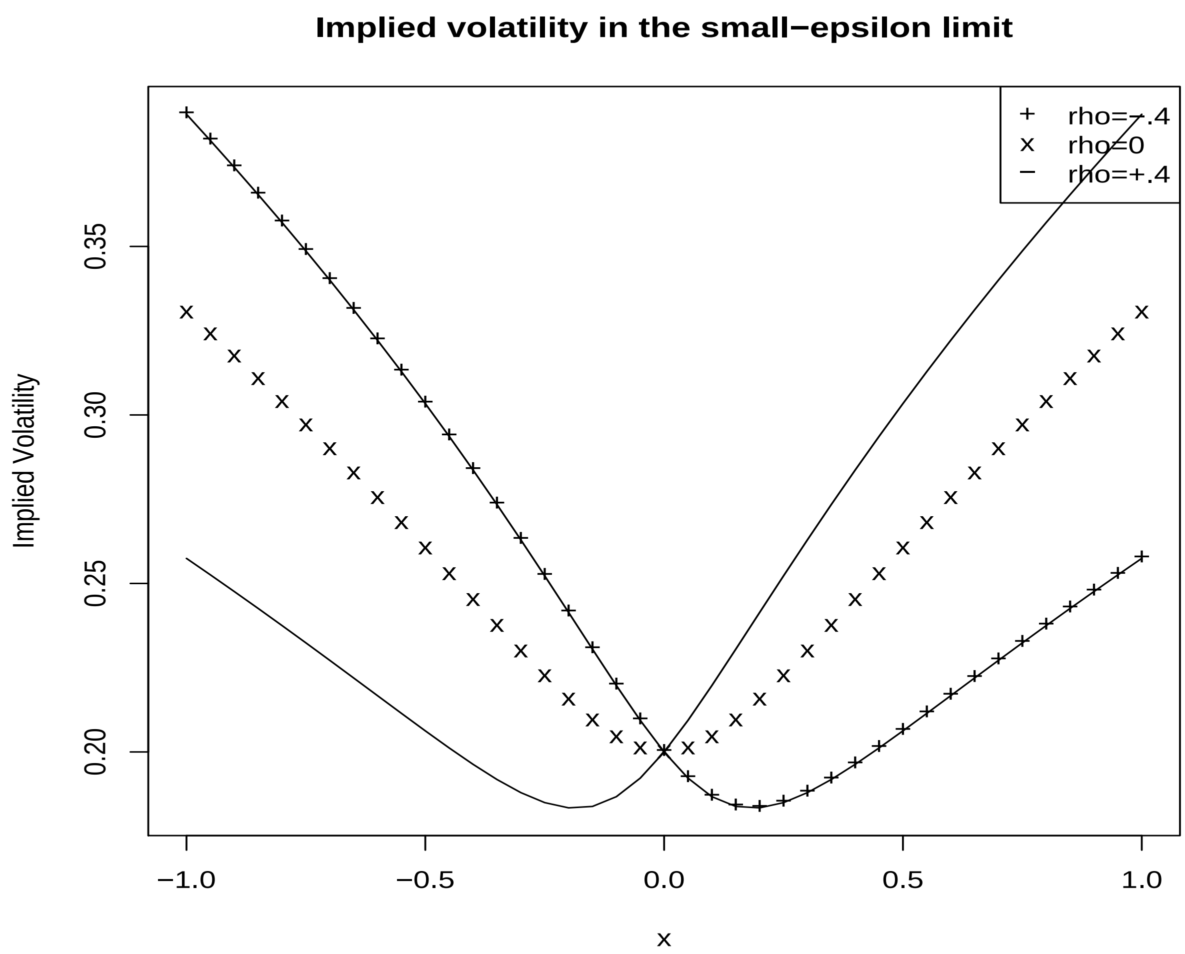
Asymptotic ATM Implied Volatility (continued)

$\bar{\sigma}^2 = \int_0^{+\infty} y\Gamma(dy)$ is the ergodic average of the square volatility $Y_{\varepsilon,\cdot}$ where Γ is the invariant distribution of the ergodic process Y. A complete proof of this result involves introducing a solution ψ of the Poisson equation $\mathcal{L}\psi(y) = y - \bar{\sigma}^2$, where $\mathcal{L}$ is the infinitesimal generator of the process Y, and using Ito's formula to show that

$$\int_0^t \left(Y_{\varepsilon,s} - \bar{\sigma}^2\right) ds = \int_0^t \mathcal{L}\psi(Y_{\varepsilon,s})ds = \varepsilon\left(\psi(Y_{\varepsilon,t}) - \psi(Y_0)\right) - \sqrt{\varepsilon}\int_0^t \sigma\psi'(Y_{\varepsilon,s})Y_{\varepsilon,s}dW_s^2$$

converges to zero (FPS 2000 for details).

In this case, the invariant distribution is a *Gamma* with mean θ and consequently $\bar{\sigma}^2 = \theta$. Therefore, the left-hand side of Eq * converges to $E[(\bar{\sigma}W_t^1)^+] = \bar{\sigma}\sqrt{t}/\sqrt{2\pi} = \sqrt{\theta t}/\sqrt{2\pi}$. By direct inspection of the right-hand side of Eq * and the relation between $d_{1,2}$ and $\sigma_\varepsilon(0)$, one deduces that $\sigma_\varepsilon(0)$ must converge to θ as $\varepsilon \to 0$.



The parameters are $t = 1$, $\kappa = 1.15$, $\theta = .04$, $\nu = .2$ *and* $\rho = -.4, 0, +.4$

Work in Progress

- The same technique applies to **long maturities**.
- Different approach using **homogenization of HJB equations** (Feng-Kurtz, 2006) to handle general stochastic volatility models.
- Compute the **next term** in the ε-small limit.
- ......

THANKS FOR YOUR ATTENTION