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A Few Geometrical Problems in Finance

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Some Problems and Tools

1. Problem: Calibration of Stochastic Volatility Models.

   Tool: Heat kernel expansion on a Riemannian manifold.

2. Problem: Large-strike behavior of the implied volatility.

   Tool: Schrödinger Semigroups Estimates.

3. Problem: Efficient discretization scheme for Monte-Carlo pricing.

   Tool: Combinatorial Hopf Algebra, Heat kernel on the Heisenberg group...

These tools arise in the Atiyah-Singer theorem...

Tool: HK expansion

Problem: Calibration of Stochastic Volatility Models.

Motivations

- The dimensionless parameter in Finance is $[\mathrm{vol}]^2\tau$ which is small $\Rightarrow$ Asymptotic expansion.
- Try to find a systematic way of doing asymptotic expansion in Finance:

  1. Asymptotic smile for the SABR model at the first-order [Hagan-al]
  2. Asymptotic smile for basket at the zero-order [Avellaneda-al]
  3. Asymptotic swaption at the zero-order for LMM [Rebonato-Hull-White freezing argument]

  $\Rightarrow$ Heat kernel expansion.

Tool: HK expansion

Heat Kernel (1)

SDEs: $dx^i = b^i(x)dt + \sigma^i(x)dW_i \ , \ dW_i dW_j = \rho_{ij}dt$
Backward Kolmogorov equation [Einstein Convention here]

$$\partial_\tau p(\tau, x|y) \ = \ Dp(\tau, x|y)$$

with $D = b^i\partial_i + \frac{1}{2}\rho_{ij}\sigma^i\sigma^j\partial_{ij}$
Under a change of coordinates $x^i \to x^{i'}$,

$$b^{i'} = \underbrace{\frac{\partial x^{i'}}{\partial x^i} b^i}_{\text{covariant}} \ + \ \underbrace{\frac{1}{2}\rho_{ij}\sigma_i\sigma_j \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}}_{\text{non cov.}}$$

Rewrite D as $D = g^{-\frac{1}{2}}(\partial_i + \mathcal{A}_i)g^{\frac{1}{2}}g^{ij}(\partial_j + \mathcal{A}_j) + Q$
By identifying the terms ∂_i and ∂_{ij}, we obtain

- Metric: $g_{ij} = 2\frac{\rho^{ij}}{\sigma_i\sigma_j}$, $g = \det[g_{ij}]$, $g^{ij} = [g_{ij}^{-1}]$
- Connection: $\mathcal{A}^i = \frac{1}{2}(b^i - g^{-\frac{1}{2}}\partial_j(g^{1/2}g^{ij}))$, $\mathcal{A}_i = g_{ij}\mathcal{A}^j$
- Section Q: $Q = g^{ij}(\mathcal{A}_i\mathcal{A}_j - b_j\mathcal{A}_i - \partial_j\mathcal{A}_i)$, $b_j \equiv g_{ji}b^i$

Heat Kernel (2)

To summarize a heat kernel equation on a Riemannian manifold M is constructed from the following three pieces of geometric data:

1. a metric g on M, which determines the second-order piece.
2. a connection $\mathcal{A}$ on a line bundle $\mathcal{L}$, which determines the first-order piece.
3. a section of the bundle $End(\mathcal{L}) \simeq \mathcal{L} \otimes \mathcal{L}^*$, which determines the zeroth-order piece.

p continuous section of the bundle $(\mathcal{L} \boxtimes \mathcal{L}^*)$ over $\mathbb{R}^+ \times M \times M$ [a]

[a] Let $\mathcal{E}_1$ and $\mathcal{E}_2$ two vector bundle on M. and let pr_1 be the projections from $M \times M$ onto the first and second factor M respectively. We denote the external product $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ the vector bundle $pr_1^* \mathcal{E}_1 \otimes pr_2^* \mathcal{E}_2$ over $M \times M$

Reduction Method (1)

The heat kernel equation can now be simplified by applying the actions of the following groups:

▷ The group of diffeomorphisms $\mathrm{Diff}(\mathcal{M})$ which acts on the metric g_{ij} and the connection $\mathcal{A}_i$ by

$$g_{ij} \xrightarrow{f \in \mathrm{Diff}(\mathcal{M})} (f^* g)_{ij} = g_{pk} \partial_i f^p(x) \partial_j f^k(x)$$
$$\mathcal{A}_i \xrightarrow{f \in \mathrm{Diff}(\mathcal{M})} (f^* \mathcal{A})_i = \mathcal{A}_p \partial_i f^p(x)$$

▷ The group of gauge transformations $\mathcal{G}$ which acts on the conditional probability (and the call option $\mathcal{C}$) by

$$p(\tau, x|y) \xrightarrow{\mathcal{G}} p'(\tau, x|y) = e^{\chi(\tau,x) - \chi(0,y)} p(\tau, x|y)$$

Then p' satisfies the same equation as p with

$$\mathcal{A}'_i \equiv \mathcal{A}_i - \partial_i \chi$$
$$Q' \equiv Q + \partial_\tau \chi$$

Reduction Method (2)

If the connection $\mathcal{A}$ is an exact form

$$\mathcal{A}_i = \partial_i \Lambda$$

then by applying a gauge transformation

$$\mathcal{A}'_i = \mathcal{A}_i - \partial_i \Lambda = \partial_i \Lambda - \partial_i \Lambda = 0$$

The HK equation reduces to

$$\partial_\tau p'(\tau, x|y) = (\triangle + Q')p'(\tau, x|y)$$

The statement "$\mathcal{A}$ is exact" is equivalent to $\mathcal{F} = 0$, where $\mathcal{F}$ is given in a specific coordinate system by

$$\mathcal{F}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i$$

Obvious($\Rightarrow$): $\partial_i \partial_j \Lambda - \partial_j \partial_i \Lambda = 0$

Heat Kernel Expansion

Let M be a Riemannian manifold without a boundary. Then for each $x \in M$, there is a complete asymptotic expansion for small τ

$$p(\tau, x|y) = \frac{\sqrt{g(x)}}{(4\pi\tau)^{\frac{n}{2}}} \sqrt{D(x,y)} \mathcal{P}(x,y) e^{-\frac{d(x,y)^2}{4\tau}} \sum_{k=0}^{\infty} a_k(x,y) \tau^k$$

▶ $d(x,y)$ is the geodesic distance: $d(x,y) = \min_C \int_0^1 \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} dt$. On a flat manifold $\mathbb{R}^n$, the geodesic curves are the straight lines and the geodesic distance is the Euclidean distance.

▶ Van Vleck-Morette determinant $D(x,y)$:

$$D(x,y) = g(x)^{-\frac{1}{2}} \det \left(- \frac{\partial^2 \frac{d(x,y)^2}{2}}{\partial x \partial y} \right) g(y)^{-\frac{1}{2}} \text{ with } g(x) = \det[g_{ij}(x,x)]$$

▶ Parallel gauge transport: $\mathcal{P}(x,y) = e^{-\int_{\mathcal{C}(x,y)} \mathcal{A}_i dx^i}$ with $\mathcal{C}(x,y)$ a geodesic from the point x to y.

Heat Kernel Coefficients

The $a_i(x, y)$ are smooth functions on M and depend on geometric invariants such as the scalar curvature R.

$$\begin{aligned} a_0(x, y) &= 1 \\ a_1(x, x) &= \frac{1}{6}R + Q \end{aligned}$$

Problem: Calibration of SVMs

SVM and Riemann Surface

A (1-factor) stochastic volatility model (SVM) depends on two SDEs, one for the asset f and one for the volatility a. In a risk-neutral measure, we have

$$
\begin{aligned}
df &= C(f)a\,dW_1 \\
da &= b(a)dt + \sigma(a)dW_2 \\
dW_1 dW_2 &= \rho dt
\end{aligned}
$$

This model corresponds in our geometrical framework to a Riemann surface Σ, endowed with a two-by-two metric.

Riemann Uniformization Theorem

Isothermal Coordinates:

$$ds^2 = F(y)(dx^2 + dy^2)$$

Name	Conformal factor	Scalar curvature	Surface
Geometric	$F(y) \sim y^{-2}$	$R = -1$	$\mathbb{H}^2$
3/2-model	$F(y) \sim e^{\frac{-2y}{\sqrt{1-\rho^2}}}$	$R = 0$	$\mathbb{R}^2$
SABR	$F(y) \sim y^{-2}$	$R = -1$	$\mathbb{H}^2$
Heston	$F(y) \sim y^{-1}$	$R = -2a^{-2} < 0$	Baby Black Hole

▷ $R = \frac{\sigma(a)^2}{a}\left(\frac{\sigma'(a)}{\sigma(a)} - \frac{2}{a}\right)$.

▷ For $a\frac{\sigma'(a)}{\sigma} \leq 2$: Σ is a Cartan-Hadamard manifold. $\longrightarrow$ The cut-locus is empty!

Metric with one Killing vector

Integrable geodesics:

$$d(x,y) \ = \ |\int_{y_1}^{y_2} \frac{F(y')dy'}{\sqrt{F(y')-C^2}}|$$

with the constant $C = C(x_1, y_1, x_2, y_2)$ determined by the equation

$$x_2 - x_1 = \int_{y_1}^{y_2} \frac{C}{\sqrt{F(y')-C^2}} dy'$$

Asymptotic Local Volatility

LV: $df_t = \sigma(t,f)dW_t$ and $df_t = a_t C(f_t)dW_t$ have the same marginals if:
We have that the local vol^2 is the mean value of the stochastic vol^2 conditional to the forward:

$$\begin{aligned}\sigma^2(t,f) &= C^2(f)\mathbb{E}[a_t^2|f_t = f] \\ &\equiv C^2(f)\frac{\int_0^\infty a^2 p(t,f,a|\alpha,f_0)\sqrt{g}da}{\int_0^\infty p(t,f,a|\alpha,f_0)\sqrt{g}da}\end{aligned}$$

- Saddle point:

$$\sigma(t,f) = C(f)a_{\min}\ ,\ a_{\min} \equiv a\,|\min_{(a,f\ fixed)} d(z,z_0)$$

- Asymptotic local volatility ($\phi(f,a) \equiv d^2(f,a)$):

$$\sigma(T,f) = \sqrt{2g^{ff}(a_{\min})}\,(1+ \frac{T}{\phi''(a_{\min})}\left(\frac{g^{ff'}(a_{\min})}{g^{ff}(a_{\min})}\left(\ln(Dg\mathcal{P}^2)'(a_{\min}) - \frac{\phi'''(a_{\min})}{\phi''(a_{\min})}\right) + \frac{g^{ff''}(a_{\min})}{g^{ff}(a_{\min})}\right))$$

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Asymptotic Implied Volatility for any SVM

Use asymptotic map between local and implied volatilities [Time-dependent HKE]:

$$\sigma_{BS}(\tau,K)=\frac{\ln(\frac{K}{f_0})}{\int_{f_0}^{K}\frac{df'}{\sigma(f')}}\left(1+\frac{\tau}{3}\left(\frac{1}{8}\left(\frac{\ln(\frac{K}{f_0})}{\int_{f_0}^{K}\frac{df'}{\sigma(f')}}\right)^2+Q(f_{\mathrm{av}})+\frac{3\mathcal{G}(f_{\mathrm{av}})}{4}\right)\right)$$

with $f_{\mathrm{av}}\equiv\frac{f_0+K}{2}$, $\sigma(f)\equiv\sigma(0,f)$ and $\mathcal{G}(f)\equiv 2\partial_t\ln\sigma(0,f)$.

$$\begin{aligned}\sigma_{\mathrm{BS}}(T,K) &= \frac{\ln\frac{K}{f_0}}{\int_{f_0}^{K}\frac{df'}{\sqrt{2g^{ff}(a_{\min})}}}(1\\ &+ \frac{g^{ff}(a_{\min})T}{12}\left(-\frac{3}{4}\left(\frac{\partial_f g^{ff}(a_{\min})}{g^{ff}(a_{\min})}\right)^2+\frac{\partial_f^2 g^{ff}(a_{\min})}{g^{ff}(a_{\min})}+\frac{1}{f_{\mathrm{av}^2}}\right)\\ &\quad+\frac{g^{ff'}(a_{\min})T}{2g^{ff}(a_{\min})\phi''(a_{\min})}\left(\ln(Dg\mathcal{P}^2)'(a_{\min})-\frac{\phi'''(a_{\min})}{\phi''(a_{\min})}+\frac{g^{ff''}(a_{\min})}{g^{ff'}(a_{\min})}\right)\Bigg)\end{aligned}$$

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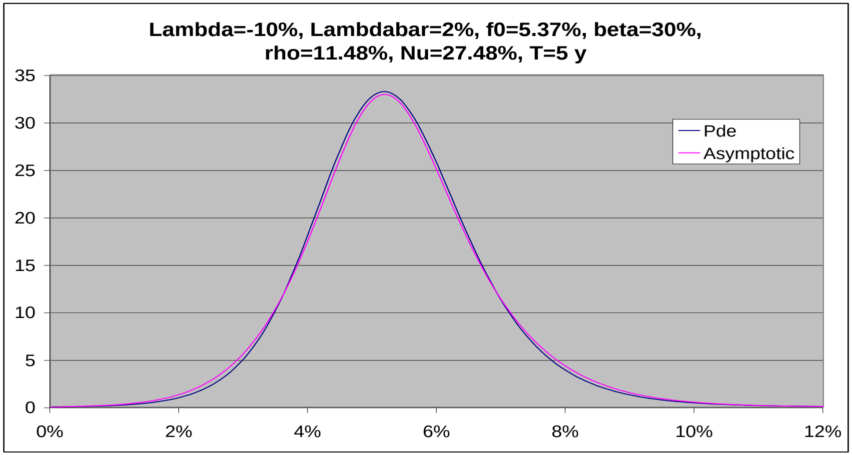
SABR with a mean-reversion term

$$
\begin{aligned}
df_t &= a_t f_t^{\beta} dW_t \\
da_t &= \lambda(a_t - \bar{\lambda})dt + \nu a_t dZ_t \\
C(f) &= f^{\beta}\ ,\ a_0 = \alpha\ ,\ f_{t=0} = f_0
\end{aligned}
$$

where W_t and Z_t are two Brownian processes with correlation $\rho \in (-1, 1)$.



SABR-BGM Model and $\mathbb{H}^{n+1}$

SABR-BGM Model given under the spot Libor measure $\mathbb{Q}$ by ($\beta(t) = m$ if $T_{m-2} < t < T_{m-1}$)

$$\begin{aligned} dF_k &= a^2 B^k(F,t)dt + \sigma_k(t) a C_k(F_k) dZ_k \ , \ \ k = 1, \cdots, n \\ da &= \nu a dZ_{n+1}; \ , \ dZ_i dZ_j = \rho_{ij} dt \ \ i,j = 1,\cdots,n+1 \end{aligned}$$

with

$$B^k(F,t) = \sum_{j=\beta(t)}^{k} \frac{\tau_j \rho_{jk} \sigma_k(t) \sigma_j(t) C_k(F_i) C_i(F_k)}{(1+\tau_j F_j)} \ , \ C_k(F_k) = F_k^{\beta_k}$$

- Bond of maturity T: $P(t,T)$
- Swap: $s_{\alpha\beta,t} = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)}$
- Libor $F_{\alpha,t} \equiv s_{\alpha(\alpha+1),t} = \frac{P(t,T_\alpha)}{\tau_{\alpha+1} P(t,T_{\alpha+1})} - 1.$

Local Volatility

▶ The forward swap rate satisfies in the forward-swap measure (associated to the numeraire $C_{\alpha\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$) the following driftless dynamics

$$ds_{\alpha\beta} = \frac{\partial s_{\alpha\beta}}{\partial F_k} \sigma_k(t) \phi_k(a, F_k) dZ_k$$

▶ The local volatility associated to the forward swap rate ($ds_{\alpha\beta} = \sigma_{loc}^{\alpha\beta}(s_{\alpha\beta}, t) dW_t$) is then by definition

$$(\sigma_{loc}^{\alpha\beta})^2(s,t) \;\equiv\; \mathbb{E}^{\alpha\beta}[\rho_{ij} \sigma_i(t) \sigma_j(t) \phi_i(a, F_i) \phi_j(a, F_j) \frac{\partial s_{\alpha\beta}}{\partial F_i} \frac{\partial s_{\alpha\beta}}{\partial F_j} | s_{\alpha\beta} = s]$$

Hyperbolic Geometry $\mathbb{H}^{n+1}$ and Geodesics

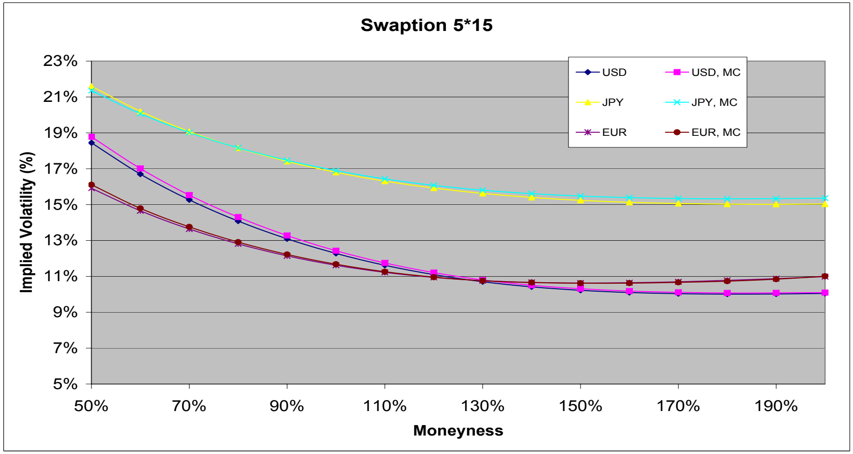
- New coordinates $[x_k]_{k=1\cdots n+1}$ (L is the Cholesky decomposition of the (reduced) correlation matrix: $[\rho]_{i,j=1\cdots n} = [\hat{L}\hat{L}^\dagger]_{i,j=1\cdots n}$)

$$
\begin{aligned}
x_k &= \sum_{i=1}^{n} \nu \hat{L}^{ki} \int_{F_i^0}^{F_i} \frac{dF_i'}{C_i(F_i')} + \sum_{i=1}^{n} \rho^{ia} \hat{L}_{ik} a \ , \ k = 1, \cdots, n \\
x_{n+1} &= (1 - \sum_{i,j}^{n} \rho^{ia} \rho^{ja} \rho_{ij})^{\frac{1}{2}} a
\end{aligned}
$$

- Metric on $\mathbb{H}^{n+1}$: $ds^2 = \frac{2(1-\sum_{i,j}^{n} \rho^{ia}\rho^{ja}\rho_{ij})}{\nu^2} \frac{\sum_{i=1}^{n} dx_i^2 + dx_{n+1}^2}{x_{n+1}^2}$
- Geodesic distance: $d(x, x^0) = \cosh^{-1}\left(1 + \frac{\sum_{i=1}^{n+1}(x_i - x_i^0)^2}{2 x_{n+1} x_{n+1}^0}\right)$

Numerical Tests

▷ $\mathbb{H}^{n+1}$ stochastic volatility LMM easily calibrated to swaption cubes.



Problem: Implied Volatility wings asymptotics.
Tool: Schrödinger Semigroups Estimates.

Problem: Implied Volatility wings asymptotics.

Benaim-Friz formula (1)

Let the strictly decreasing function $\Psi : [0, \infty] \rightarrow [0, 2]$ be

$$\Psi(x) = 2 - 4(\sqrt{x^2 + x} - x)$$

and let's define the class $\mathbb{R}_\alpha$:

A positive real-valued measurable function g is regularly varying with index α, in symbols $g \in \mathbb{R}_\alpha$, if

$$g \in \mathbb{R}_\alpha \Leftrightarrow \lim_{x \to \infty} \frac{g(\lambda x)}{g(x)} = \lambda^\alpha$$

Ex: $g(x) = x^\alpha \in \mathbb{R}_\alpha$

We assume the integrability condition (IC) on the right tail [a]

$$\exists\, \epsilon > 0 \;:\; \mathbb{E}^{\mathbb{P}}[f_T^{(1+\epsilon)}] < \infty$$

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[a] f_T: forward

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Benaim-Friz formula (2)

Assuming the IC, then if $-\ln \mathcal{C}(\tau,k)$ [a] is a regularly varying function in k (or in K) with a positive index, we have

$$\frac{\sigma_{BS}(\tau,k)^2\tau}{k} \sim \Psi\left(-\frac{\ln \mathcal{C}(\tau,k)}{k}\right)$$

[a] Here $\mathcal{C}(\tau,k)$ means a call option with moneyness k and maturity τ

Tool: Schrödinger Semigroups Estimates.

Separable Local Volatility

Under the risk-neutral measure $\mathbb{P}$: $df = A(t)C(f)dW_t$

- ▷ Connection: $\mathcal{A}_f = -\frac{1}{2}\partial_f \ln C(f)$
- ▷ Introducing the new coordinate $s = \sqrt{2}\int_{f_0}^{f} \frac{df'}{C(f')}$ and the new time $t' = \int_0^t A(s)^2 ds$, the new function $p'(t', s)$ defined by

$$p(t, f|f_0)df = p'(t', s(f)) \underbrace{\sqrt{\frac{C(f_0)}{C(f)}}} \underbrace{ds}$$

satisfies an (Euclidean) one-dimensional Schrödinger equation

$$\partial_{t'} p'(t', s) = (\partial_s^2 - Q(s))p'(t', s)$$

Time-homogeneous potential [a]: $Q(s) = -\frac{1}{2}(\ln C)''(s) + \frac{1}{4}((\ln C)'(s))^2$

[a] where the prime $'$ indicates a derivative according to s.

Time-homogeneous Potentials

LV Model	$C(f)$	Potential
BS	f	$Q(s) = -\frac{1}{8}$
Quad.	$af^2 + bf + c$	$Q(s) = -\frac{1}{8}(b^2 - 4ac)$
CEV	f^β, $0 \le \beta < 1$	$Q_{CEV}(s) = \frac{\beta(\beta-2)}{8(f^{1-\beta}+\frac{s(1-\beta)}{\sqrt{2}})^2}$
LCEV	$f \min(f^{\beta-1}, \epsilon^{\beta-1})$ with $\epsilon > f_0$	$Q_{LCEV}(s) = Q_{CEV}(s)\ \forall\ s \ge s_\epsilon \equiv \sqrt{2}(\frac{\epsilon^{1-\beta}-f^{1-\beta}}{(1-\beta)})$ $= -\frac{1}{8}\epsilon^{2(\beta-1)}\ \forall\ s < s_\epsilon$

Dupire LV Model

LV model: $df = C(t, f)dW$

Introducing the new coordinate $s_t = \sqrt{2}\int_{f_0}^{f} \frac{df'}{C(t,f')}$, the new function $p'(t, s(f))$

$$p(t, f|f_0)df = p'(t, s(f)) \underbrace{\sqrt{\frac{C(t, f_0)}{C(t, f)}} e^{\frac{1}{\sqrt{2}}\int_{f_0}^{f} \frac{df'}{C(t,f')}\partial_t \int_{f_0}^{f'} \frac{df''}{C(t,f'')}}} \underbrace{ds}$$

satisfies a one-dimensional (Euclidean) Schrödinger equation

$$\color{red}{\partial_t p'(t, s) = (\partial_{s^2} + Q(t, s))p'(t, s)}$$

with the time-dependent scalar potential [a]

$$Q(t, s) = -(\partial_s \mu(t, s) + \frac{1}{2}\mu(t, s)^2) - \int_0^s \partial_t \mu(t, s')ds'$$

[a] $\mu(t, s) = \frac{1}{\sqrt{2}}\partial_t(\int_{f_0}^{f(s)} \frac{df'}{C(t,f')}) - \frac{1}{2}\partial_s \ln(C(t, s))$

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Gauge free Stochastic Volatility Models

We impose that $\mathcal{A}$ is flat:

$$\mathcal{F} = 0$$

$\Rightarrow$ Reduction: $\partial_\tau p'(x, \alpha, \tau) = (\triangle_\Sigma + Q(x))p'(x, \alpha, \tau)$
Classification:

$$\begin{aligned} df &= (\mu f + \nu)dW \\ da &= a\sigma(a)(\gamma + \frac{1}{2}\partial_a \frac{\sigma(a)}{a})dt + \sigma(a)dZ \;,\; dW dZ = \rho dt \end{aligned}$$

name	$\sigma(a)$	SDE
Heston	$\sigma(a) = \eta$	$dv = \sqrt{\delta}(2v\gamma + \eta(\eta - 1))dt + 2\eta\sqrt{\delta}\sqrt{v}dW_2$
GB-SABR	$\sigma(a) = \eta a$	$dv = \sqrt{\delta}(2\eta\gamma v^{\frac{3}{2}} + \eta^2 v)dt + 2\sqrt{\delta}\eta v dW_2$
3/2-model	$\sigma(a) = \eta a^2$	$dv = 2\sqrt{\delta}\eta(\eta + \gamma)v^2 dt + 2\sqrt{\delta}\eta v^{\frac{3}{2}} dW_2$

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Gaussian Estimates of Heat Kernel Semi-groups: A Famous Problem

$$c_1 p_N(c_2 t, x|\alpha) \leq p'(t, x|\alpha) \leq C_1 p_N(C_2 t, x|\alpha)$$

- J. Nash, Aronsov (58) Geometry: $\partial_t p'(t, x|\alpha) = \triangle_M p'(t, x|\alpha)$
- B. Simon (82) Mathematical Physics: $\partial_t p'(t, x|\alpha) = (\partial_x^2 + Q(x))p'(t, x|\alpha)$
- Yau (78) Geometry: $\partial_t p'(t, x|\alpha) = (\triangle_M + Q(x))p'(t, x|\alpha)$
- Norris-Stroock (Malliavin calculus) (83) Probability: $\partial_t p'(t, x|\alpha) = Dp'(t, x|\alpha)$
- Q. Zhang (00) Functional Analysis: $\partial_t p'(t, x|\alpha) = (\partial_x^2 + Q(x, t))p'(t, x|\alpha)$

Autonomous Kato class

$$\partial_t p'(t,s) = (\partial_s^2 - Q(s))p'(t,s)$$

We say that Q is in the Kato class K if

$$Q \in K \Leftrightarrow \lim_{\delta \to 0} \sup_{x \in \mathbb{R}} \int_0^\delta dt \int_{\mathbb{R}} dy |Q(y)| p_G(t,y|x) = 0$$

with the free heat kernel $p_G(t,y|x) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \exp(-\frac{(y-x)^2}{4t})$

Examples of Potentials in the Kato class

- Black-Scholes: $Q(s) = -\frac{1}{8} \Rightarrow$ In the Kato class
- LCEV: $Q(s) = -\frac{1}{(s^2+1)} \Rightarrow$ In the Kato class
- Vasicek: $Q(s) = s^2 \Rightarrow$ Not in the Kato class.

$$Q(y) \equiv y^2 = (y-x)^2 + 2(y-x)x + x^2$$

$$\int_{\mathbb{R}} dy |Q(y)| p_G(t, y|x) \cong \sqrt{t} + x^2$$

$$\int_0^{\delta} dt \int_{\mathbb{R}} dy |Q(y)| p_G(t, y|x) \cong \frac{2}{3}\delta^{\frac{3}{2}} + x^2\delta$$

$$\sup_{x\in\mathbb{R}} \int_0^{\delta} dt \int_{\mathbb{R}} dy |Q(y)| p_G(t, y|x) = \infty$$

$$\lim_{\delta\to 0} \sup_{x\in\mathbb{R}} \int_0^{\delta} dt \int_{\mathbb{R}} dy |Q(y)| p_G(t, y|x) = \infty$$

Gaussian Estimates of Schrödinger Semi-groups

Let $Q^+ \equiv \max(Q, 0)$ and $Q^- = \max(-Q, 0)$.

- If $Q^+ \in K_{loc}$ [a] and $Q^- \in K$. Then we have an upper bound

  $$p'(t, y|x) \leq C_1 e^{C_2 t} p_G(t, y|x), \; t > 0, \; x, y \in \mathbb{R}$$

  with two constants C_1, C_2. Note that the constant $C_2 = 0$ if $Q^+ = 0$.

- Assuming that Q^+ and Q^- are both in the Kato class K, we have also a lower bound

  $$c_1 e^{c_2 t} p_G(t, y|x) \leq p'(t, x|y), \; t > 0, \; x, y \in \mathbb{R}$$

  with two constants c_1 and c_2.

[a] $Q^+(y)1(y \leq N) \in K$

Gaussian Estimates

Providing that the scalar potential associated to a local volatility function belongs to the Kato class, we have the Gaussian bounds on the function $p'(t,s)$

$$c_1 e^{c_2 t} p_G(t,s) \leq p'(t,s) \leq C_1 e^{C_2 t} p_G(t,s)$$

This inequality translates directly on an estimation of the conditional probability $p(t,f|f_0)$

$$c_1 e^{c_2 t} p_G(t,s) \leq p'(t',s(f)) = \frac{C(f)^{\frac{3}{2}}}{\sqrt{2C(f_0)}} p(t,f|f_0) \leq C_1 e^{C_2 t} p_G(t,s)$$

Gaussian Estimates of European Options

We can directly translate the Gaussian bounds on the conditional probability $p'(t', s(K))$ into bounds on the implied volatility as

$$\mathcal{C}(\tau, k) = \max(f_0 - K, 0) + \frac{\sqrt{C(K)C(f_0)}}{\sqrt{2}} \int_0^{\tau} p'(t', s(K))dt'$$

The large strike behavior of the implied volatility:

$$\frac{\sigma_{BS}(\tau, k)^2 \tau}{k} \sim_{k \to \infty} \Psi\left(\frac{-\frac{1}{2}\ln(C(K)) + \frac{(\int_{f_0}^{K} \frac{df'}{C(f')})^2}{2\tau}}{k} \right)$$

If $s(K)$ is the leading term, $\sigma_{BS}(\tau, k) \sim_{k \to \infty} \frac{k}{\int_{f_0}^{K} \frac{df'}{C(f')}}$

Short-time limit of the IV $\tau << 1$, $f_{\text{av}} = \frac{f_0 + K}{2}$:

$$\sigma_{BS}(\tau, k) = \frac{k}{\int_{f_0}^{K} \frac{df'}{C(f')}} \left(1 + \frac{\tau}{3} \left(\frac{1}{8} \left(\frac{k}{\int_{f_0}^{K} \frac{df'}{C(f')}} \right)^2 + Q(f_{av}) \right) \right) + o(\tau^2)$$

Example: CEV model

For $0 \leq \beta < 1$,we have

$$\sigma_{BS}(k,\tau) \sim_{k \to \infty} \frac{k(1-\beta)}{K^{1-\beta}}$$

and for $\beta = 1$, we have $\sigma_{BS}(\tau, k) \sim 1$.
This result should be compared with the result obtained using the Lee moment formula:

$$\limsup_{k \to \infty} \frac{\sigma_{BS}(\tau,k)^2 \tau}{k} \;=\; 0$$

as all the moments exist.

Extensions and Questions

- Dupire Local volatility: Gaussian estimates of SE with potentials $Q(t,s)$ belonging to the non-autonomous Kato class.
- General SVMs: R is not bounded from below by $-K, K > 0$ and/or Q is unbounded. Need Generalization of the Li-Yau estimates.

$$
\begin{aligned}
Q_{\mathrm{LN-SABR}}(a) &= -\frac{a^2}{8(1-\rho^2)} \\
Q_{\mathrm{LN-SABR}}(a_{\min}) &= -\frac{\nu^2 k^2}{8(1-\rho^2)} \to_{k\to\infty} \infty!
\end{aligned}
$$

Numerical Methods in Finance

Available Numerical Methods in Finance

- PDE: Only when the number of assets is small.
- Monte-Carlo, Quasi Monte-Carlo: Euler, Milstein, stochastic Runge-Kutta, Ninomiya-Victoir, Cubature...

$\Rightarrow$ Combinatorial Hopf algebra. Similar structure in

- Renormalization in Quantum Field Theory: Connes-Kreimer
- Butcher Hopf algebra in deterministic Runge-Kutta methods
- Combinatorics of multi-zeta Riemann functions (polylogarithm): Zagier, Cartier, Kontsevitch, ..

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Taylor-Stratonovich Expansion

$$dx_t = \sum_{i=0}^{n} V_i \diamond dW_t^i \ , \ dW_t^0 \equiv dt$$

$$f(x_t) \ = \sum_{(i_1,..,i_k) \in \mathcal{A}_r} V_{i_1} ... V_{i_k} f(x) \int_{0<t_1<...<t_k<t} o dW_{t_1}^{i_1} ... {}_o dW_{t_k}^{i_k} + R_r$$

with $V_i = V_i^k \partial_k$.

$\triangleright$ Graduation : $[dW^0] = 1$ and $[dW^i] = \frac{1}{2} \ \ i = 1, \cdots n$.

Replace the vector fields $V_0, V_1, ..., V_d$ by letters $\varepsilon_0, \varepsilon_1, ..., \varepsilon_d$:

$$X_{0,1}(\omega) = \sum_{(i_1,...,i_k) \in \mathcal{A}_r} \varepsilon_{i_1} ... \varepsilon_{i_k} \int_{0<t_1<...<t_k<1} d\omega^{i_1}(t_1) ... d\omega^{i_k}(t_k)$$

$\rightarrow$ Nice element of a (graded) non-commutative Hopf algebra.

(Graded) Hopf algebra of words (1)

Define operations on $\mathcal{H}_r$

- a scalar multiplication $\times$
- sum $+$
- concatenation .
- coproduct $\Delta : \mathcal{H}_r \to \mathcal{H}_r \otimes \mathcal{H}_r$
- unit $\varepsilon : \mathcal{H}_r \to k$
- counit η
- antipode $a : \mathcal{H}_r \to \mathcal{H}_r$

(Graded) Hopf algebra of words (2)

-Hopf algebra $\mathcal{H}_r$ = Algebra + Bialgebra + antipode

$\triangleright$ Primitive elements : $\Delta(x) = x \otimes 1 + 1 \otimes x$

$\Rightarrow$ Lie algebra $\mathcal{G}_r$: $a(\mathcal{L}) = -\mathcal{L}$

$\triangleright$ $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$\triangleright$ Group-like elements $\Delta(g) = g \otimes g$: $G_r = exp(\mathcal{G}_r)$:

$\triangleright$ $a(g) = g^{-1}$

$\triangleright$ $\log(\exp) = 1$

Group-like element

Theorem 0.1 (Chen) $X_{0,1}(\omega)$ is a group-like element of $\mathcal{H}_r$

$$X_{0,1}(\omega) = \exp(\mathcal{L})$$

for a primitive element (Lie polynomial) $\mathcal{L}$ in $\mathcal{G}_r$

Yamato thm. and Carr-Madan Brownian rep.

Theorem 0.2 (Yamato)

$$\mathcal{L}_t = tV_0 + \cdots + W_t^m V_m + \sum_{r=2} \sum_{J \in \mathcal{A}_r} c_J W_t^J V^J$$

with the iterated Brownian integrals

$$W_t^J = \int_{0 \le t_{j_1} < \cdots < t_{j_m} \le t} dW_{u_1}^{j_1} \cdots \diamond dW_{u_m}^{j_m} \ , \ V^J = [\cdots [V_{j_1}, V_{j_2}] \cdots V_{j_m}] \ , \ J = (j_1, \cdots, j_m)$$

c_J are some constants.

- Carr-Madan: Classify models that can be written as a functional of a BM: $\Rightarrow$ Abelian Lie algebra.
- Classify models that can be written as a functional of W_i, $\int_0^1 W_t^{[i} dW_t^{j]}$ (Levy area): $V_0 = 0$, $\{V_i\}$ 1-step nilpotent Lie algebra: Heisenberg Lie algebra.

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Discretization scheme à la Ninomiya-Victoir

Weak order discretization scheme [a]

$$\Pi[\exp(\varepsilon_0 + \frac{1}{2}\sum_{i=1}^{d} \varepsilon_i \varepsilon_i)] = \sum_{p=1}^{P} \lambda_p \Pi[\mathbb{E}[\exp(\mathcal{L}_p)]]$$

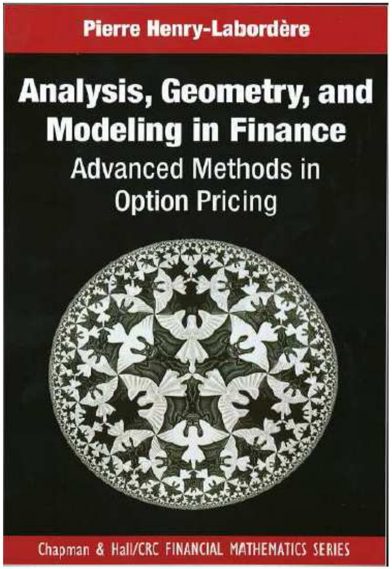
Weak order 3.0 at $d = 1$ (Denuelle-PHL 2007):

$$\mathcal{L}_\pm = e^{\frac{1}{24}[\epsilon_1,\epsilon_1,\epsilon_0]} e^{\pm\frac{1}{2\sqrt{12}}[\epsilon_1,\epsilon_0]} e^{\Delta W_n^1 \epsilon_1 + \epsilon_0} e^{\pm\frac{1}{2\sqrt{12}}[\epsilon_1,\epsilon_0]} e^{\frac{1}{24}[\epsilon_1,\epsilon_1,\epsilon_0]} e^{-\frac{1}{240}[\epsilon_1,\epsilon_1,\epsilon_1,\epsilon_1\epsilon_0]}$$

with $\lambda_\pm = \frac{1}{2}$. Use the Hopf algebra structure to simplify the computations!

[a] Π: truncation operator with respect to the grading.

Book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