Asymptotic Formulas in Analytically Tractable Stochastic Volatility Models

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Samuelson’s model of the stock price

In their celebrated work on pricing of options, Black and Scholes used Samuelson’s model of the stock price. Samuelson suggested to describe the random behavior of the stock price by a diffusion process $X_t$ satisfying the following stochastic differential equation:

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x_0,$$

where $\mu$ is a real constant, $\sigma$ is a positive constant, and $W$ is a standard Brownian motion. The constants $\mu \in \mathbb{R}^1$ and $\sigma > 0$ are called the drift and the volatility of the stock, respectively.

**Explicit formula**

The following formula holds for the stock price process in Samuelson’s model:

$$X_t = x_0 \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\}.$$

The process $X$ is called a geometric Brownian motion.
The distribution function of the stock price $X_t$ is given by

$$r \mapsto \mathbb{P}(X_t < r), \quad 0 \leq r < \infty,$$

where $\mathbb{P}$ stands for the Wiener measure.

The distribution density of $X_t$ (if it exists) is a function $D_t$ on $[0, \infty)$ such that

$$\mathbb{P}(X_t < r) = \int_0^r D_t(u)du$$

for all $r \geq 0$.

The distribution density of the stock price process $X_t$ in Samuelson’s model can be computed explicitly. We have

$$D_t(x) = \frac{1}{\sqrt{2\pi t} \sigma x} \exp\left\{-\frac{\left(\log \frac{x}{x_0} - (\mu - \frac{1}{2} \sigma^2) t\right)^2}{2t\sigma^2}\right\}.$$ 

Such densities are called log-normal.
Black-Scholes model

The Black-Scholes formula for the price of a European call option at $t = 0$ under the risk-free measure $\mathbb{P}^*$ is the following:

$$C_{BS} = x_0 N (d_1) - Ke^{-rT} N (d_2),$$

where

$$d_1 = \frac{\log x_0 - \log K + (r + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}},$$

$$d_2 = d_1 - \sigma \sqrt{T},$$

and

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp \left\{ -\frac{y^2}{2} \right\} dy.$$  

Here $T$ stands for the expiration date, $K$ is the strike price, and $r$ denotes the interest rate.
**Constant volatility assumption**

It is assumed in the Black-Scholes model that the volatility of the stock is constant. However, it has been established that the implied volatility \( K \mapsto I(K) \) for a European call option in uncorrelated stochastic volatility models is decreasing on the interval \([0, x_0 e^{rT}]\) and increasing on the interval \([x_0 e^{rT}, \infty)\) (Renault-Touzi). This is the so-called “volatility smile” effect. It contradicts the constant volatility assumption.

**Stochastic volatility models**

To improve the Black-Scholes model, various stochastic volatility models have been developed during the last decades, e.g., the Hull-White model where the volatility is a geometric Brownian motion, the Stein-Stein model where the absolute value of an Ornstein-Uhlenbeck process is used as the volatility process, and the Heston model where the volatility is a Cox-Ingersoll-Ross process (a Feller process).

**General stochastic volatility models**

\[
\begin{align*}
\left\{ 
  dX_t &= \mu X_t dt + f (Y_t) X_t dW_t \\
  dY_t &= b (t, Y_t) dt + \sigma (t, Y_t) dZ_t
\end{align*}
\]

Here, \( \mu \in \mathbb{R}^1; \) \( b \) and \( \sigma \) are continuous functions on \([0, T] \times \mathbb{R}^1; \) \( W \) and \( Z \) are independent one-dimensional standard Brownian motions; and \( f \) is a nonnegative function on \( \mathbb{R}^1. \) The process \( X \) plays the role of the stock price process, while \( f (Y_t), t \geq 0, \) is the volatility process. The initial conditions for the processes \( X \) and \( Y \) will be denoted by \( x_0 \) and \( y_0, \) respectively. We also assume that the second equation in the model above has a unique strong solution \( Y. \)
**Explicit formula**

Under certain restrictions, the following formula holds for the stock price process $X_t$:

$$X_t = x_0 \exp \left\{ \mu t - \frac{1}{2} \int_0^t f(Y_s)^2 ds + \int_0^t f(Y_s) dW_s \right\}.$$

**Distribution densities**

We will denote by $D_t$ the distribution density of the stock price $X_t$, and by $m_t$ the mixing distribution density associated with the volatility process $f(Y_t)$, that is, the distribution density of the random variable

$$\alpha_t = \left\{ \frac{1}{t} \int_0^t f(Y_s)^2 ds \right\}^{\frac{1}{2}}.$$

**Special models. The Hull-White model**

The stock price process $X$ and the volatility process $Y$ in the Hull-White model satisfy the following system of stochastic differential equations:

$$\begin{cases} dX_t = \mu X_t dt + Y_t X_t dW_t \\ dY_t = \nu Y_t dt + \xi Y_t dZ_t. \end{cases}$$
Special models. The Heston model

The stock price process $X$ and the volatility process $Y$ in the Heston model satisfy the following system of stochastic differential equations:

\[
\begin{align*}
    dX_t &= \mu X_t \, dt + \sqrt{Y_t} X_t \, dW_t \\
    dY_t &= (a + b Y_t) \, dt + c \sqrt{Y_t} \, dZ_t.
\end{align*}
\]

We consider the case where $\mu \in \mathbb{R}$, $a \geq 0$, $b \leq 0$, and $c > 0$. It is often assumed that the volatility equation in the model above is written in a mean-reverting form. Then the Heston model becomes

\[
\begin{align*}
    dX_t &= \mu X_t \, dt + \sqrt{Y_t} X_t \, dW_t \\
    dY_t &= r \left( m - Y_t \right) \, dt + c \sqrt{Y_t} \, dZ_t,
\end{align*}
\]

where $r \geq 0$, $m \geq 0$, and $c > 0$. The volatility process in the Heston model is called a Cox-Ingersoll-Ross process (a CIR process).
Special models. The Stein-Stein model

The stock price process $X$ and the volatility process $Y$ in the Stein-Stein model satisfy the following system of stochastic differential equations:

\[
\begin{aligned}
    dX_t &= \mu X_t \, dt + |Y_t| \, X_t \, dW_t \\
    dY_t &= q (m - Y_t) \, dt + \sigma \, dZ_t.
\end{aligned}
\]

An important special case is the following: $\mu \in \mathbb{R}$, $q \geq 0$, $m = 0$, and $\sigma > 0$. Then the Stein-Stein model becomes

\[
\begin{aligned}
    dX_t &= \mu X_t \, dt + |Y_t| \, X_t \, dW_t \\
    dY_t &= -q Y_t \, dt + \sigma \, dZ_t.
\end{aligned}
\]

The solution to the second stochastic differential equation in the model above is an Ornstein-Uhlenbeck process, for which the long run mean equals zero. Such Ornstein-Uhlenbeck processes and CIR-processes and can be dealt with in a similar way, using Bessel processes.
Explicit formula

The following formula holds for the stock price process in the general stochastic volatility model described above:

\[ X_t = x_0 \exp \left\{ \mu t - \frac{1}{2} \int_0^t f(Y_s)^2 \, ds + \int_0^t f(Y_s) \, dW_s \right\}. \]

Integral representation

\[ D_t(x) = \frac{1}{x_0 e^{\mu t}} \int_0^\infty L \left( t, y, \frac{x}{x_0 e^{\mu t}} \right) m_t(y) \, dy, \]

where \( L \) is the log-normal density defined by

\[ L(t, y, v) = \frac{1}{\sqrt{2\pi tyv}} \exp \left\{ -\frac{\left( \log v + \frac{ty^2}{2} \right)^2}{2ty^2} \right\}. \]

It follows that

\[ D_t(x) = \frac{\sqrt{x_0 e^{\mu t}}}{\sqrt{2\pi t}} x^{-\frac{3}{2}} \int_0^\infty y^{-1} m_t(y) \exp \left\{ -\left[ \frac{1}{2ty^2} \log^2 \frac{x}{x_0 e^{\mu t}} + \frac{ty^2}{8} \right] \right\} \, dy. \]
Symmetry property

\[ D_t \left( x_0 e^{\mu t} x \right) = x^{-3} D_t \left( \frac{x_0 e^{\mu t}}{x} \right). \]

Asymptotic behavior of the mixing distribution

The Hull-White model. A special case:

\[
m_t \left( y; \frac{1}{2}, 1, 1 \right) = c_1 y c_2 \left( \log y \right)^{c_3} \exp \left\{ -\frac{1}{2t} \left( \log y + \frac{1}{2} \log \log y \right)^2 \right\} \left( 1 + O \left( \left( \log y \right)^{-\frac{1}{2}} \right) \right), \quad y \to \infty.
\]

Formulas for the constants:

\[
c_1 = \frac{1}{\sqrt{\pi t}} 2^{-\frac{1}{4}} \exp \left\{ -\frac{(\log 2)^2}{2t} \right\}, \quad c_2 = -1 - \frac{1 - 2 \log 2}{2t}, \quad c_3 = -\frac{1 + 2 \log 2}{4t}.
\]
The law of the time integral. A special case:

\( \tilde{m}_T^{(0)} \) denotes the distribution density of the following time integral of a geometric Brownian motion:

\[
A_T^{(\rho)} = \int_0^T \exp \{2 (\rho u + Z_u)\} \, du.
\]

The next formula characterizes the asymptotic behavior of this density:

\[
\tilde{m}_T^{(0)}(y) = c_1 2^{-1-c_3} T^{-\frac{c_2+1}{2}} y^{\frac{c_2-1}{2}} \left( \log \frac{y}{T} \right)^{c_3} \exp \left\{ -\frac{1}{2T} \left( \log \sqrt{\frac{y}{T}} + \frac{1}{2} \log \log \sqrt{\frac{y}{T}} \right)^2 \right\} \left( 1 + O \left( \left( \log y \right)^{-\frac{1}{2}} \right) \right), \quad y \to \infty,
\]

where \( c_1, c_2, \) and \( c_3 \) are as above with \( t \) replaced by \( T \).
The Hull-White model. General case:

\[
m_t (y; \nu, \xi, y_0) = c_1 y^{c_2} (\log y)^{c_3} \exp \left\{- \frac{1}{2t\xi^2} \left( \log \frac{y}{y_0} + \frac{1}{2} \log \log \frac{y}{y_0} \right)^2 \right\} \\
\left(1 + O \left( (\log y)^{-\frac{1}{2}} \right) \right), \quad y \to \infty.
\]

Formulas for the constants:

\[
c_1 = \frac{1}{\xi \sqrt{\pi t}} \frac{2 \log 2 - 1}{2t\xi^2} \frac{1}{y_0} \exp \left\{- \frac{(\log 2)^2}{2t\xi^2} \right\} \exp \left\{- \frac{\alpha^2 \xi^2 t}{2} \right\},
\]

\[
c_2 = \alpha - 1 + \frac{1 - 2 \log 2}{2t\xi^2},
\]

\[
c_3 = \frac{\alpha}{2} - \frac{1 + 2 \log 2}{4t\xi^2},
\]

where

\[
\alpha = \frac{2\nu - \xi^2}{2\xi^2}.
\]
The law of the time integral. General case:

The following formula holds for the distribution density of the time integral of a geometric Brownian motion:

\[
\tilde{m}_T^{(\rho)}(y) = C_1 2^{-\frac{1}{2}C_3 T} y^{\frac{C_2 - 1}{2}} \left( \log \left( \frac{y}{T} \right) \right)^{C_3} \\
\exp \left\{ -\frac{1}{2T} \left( \log \sqrt{\frac{y}{T}} + \frac{1}{2} \log \log \sqrt{\frac{y}{T}} \right)^2 \right\} \\
\left( 1 + O \left( (\log y)^{-\frac{1}{2}} \right) \right), \quad y \to \infty.
\]

Constants:

\[
C_1 = \frac{1}{\sqrt{\pi T}} 2^{-\frac{1}{2}} \exp \left\{ -\frac{(\log 2)^2}{2T} \right\} \exp \left\{ -\frac{\rho^2 T}{2} \right\},
\]

\[
C_2 = \rho - 1 + \frac{1 - 2 \log 2}{2T},
\]

\[
C_3 = \frac{\rho}{2} - \frac{1 + 2 \log 2}{4T}.
\]
Asymptotic behavior of the stock price density

Hull-White model. Special case:

\[ D_t \left( x; 0, \frac{1}{2}, 1, 1, 1 \right) = c_1 2^{\frac{c_2 - 2c_3 - 1}{2} t} x^{-2} \]

\[ (\log x)^{\frac{c_2 - 1}{2}} \left( \log \left[ \frac{2 \log x}{t} \right] \right)^{c_3} \]

\[ \exp \left\{ -\frac{1}{2t} \left( \log \sqrt{\frac{2 \log x}{t}} + \frac{1}{2} \log \log \sqrt{\frac{2 \log x}{t}} \right)^2 \right\} \]

\[ \left( 1 + O \left( (\log \log x)^{-\frac{1}{2}} \right) \right), \quad x \to \infty, \]

where the constants \( c_1, c_2, \) and \( c_3 \) are as above.
Hull-White model. General case:

Let $-\infty < \mu < \infty$, $-\infty < \nu < \infty$, $\xi > 0$, $x_0 > 0$, $y_0 > 0$, and $t > 0$. Then

$$D_t \left( x_0 e^{\mu t} x; \mu, \nu, \xi, x_0, y_0 \right) = \frac{c_1}{x_0 e^{\mu t}} 2^{c_2 - 1/2} t^{-c_2 + 1/2} x^{-2}$$

$$(\log x)^{c_3 - 1/2} \left( \log \left[ \frac{2 \log x}{t} \right] \right)^{c_3}$$

$$\exp \left\{ -\frac{1}{2t\xi^2} \left( \log \left[ \frac{1}{y_0} \sqrt{2 \log x} \right] + \frac{1}{2} \log \log \left[ \frac{1}{y_0} \sqrt{2 \log x} \right] \right)^2 \right\}$$

$$\left( 1 + O \left( (\log \log x)^{-1/2} \right) \right)$$

as $x \to \infty$. 
Asymptotic behavior of the mixing distribution

Heston model:

For $a \geq 0$, $b \leq 0$, and $c > 0$, there exist $A > 0$, $B > 0$, and $C > 0$ such that

$$m_t(y; a, b, c, y_0) = Ay^{-\frac{1}{2} + \frac{2a}{c^2}}e^{By}e^{-Cy^2} \left(1 + O\left(y^{-\frac{1}{2}}\right)\right), \ y \to \infty.$$ 

Formulas for the constants in the case $b = 0$:

$$A = \frac{2^{1 + \frac{a}{c^2}}y_0^{\frac{1}{4} - \frac{a}{c^2}}}{c\sqrt{t}} \exp\left\{\frac{4y_0}{c^2t}\right\},$$

$$B = \frac{2\sqrt{2y_0\pi}}{c^2t}, \text{ and } C = \frac{\pi^2}{2c^2t}.$$ 

Asymptotic behavior of the stock price distribution

Heston model.

For $a \geq 0$, $b \leq 0$, and $c > 0$, there exist $A_1 > 0$, $A_2 > 0$, and $A_3 > 0$ such that for every $\delta$ with $0 < \delta < \frac{1}{2}$,

$$D_t\left(x_0e^{\mu t}x\right) = A_1(\log x)^{-\frac{3}{4} + \frac{a}{c^2}}e^{A_2\sqrt{\log x}x}x^{-A_3}$$

$$\left(1 + O\left((\log x)^{-\delta}\right)\right), \ x \to \infty.$$ 

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Asymptotic behavior of the stock price distribution

Stein-Stein model.

For \( q \geq 0, m = 0, \) and \( \sigma > 0, \) there exist \( B_1 > 0, B_2 > 0, \) and \( B_3 > 0 \) such that for every \( \delta \) with \( 0 < \delta < \frac{1}{2}, \)

\[
D_t \left( x_0 e^{\mu t} x \right) = B_1 (\log x)^{-\frac{1}{2}} e^{B_2 \sqrt{\log x} x^{-B_3}}
\]

\[
(1 + O \left( (\log x)^{-\delta} \right)) , \ x \to \infty.
\]

Conclusions. Fat tails.

**Hull-White:** The distribution density \( D_t \) of the stock price in the Hull-White model decays extremely slowly. This is the so-called “fat tail” effect. The function \( x \mapsto D_t(x) \) behaves near infinity like \( \frac{1}{x^2} \) times certain logarithmic factors. In addition, the function \( x \mapsto D_t(x) \) behaves near zero like \( \frac{1}{x} \) times logarithmic factors making the density integrable near zero. In a sense, the stock price distribution density in the Hull-White model has the slowest decay among stock price distribution densities in similar stochastic volatility models.

**Heston:** The density \( D_t(x) \) behaves at infinity roughly like the function \( x^{-A_3} \) and at zero as the function \( x^{A_3-3} \). The number \( A_3 \) is strictly greater than 2.

**Stein-Stein:** The density \( D_t(x) \) behaves at infinity roughly like the function \( x^{-B_3} \) and at zero like the function \( x^{B_3-3} \). The number \( B_3 \) is strictly greater than 2.
Pricing functions for European call options

Stochastic volatility model:
\[
\begin{align*}
\frac{dX_t}{t} &= rX_t dt + f(Y_t) X_t dW^*_t \\
\frac{dY_t}{t} &= b(t, Y_t) dt + \sigma(t, Y_t) dZ^*_t
\end{align*}
\]
under a risk-free measure \( \mathbb{P}^* \).

The price of a European call option at \( t = 0 \) is given by the following formula:
\[
C(K) = \mathbb{E}^* \left[ e^{-rT} (X_T - K)_+ \right].
\]

It is clear that
\[
C(K) = e^{-rT} \int_K^\infty x D_T(x) dx - e^{-rT} K \int_K^\infty D_T(x) dx.
\]

**Implied volatility**

The implied volatility in an option pricing model is the volatility in the Black-Scholes model for which the corresponding Black-Scholes price of the option is equal to its price in the model under consideration. More precisely, for \( K > 0 \) the implied volatility \( I(K) \) is determined from the equality
\[
C_{BS}(K, I(K)) = C(K).
\]
The implied volatility can be considered as a function of the log-strike \( k = \log \frac{K}{x_0 e^{rT}} \). In terms of \( k \), the definition is as follows:
\[
\hat{I}(k) = I(K), \quad -\infty < k < \infty, \quad 0 < K < \infty.
\]
Asymptotic behavior of the implied volatility

Suppose there exist positive increasing continuous functions $\psi$ and $\phi$ such that

$$\lim_{K \to \infty} \psi(K) = \lim_{K \to \infty} \phi(K) = \infty$$

and

$$C(K) \approx \frac{\psi(K)}{\phi(K)} \exp \left\{-\frac{\phi(K)^2}{2}\right\}.$$ 

Then the following asymptotic formula holds:

$$I(K) = \frac{1}{\sqrt{T}} \left( \sqrt{2 \log K + \phi(K)^2} - \phi(K) \right) + O\left(\frac{\psi(K)}{\phi(K)}\right)$$

as $K \to \infty$. 
Hull-White model

Let $\psi$ be a positive increasing continuous function such that 
\[ \lim_{K \to \infty} \psi(K) = \infty. \]
Then

\[
\hat{I}(k) = \frac{\sqrt{2}}{\sqrt{T}} k^{\frac{1}{2}} - \frac{1}{2T \xi} \log k - \frac{1}{2T \xi} \log \log k \\
+ (\alpha + 2A + 2C) \xi - \frac{1}{2T \xi} \frac{\log \log k}{\log k} + O\left(\frac{\psi(k)}{\log k}\right)
\]

as $k \to \infty$, where

\[
\alpha = \frac{2\nu - \xi^2}{2\xi^2}, \quad A = \frac{1 - 2 \log 2}{4T \xi^2},
\]

and

\[
C = \frac{2 \log y_0 + \log T}{2T \xi^2}.
\]
**Heston model**

Let $\psi$ be a positive increasing continuous function such that
$$\lim_{K \to \infty} \psi(K) = \infty.$$ Then
$$\hat{I}(k) = \beta_1 k^{\frac{1}{2}} + \beta_2 + \beta_3 \frac{\log k}{k^{\frac{1}{2}}} + O \left( \frac{\psi(k)}{k^{\frac{1}{2}}} \right)$$
as $k \to \infty$, where

$$\beta_1 = \frac{\sqrt{2}}{\sqrt{T} \left( \sqrt{A_3} - 1 - \sqrt{A_3 - 2} \right)},$$

$$\beta_2 = \frac{A_2}{\sqrt{2T}} \left( \frac{1}{\sqrt{A_3 - 2}} - \frac{1}{\sqrt{A_3 - 1}} \right),$$

$$\beta_3 = \frac{1}{\sqrt{2T}} \left( \frac{1}{4} - \frac{a}{c^2} \right) \left( \frac{1}{\sqrt{A_3} - 1} - \frac{1}{\sqrt{A_3 - 2}} \right),$$

and $A_3$ is the constant that appears in the asymptotic formula for the distribution density of the stock price in the Heston model.
Stein-Stein model

Let $\psi$ be a positive increasing continuous function such that
$$\lim_{K \to \infty} \psi(K) = \infty.$$ Then
$$\hat{I}(k) = \gamma_1 k^{\frac{1}{2}} + \gamma_2 + O\left(\frac{\psi(k)}{k^{\frac{1}{2}}}\right)$$
as $k \to \infty$, where

$$\gamma_1 = \frac{\sqrt{2}}{\sqrt{T}} \left(\sqrt{B_3 - 1} - \sqrt{B_3 - 2}\right),$$

$$\gamma_2 = \frac{B_2}{\sqrt{2T}} \left(\frac{1}{\sqrt{B_3 - 2}} - \frac{1}{\sqrt{B_3 - 1}}\right),$$

and $B_3$ is the constant that appears in the asymptotic formula for the distribution density of the stock price in the Stein-Stein model.

**Remark**

The previous theorems contain asymptotic formulas with error estimates for the implied volatility in analytically tractable stochastic volatility models. These formulas are sharper than the asymptotic formulas which can be derived from general results due to Lee, Friz, and Benaim.