A Few Geometrical Problems in Finance

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Some Problems and Tools

   Tool: Heat kernel expansion on a Riemannian manifold.

2. Problem: Large-strike behavior of the implied volatility.
   Tool: Schrödinger Semigroups Estimates.

   Tool: Combinatorial Hopf Algebra, Heat kernel on the Heisenberg group...

These tools arise in the Atiyah-Singer theorem...
Tool: HK expansion
Problem: Calibration of Stochastic Volatility Models.
Motivations

- The dimensionless parameter in Finance is \([\text{vol}]^2 \tau\) which is small \(\Rightarrow\) Asymptotic expansion.

- Try to find a systematic way of doing asymptotic expansion in Finance:
  1. Asymptotic smile for the SABR model at the first-order [Hagan-al]
  2. Asymptotic smile for basket at the zero-order [Avellaneda-al]
  3. Asymptotic swaption at the zero-order for LMM [Rebonato-Hull-White freezing argument]

\(\Rightarrow\) Heat kernel expansion.
Tool: HK expansion
Heat Kernel (1)

SDEs: \( dx^i = b^i(x)dt + \sigma^i(x)dW_i \), \( dW_i dW_j = \rho_{ij} dt \)

Backward Kolmogorov equation [Einstein Convention here]

\[
\partial_\tau p(\tau, x|y) = D p(\tau, x|y)
\]

with \( D = b^i \partial_i + \frac{1}{2} \rho_{ij} \sigma^i \sigma^j \partial_{ij} \)

Under a change of coordinates \( x^i \rightarrow x'^i \),

\[
b'^i = \frac{\partial x'^i}{\partial x^i} b^i + \frac{1}{2} \rho_{ij} \sigma^i \sigma^j \frac{\partial^2 x'^i}{\partial x^i \partial x^j}
\]

covariant non cov.

Rewrite \( D \) as \( D = g^{-\frac{1}{2}}(\partial_i + A_i) g^{\frac{1}{2}} g^{ij} (\partial_j + A_j) + Q \)

By identifying the terms \( \partial_i \) and \( \partial_{ij} \), we obtain

- Metric: \( g_{ij} = 2 \rho_{ij} \sigma_i \sigma_j \), \( g = \det[g_{ij}] \), \( g^{ij} = [g_{ij}]^{-1} \)
- Connection: \( A^i = \frac{1}{2} (b^i - g^{-\frac{1}{2}} \partial_j (g^{1/2} g^{ij})) \), \( A_i = g_{ij} A^j \)
- Section Q: \( Q = g^{ij} (A_i A_j - b_j A_i - \partial_j A_i) \), \( b_j \equiv g_{ji} b^i \)
Heat Kernel (2)

To summarize a heat kernel equation on a Riemannian manifold $M$ is constructed from the following three pieces of geometric data:

1. a metric $g$ on $M$, which determines the second-order piece.
2. a connection $\mathcal{A}$ on a line bundle $\mathcal{L}$, which determines the first-order piece.
3. a section of the bundle $\text{End}(\mathcal{L}) \simeq \mathcal{L} \otimes \mathcal{L}^*$, which determines the zeroth-order piece.

$p$ continuous section of the bundle $(\mathcal{L} \boxtimes \mathcal{L}^*)$ over $\mathbb{R}^+ \times M \times M$ \footnote{Let $\mathcal{E}_1$ and $\mathcal{E}_2$ two vector bundle on $M$, and let $pr_1$ be the projections from $M \times M$ onto the first and second factor $M$ respectively. We denote the external product $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ the vector bundle $pr_1^*\mathcal{E}_1 \otimes pr_2^*\mathcal{E}_2$ over $M \times M$.}
Reduction Method (1)

The heat kernel equation can now be simplified by applying the actions of the following groups:

- The group of diffeomorphisms $\text{Diff}(\mathcal{M})$ which acts on the metric $g_{ij}$ and the connection $\mathcal{A}_i$ by

\[
\begin{align*}
g_{ij} & \xrightarrow{f \in \text{Diff}(\mathcal{M})} (f^*g)_{ij} = g_{pk} \partial_i f^p(x) \partial_j f^k(x) \\
\mathcal{A}_i & \xrightarrow{f \in \text{Diff}(\mathcal{M})} (f^*\mathcal{A})_i = \mathcal{A}_p \partial_i f^p(x)
\end{align*}
\]

- The group of gauge transformations $\mathcal{G}$ which acts on the conditional probability (and the call option $\mathcal{C}$) by

\[\begin{align*}
p(\tau, x|y) & \xrightarrow{G} p'(\tau, x|y) = e^{\chi(\tau, x) - \chi(0, y)} p(\tau, x|y)
\end{align*}\]

Then $p'$ satisfies the same equation as $p$ with

\[
\begin{align*}
\mathcal{A}'_i & \equiv \mathcal{A}_i - \partial_i \chi \\
Q' & \equiv Q + \partial_\tau \chi
\end{align*}
\]
Reduction Method (2)

If the connection $\mathcal{A}$ is an exact form

$$\mathcal{A}_i = \partial_i \Lambda$$

then by applying a gauge transformation

$$\mathcal{A}_i' = \mathcal{A}_i - \partial_i \Lambda = \partial_i \Lambda - \partial_i \Lambda = 0$$

The HK equation reduces to

$$\partial_\tau p'(\tau, x|y) = (\Delta + Q')p'(\tau, x|y)$$

The statement "$\mathcal{A}$ is exact" is equivalent to $\mathcal{F} = 0$, where $\mathcal{F}$ is given in a specific coordinate system by

$$\mathcal{F}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i$$

Obvious($\Rightarrow$): $\partial_i \partial_j \Lambda - \partial_j \partial_i \Lambda = 0$
Heat Kernel Expansion

Let $M$ be a Riemannian manifold without a boundary. Then for each $x \in M$, there is a complete asymptotic expansion for small $\tau$

$$p(\tau, x|y) = \frac{\sqrt{g(x)}}{(4\pi \tau)^{n/2}} \sqrt{D(x, y)} \mathcal{P}(x, y) e^{-\frac{d(x, y)^2}{4\tau}} \sum_{k=0}^{\infty} a_k(x, y) \tau^k$$

$\Box$ $d(x, y)$ is the geodesic distance: $d(x, y) = \min_C \int_0^1 \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} dt$.

On a flat manifold $\mathbb{R}^n$, the geodesic curves are the straight lines and the geodesic distance is the Euclidean distance.

$\Box$ Van Vleck-Morette determinant $D(x, y)$:

$$D(x, y) = g(x)^{-\frac{1}{2}} \det \left( -\frac{\partial^2 d(x, y)^2}{\partial x \partial y} \right) g(y)^{-\frac{1}{2}} \text{ with } g(x) = \det [g_{ij}(x, x)]$$

$\Box$ Parallel gauge transport: $\mathcal{P}(x, y) = e^{-\int_{C(x, y)} A_i dx^i}$ with $C(x, y)$ a geodesic from the point $x$ to $y$. 
Heat Kernel Coefficients

The $a_i(x, y)$ are smooth functions on $M$ and depend on geometric invariants such as the scalar curvature $R$.

$$a_0(x, y) = 1$$

$$a_1(x, x) = \frac{1}{6}R + Q$$
Problem: Calibration of SVMs
SVM and Riemann Surface

A (1-factor) stochastic volatility model (SVM) depends on two SDEs, one for the asset $f$ and one for the volatility $a$. In a risk-neutral measure, we have

$$
\begin{align*}
    df &= C'(f) adW_1 \\
    da &= b(a) dt + \sigma(a) dW_2 \\
    dW_1 dW_2 &= \rho dt
\end{align*}
$$

This model corresponds in our geometrical framework to a Riemann surface $\Sigma$, endowed with a two-by-two metric.
Riemann Uniformization Theorem

Isothermal Coordinates:
\[ ds^2 = F(y)(dx^2 + dy^2) \]

<table>
<thead>
<tr>
<th>Name</th>
<th>Conformal factor</th>
<th>Scalar curvature</th>
<th>Surface</th>
</tr>
</thead>
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<tr>
<td>Geometric</td>
<td>( F(y) \sim y^{-2} )</td>
<td>( R = -1 )</td>
<td>( \mathbb{H}^2 )</td>
</tr>
<tr>
<td>3/2-model</td>
<td>( F(y) \sim e^{-2y} )</td>
<td>( R = 0 )</td>
<td>( \mathbb{R}^2 )</td>
</tr>
<tr>
<td>SABR</td>
<td>( F(y) \sim y^{-2} )</td>
<td>( R = -1 )</td>
<td>( \mathbb{H}^2 )</td>
</tr>
<tr>
<td>Heston</td>
<td>( F(y) \sim y^{-1} )</td>
<td>( R = -2a^{-2} &lt; 0 )</td>
<td>Baby Black Hole</td>
</tr>
</tbody>
</table>

\[ \triangleright R = \frac{\sigma(a)^2}{a} \left( \frac{\sigma'(a)}{\sigma(a)} - \frac{2}{a} \right). \]

\[ \triangleright \text{For } a \frac{\sigma'(a)}{\sigma} \leq 2: \text{ } \Sigma \text{ is a Cartan-Hadamard manifold. } \rightarrow \text{ The cut-locus is empty!} \]
Metric with one Killing vector

Integrable geodesics:

\[ d(x, y) = \left| \int_{y_1}^{y_2} \frac{F(y')dy'}{\sqrt{F(y') - C^2}} \right| \]

with the constant \( C = C(x_1, y_1, x_2, y_2) \) determined by the equation

\[ x_2 - x_1 = \int_{y_1}^{y_2} \frac{C}{\sqrt{F(y') - C^2}}dy' \]
Asymptotic Local Volatility

LV: \( df_t = \sigma(t, f) dW_t \) and \( df_t = a_t C(f_t) dW_t \) have the same marginals if:

We have that the local \( \text{vol}^2 \) is the mean value of the stochastic \( \text{vol}^2 \) conditional to the forward:

\[
\sigma^2(t, f) = C^2(f) \mathbb{E}[a_t^2 | f_t = f] \\
\equiv C^2(f) \frac{\int_0^\infty a^2 p(t, f, a | \alpha, f_0) \sqrt{g} da}{\int_0^\infty p(t, f, a | \alpha, f_0) \sqrt{g} da}
\]

► Saddle point:

\[\sigma(t, f) = C(f) a_{\text{min}}, \ a_{\text{min}} \equiv a \left| \min_{(a, f \text{ fixed})} d(z, z_0) \right.\]

► Asymptotic local volatility \( (\hat{\sigma}(f, a) \equiv d^2(f, a)):\)

\[
\sigma(T, f) = \sqrt{2g^f f (a_{\text{min}})} (1+ \\
\frac{T}{\phi''(a_{\text{min}})} \left( \frac{g^{ff}(a_{\text{min}})}{g^f f(a_{\text{min}})} \left( \ln(DgP^2)'(a_{\text{min}}) - \frac{\phi'''(a_{\text{min}})}{\phi''(a_{\text{min}})} \right) + \frac{g^{ff'}(a_{\text{min}})}{g^f f(a_{\text{min}})} \right))
\]
Asymptotic Implied Volatility for any SVM

Use asymptotic map between local and implied volatilities [Time-dependent HKE]:

\[
\sigma_{BS}(\tau, K) = \frac{\ln(K_f)}{\int_{f_0}^{K} \frac{df'}{\sigma(f')}} \left( 1 + \frac{\tau}{3} \left( \frac{\ln(K_f)}{\int_{f_0}^{K} \frac{df'}{\sigma(f')}} \right)^2 + Q(f_{av}) + \frac{3G(f_{av})}{4} \right)
\]

with \( f_{av} \equiv \frac{f_0 + K}{2} \), \( \sigma(f) \equiv \sigma(0, f) \) and \( G(f) \equiv 2\partial_t \ln \sigma(0, f) \).

\[
\sigma_{BS}(T, K) = \frac{\ln(K_f)}{\int_{f_0}^{K} \frac{df'}{\sqrt{2gf'(a_{min})}}} (1)
\]

\[
+ \frac{gf'(a_{min})T}{12} \left( -\frac{3}{4} \left( \frac{\partial_f gf'(a_{min})}{gf'(a_{min})} \right)^2 + \frac{\partial^2_f gf'(a_{min})}{gf'(a_{min})} + \frac{1}{f_{av}^2} \right)
\]

\[
+ \frac{gf''(a_{min})T}{2gf'(a_{min})\phi''(a_{min})} \left( \ln(DgP^2)'(a_{min}) - \frac{\phi'''(a_{min})}{\phi''(a_{min})} + \frac{gf''(a_{min})}{gf'(a_{min})} \right)
\]


SABR with a mean-reversion term

\[ df_t = a_t f_t^\beta dW_t \]
\[ da_t = \lambda (a_t - \bar{\lambda}) dt + \nu a_t dZ_t \]
\[ C(f) = f^\beta, \quad a_0 = \alpha, \quad f_{t=0} = f_0 \]

where \( W_t \) and \( Z_t \) are two Brownian processes with correlation \( \rho \in (-1, 1) \).
SABR-BGM Model and $\mathbb{H}^{n+1}$

SABR-BGM Model given under the spot Libor measure $\mathbb{Q}$ by ($\beta(t) = m$ if $T_{m-2} < t < T_{m-1}$)

$$dF_k = a^2 B^k(F, t) dt + \sigma_k(t) a C_k(F_k) dZ_k, \quad k = 1, \cdots, n$$

$$da = \nu a dZ_{n+1}, \quad dZ_i dZ_j = \rho_{ij} dt \quad i, j = 1, \cdots, n + 1$$

with

$$B^k(F, t) = \sum_{j=\beta(t)}^k \frac{\tau_j \rho_{jk} \sigma_k(t) \sigma_j(t) C_k(F_i) C_i(F_k)}{(1 + \tau_j F_j)} \cdot C_k(F_k) = F_k^{\beta_k}$$

- Bond of maturity $T$: $P(t, T)$
- Swap: $s_{\alpha\beta,t} = \frac{P(t,T_{\alpha})-P(t,T_{\beta})}{\sum_{i=\alpha+1}^\beta \tau_i P(t,T_i)}$
- Libor $F_{\alpha,t} \equiv s_{\alpha(\alpha+1),t} = \frac{P(t,T_{\alpha})}{\tau_{\alpha+1} P(t,T_{\alpha+1})} - 1$. 

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Local Volatility

The forward swap rate satisfies in the forward-swap measure (associated to the numeraire $C_{\alpha\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$) the following driftless dynamics

$$ds_{\alpha\beta} = \frac{\partial s_{\alpha\beta}}{\partial F_k} \sigma_k(t) \phi_k(a, F_k) dZ_k$$

The local volatility associated to the forward swap rate ($ds_{\alpha\beta} = \sigma_{loc}^{\alpha\beta}(s_{\alpha\beta}, t)dW_t$) is then by definition

$$(\sigma_{loc}^{\alpha\beta})^2(s, t) \equiv \mathbb{E}^{\alpha\beta}[\rho_{ij} \sigma_i(t) \sigma_j(t) \phi_i(a, F_i) \phi_j(a, F_j) \frac{\partial s_{\alpha\beta}}{\partial F_i} \frac{\partial s_{\alpha\beta}}{\partial F_j} | s_{\alpha\beta} = s]$$
Hyperbolic Geometry $\mathbb{H}^{n+1}$ and Geodesics

- New coordinates $[x_k]_{k=1\ldots n+1}$ ($L$ is the Cholesky decomposition of the (reduced) correlation matrix: $[\rho]_{i,j=1\ldots n} = [\hat{L}\hat{L}^\dagger]_{i,j=1\ldots n}$)

$$x_k = \sum_{i=1}^{n} \nu \hat{L}^k i \int_{F_i^0}^{F_i} \frac{dF_i'}{C_i(F_i')} + \sum_{i=1}^{n} \rho^i a \hat{L}_{ik} a, \ k = 1, \ldots, n$$

$$x_{n+1} = (1 - \sum_{i,j}^{n} \rho^i a \rho^j a \rho_{ij})^{\frac{1}{2}} a$$

- Metric on $\mathbb{H}^{n+1}$: $ds^2 = \frac{2(1 - \sum_{i,j}^{n} \rho^i a \rho^j a \rho_{ij})}{\nu^2} \sum_{i=1}^{n} dx_i^2 + dx_{n+1}^2$

- Geodesic distance: $d(x, x^0) = \cosh^{-1} \left( 1 + \frac{\sum_{i=1}^{n+1} (x_i - x^0_i)^2}{2x_{n+1}x^0_{n+1}} \right)$
Numerical Tests

▷ $H^{n+1}$ stochastic volatility LMM easily calibrated to swaption cubes.
Problem: Implied Volatility wings asymptotics.
Tool: Schrödinger Semigroups Estimates.
Problem: Implied Volatility wings asymptotics.
Benaim-Friz formula (1)

Let the strictly decreasing function $\Psi : [0, \infty] \to [0, 2]$ be

$$\Psi(x) = 2 - 4(\sqrt{x^2 + x} - x)$$

and let’s define the class $\mathbb{R}_\alpha$:

A positive real-valued measurable function $g$ is regularly varying with index $\alpha$, in symbols $g \in \mathbb{R}_\alpha$, if

$$g \in \mathbb{R}_\alpha \iff \lim_{x \to \infty} \frac{g(\lambda x)}{g(x)} = \lambda^\alpha$$

Ex: $g(x) = x^\alpha \in \mathbb{R}_\alpha$

We assume the integrability condition (IC) on the right tail $^a$

$$\exists \epsilon > 0 : \mathbb{E}^p[f_T^{(1+\epsilon)}] < \infty$$

$^a f_T$: forward
Benaim-Friz formula (2)

Assuming the IC, then if $-\ln C(\tau, k)^{a}$ is a regularly varying function in $k$ (or in $K$) with a positive index, we have

$$\frac{\sigma_{BS}(\tau, k)^{2}\tau}{k} \sim \Psi \left( -\frac{\ln C(\tau, k)}{k} \right)$$

\(^{a}\)Here $C(\tau, k)$ means a call option with moneyness $k$ and maturity $\tau$
Tool: Schrödinger Semigroups Estimates.
Separable Local Volatility

Under the risk-neutral measure $\mathbb{P}$: $df = A(t)C(f)dW_t$

$\triangleright$ Connection: $A_f = -\frac{1}{2} \partial_f \ln C(f)$

$\triangleright$ Introducing the new coordinate $s = \sqrt{2} \int_{f_0}^{f} \frac{df'}{C(f')}$ and the new time $t' = \int_{0}^{t} A(s)^2 ds$, the new function $p'(t', s)$ defined by

$$p(t, f|f_0)df = p'(t', s(f)) \sqrt{\frac{C(f_0)}{C(f)}} ds$$

satisfies an (Euclidean) one-dimensional Schrödinger equation

$$\partial_t p'(t', s) = (\partial_s^2 - Q(s))p'(t', s)$$

Time-homogeneous potential $^a$: $Q(s) = -\frac{1}{2}(\ln C)''(s) + \frac{1}{4}((\ln C)'(s))^2$

$^a$where the prime $'$ indicates a derivative according to $s$. 
## Time-homogeneous Potentials

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<tr>
<th>LV Model</th>
<th>$C'(f)$</th>
<th>Potential</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
<td>$f$</td>
<td>$Q(s) = -\frac{1}{8}$</td>
</tr>
<tr>
<td>Quad.</td>
<td>$af^2 + bf + c$</td>
<td>$Q(s) = -\frac{1}{8}(b^2 - 4ac)$</td>
</tr>
<tr>
<td>CEV</td>
<td>$f^\beta$, $0 \leq \beta &lt; 1$</td>
<td>$Q_{CEV}(s) = \frac{\beta(\beta-2)}{8(f^{1-\beta} + \frac{s(1-\beta)}{\sqrt{2}})^2}$</td>
</tr>
<tr>
<td>LCEV</td>
<td>$f \min(f^{\beta-1}, \epsilon^{\beta-1})$ with $\epsilon &gt; f_0$</td>
<td>$Q_{LCEV}(s) = Q_{CEV}(s) \forall s \geq s_\epsilon \equiv \sqrt{2(\frac{\epsilon^{1-\beta} - f^{1-\beta}}{(1-\beta)})}$ $= -\frac{1}{8}\epsilon^{2(\beta-1)} \forall s &lt; s_\epsilon$</td>
</tr>
</tbody>
</table>
Dupire LV Model

LV model: \( df = C(t, f) \, dW \)

Introducing the new coordinate \( s_t = \sqrt{2} \int_{f_0}^f \frac{df'}{C(t, f')} \), the new function \( p'(t, s(f)) \)

\[
p(t, f|f_0) \, df = p'(t, s(f)) \sqrt{\frac{C(t, f_0)}{C(t, f)}} e^{\frac{1}{\sqrt{2}} \int_{f_0}^f \frac{df'}{C(t, f')} \partial_t \int_{f_0}^{f'} \frac{df''}{C(t, f'')}} \\ ds
\]

satisfies a one-dimensional (Euclidean) Schrödinger equation

\[
\partial_t p'(t, s) = (\partial_s^2 + Q(t, s)) p'(t, s)
\]

with the time-dependent scalar potential \(^a\)

\[
Q(t, s) = -(\partial_s \mu(t, s) + \frac{1}{2} \mu(t, s)^2) - \int_0^s \partial_t \mu(t, s') \, ds'
\]

\(^a\mu(t, s) = \frac{1}{\sqrt{2}} \partial_t (\int_{f_0}^{f(s)} \frac{df'}{C(t, f')}) - \frac{1}{2} \partial_s \ln(C(t, s)) \)
The gauge freedom can be imposed by setting $\mathcal{A} = 0$.

\[
\mathcal{F} = 0
\]

Therefore, the reduction equation is:

\[
\frac{\partial}{\partial \tau} p'(x, \alpha, \tau) = (\triangle \Sigma + Q(x)) p'(x, \alpha, \tau)
\]

Classification:

\[
\begin{align*}
    df &= (\mu f + \nu) dW \\
    da &= a \sigma(a) \left( \gamma + \frac{1}{2} \partial_a \frac{\sigma(a)}{a} \right) dt + \sigma(a) dZ, \quad dW dZ = \rho dt
\end{align*}
\]

<table>
<thead>
<tr>
<th>name</th>
<th>$\sigma(a)$</th>
<th>SDE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heston</td>
<td>$\sigma(a) = \eta$</td>
<td>$dv = \sqrt{\delta}(2\eta \gamma + \eta(\eta - 1)) dt + 2\eta \sqrt{\delta} \sqrt{v} dW_2$</td>
</tr>
<tr>
<td>GB-SABR</td>
<td>$\sigma(a) = \eta a$</td>
<td>$dv = \sqrt{\delta}(2\eta \gamma v^{\frac{3}{2}} + \eta^2 v) dt + 2\sqrt{\delta} \eta v dW_2$</td>
</tr>
<tr>
<td>3/2-model</td>
<td>$\sigma(a) = \eta a^2$</td>
<td>$dv = 2\sqrt{\delta} \eta (\eta + \gamma) v^2 dt + 2\sqrt{\delta} \eta v^{\frac{3}{2}} dW_2$</td>
</tr>
</tbody>
</table>
Gaussian Estimates of Heat Kernel Semigroups: A Famous Problem

\[ c_1 p_N(c_2 t, x|\alpha) \leq p'(t, x|\alpha) \leq C_1 p_N(C_2 t, x|\alpha) \]

- J. Nash, Aronsov (58) Geometry: \( \partial_t p'(t, x|\alpha) = \triangle_mp'(t, x|\alpha) \)
- B. Simon (82) Mathematical Physics: \( \partial_t p'(t, x|\alpha) = (\partial_x^2 + Q(x))p'(t, x|\alpha) \)
- Yau (78) Geometry: \( \partial_t p'(t, x|\alpha) = (\triangle_M + Q(x))p'(t, x|\alpha) \)
- Norris-Stroock (Malliavin calculus) (83) Probability: \( \partial_t p'(t, x|\alpha) = Dp'(t, x|\alpha) \)
- Q. Zhang (00) Functional Analysis: \( \partial_t p'(t, x|\alpha) = (\partial_x^2 + Q(x, t))p'(t, x|\alpha) \)
Autonomous Kato class

\[
\partial_t p'(t, s) = (\partial_s^2 - Q(s)) p'(t, s)
\]

We say that \( Q \) is in the Kato class \( K \) if

\[
Q \in K \iff \lim_{\delta \to 0} \sup_{x \in \mathbb{R}} \int_0^\delta dt \int_{\mathbb{R}} dy |Q(y)| p_G(t, y | x) = 0
\]

with the free heat kernel \( p_G(t, y | x) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \exp\left(-\frac{(y-x)^2}{4t}\right) \)
Examples of Potentials in the Kato class

- Black-Scholes: $Q(s) = -\frac{1}{8} \Rightarrow$ In the Kato class
- LCEV: $Q(s) = -\frac{1}{(s^2+1)} \Rightarrow$ In the Kato class
- Vasicek: $Q(s) = s^2 \Rightarrow$ Not in the Kato class.

\[
Q(y) \equiv y^2 = (y - x)^2 + 2(y - x)x + x^2
\]
\[
\int_{\mathbb{R}} dy|Q(y)|p_G(t, y|x) \approx \sqrt{t} + x^2
\]
\[
\int_{0}^{\delta} dt \int_{\mathbb{R}} dy|Q(y)|p_G(t, y|x) \approx \frac{2}{3} \delta^{\frac{3}{2}} + x^2 \delta
\]
\[
\sup_{x \in \mathbb{R}} \int_{0}^{\delta} dt \int_{\mathbb{R}} dy|Q(y)|p_G(t, y|x) = \infty
\]
\[
\limsup_{\delta \to 0} \sup_{x \in \mathbb{R}} \int_{0}^{\delta} dt \int_{\mathbb{R}} dy|Q(y)|p_G(t, y|x) = \infty
\]
Gaussian Estimates of Schrödinger Semigroups

Let $Q^+ \equiv \max(Q, 0)$ and $Q^- = \max(-Q, 0)$.

- If $Q^+ \in K_{loc}$ and $Q^- \in K$. Then we have an upper bound
  $$p'(t, y|x) \leq C_1 e^{C_2 t} p_G(t, y|x), \ t > 0, \ x, y \in \mathbb{R}$$
  with two constants $C_1, C_2$. Note that the constant $C_2 = 0$ if $Q^+ = 0$.

- Assuming that $Q^+$ and $Q^-$ are both in the Kato class $K$, we have also a lower bound
  $$c_1 e^{c_2 t} p_G(t, y|x) \leq p'(t, x|y), \ t > 0, \ x, y \in \mathbb{R}$$
  with two constants $c_1$ and $c_2$.

---

\(^a\) $Q^+(y)1(y \leq N) \in K$
Gaussian Estimates

Providing that the scalar potential associated to a local volatility function belongs to the Kato class, we have the Gaussian bounds on the function $p'(t, s)$

$$c_1 e^{c_2 t} p_G(t, s) \leq p'(t, s) \leq C_1 e^{C_2 t} p_G(t, s)$$

This inequality translates directly on an estimation of the conditional probability $p(t, f | f_0)$

$$c_1 e^{c_2 t} p_G(t, s) \leq p'(t', s(f)) = \frac{C(f)^{3/2}}{\sqrt{2 C(f_0)}} p(t, f | f_0) \leq C_1 e^{C_2 t} p_G(t, s)$$
Gaussian Estimates of European Options

We can directly translate the Gaussian bounds on the conditional probability $p'(t', s(K))$ into bounds on the implied volatility as

$$C(\tau, k) = \max(f_0 - K, 0) + \frac{\sqrt{C(K)C(f_0)}}{\sqrt{2}} \int_0^\tau p'(t', s(K))dt'$$

The large strike behavior of the implied volatility:

$$\frac{\sigma_{BS}(\tau, k)^2 \tau}{k} \sim_{k \to \infty} \Psi \left( \frac{-\frac{1}{2} \ln(C(K)) + \left( \int_{f_0}^K \frac{df'}{C(f')} \right)^2}{2\tau} \right)$$

If $s(K)$ is the leading term, $\sigma_{BS}(\tau, k) \sim_{k \to \infty} \frac{k}{f_0} \int_{f_0}^K \frac{df'}{C(f')}$

Short-time limit of the IV $\tau << 1$, $f_{av} = \frac{f_0 + K}{2}$:

$$\sigma_{BS}(\tau, k) = \frac{k}{\int_{f_0}^K \frac{df'}{C(f')}} \left( 1 + \frac{\tau}{3} \left( \frac{1}{8} \left( \frac{k}{\int_{f_0}^K \frac{df'}{C(f')}} \right)^2 + Q(f_{av}) \right) \right) + o(\tau^2)$$
Example: CEV model

For $0 \leq \beta < 1$, we have

$$\sigma_{BS}(k, \tau) \sim_{k \to \infty} \frac{k(1 - \beta)}{K^{1-\beta}}$$

and for $\beta = 1$, we have $\sigma_{BS}(\tau, k) \sim 1$. This result should be compared with the result obtained using the Lee moment formula:

$$\limsup_{k \to \infty} \frac{\sigma_{BS}(\tau, k)^2 \tau}{k} = 0$$

as all the moments exist.
Extensions and Questions

- Dupire Local volatility: Gaussian estimates of SE with potentials $Q(t, s)$ belonging to the non-autonomous Kato class.
- General SVMs: $R$ is not bounded from below by $-K, K > 0$ and/or $Q$ is unbounded. Need Generalization of the Li-Yau estimates.

$$Q_{LN-SABR}(a) = -\frac{a^2}{8(1 - \rho^2)}$$

$$Q_{LN-SABR}(a_{\text{min}}) = -\frac{\nu^2 k^2}{8(1 - \rho^2)} \rightarrow_{k \rightarrow \infty} \infty!$$
Numerical Methods in Finance

Available Numerical Methods in Finance

- PDE: Only when the number of assets is small.
- Monte-Carlo, Quasi Monte-Carlo: Euler, Milstein, stochastic Runge-Kutta, Ninomiya-Victoir, Cubature...

⇒ Combinatorial Hopf algebra. Similar structure in

- Renormalization in Quantum Field Theory: Connes-Kreimer
- Butcher Hopf algebra in deterministic Runge-Kutta methods
- Combinatorics of multi-zeta Riemann functions (polylogarithm): Zagier, Cartier, Kontsevitch, ..
Taylor-Stratonovich Expansion

\[ dx_t = \sum_{i=0}^{n} V_i \circ dW_t^i , \quad dW_t^0 \equiv dt \]

\[ f(x) = \sum_{(i_1, \ldots, i_k) \in A_r} V_{i_1} \ldots V_{i_k} f(x) \int_{0 < t_1 < \ldots < t_k < t} odW_{t_1}^i \ldots dW_{t_k}^i + R_r \]

with \( V_i = V_k^i \partial_k \).

▷ Graduation: \([dW^0] = 1\) and \([dW^i] = \frac{1}{2} \quad i = 1, \ldots, n\).

Replace the vector fields \( V_0, V_1, \ldots, V_d \) by letters \( \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_d \):

\[ X_{0,1}(\omega) = \sum_{(i_1, \ldots, i_k) \in A_r} \varepsilon_{i_1} \ldots \varepsilon_{i_k} \int_{0 < t_1 < \ldots < t_k < 1} d\omega^{i_1}(t_1) \ldots d\omega^{i_k}(t_k) \]

→ Nice element of a (graded) non-commutative Hopf algebra.
(Graded) Hopf algebra of words (1)

Define operations on $\mathcal{H}_r$

- a scalar multiplication $\times$
- sum $+$
- concatenation $\cdot$
- coproduct $\Delta : \mathcal{H}_r \rightarrow \mathcal{H}_r \otimes \mathcal{H}_r$
- unit $\varepsilon : \mathcal{H}_r \rightarrow k$
- counit $\eta$
- antipode $a : \mathcal{H}_r \rightarrow \mathcal{H}_r$
(Graded) Hopf algebra of words (2)

Hopf algebra $\mathcal{H}_r = \text{Algebra} + \text{Bialgebra} + \text{antipode}$

▷ Primitive elements: $\Delta(x) = x \otimes 1 + 1 \otimes x$
Rightarrow Lie algebra $\mathcal{G}_r$: $a(\mathcal{L}) = -\mathcal{L}$

▷ $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

▷ Group-like elements $\Delta(g) = g \otimes g$: $G_r = \exp(\mathcal{G}_r)$:
⇒ $a(g) = g^{-1}$

⇒ $\log(\exp) = 1$
Group-like element

Theorem 0.1 (Chen) \( X_{0,1}(\omega) \) is a group-like element of \( \mathcal{H}_r \)

\[
X_{0,1}(\omega) = \exp(\mathcal{L})
\]

for a primitive element (Lie polynomial) \( \mathcal{L} \) in \( \mathcal{G}_r \)
Yamato thm. and Carr-Madan Brownian rep.

Theorem 0.2 (Yamato)

\[ \mathcal{L}_t = tV_0 + \cdots + W_t^m V_m + \sum_{r=2} \sum_{J \in A_r} c_J W_t^J V^J \]

with the iterated Brownian integrals

\[ W_t^J = \int_{0 \leq t_1 < \cdots < t_m \leq t} dW_{u_1}^j \cdots dW_{u_m}^j, \ V^J = \left[ \cdots \left[ V_{j_1}, V_{j_2} \right] \cdots \left[ V_{j_m} \right] \right], \ J = (j_1, \cdots, j_m) \]

c_J are some constants.

- Carr-Madan: Classify models that can be written as a functional of a BM: \( \Rightarrow \) Abelian Lie algebra.

- Classify models that can be written as a functional of \( W_i \), \( \int_0^1 W_t^i dW_t^j \) (Levy area): \( V_0 = 0 \), \( \{V_i\} \) 1-step nilpotent Lie algebra: Heisenberg Lie algebra.
Discretization scheme à la Ninomiya-Victoir

Weak order discretization scheme

\[ \Pi[\exp(\varepsilon_0 + \frac{1}{2}\sum_{i=1}^{d}\varepsilon_i\varepsilon_i)] = \sum_{p=1}^{P} \lambda_p \Pi[\mathbb{E}[\exp(\mathcal{L}_p)]] \]

Weak order 3.0 at \(d = 1\) (Denuelle-PHL 2007):

\[ \mathcal{L}_\pm = e^{\frac{1}{24}[\varepsilon_1,\varepsilon_1,\varepsilon_0]}e^{\pm\frac{1}{2\sqrt{12}}[\varepsilon_1,\varepsilon_0]}e^{\Delta W_1^1\varepsilon_1+\varepsilon_0}e^{\pm\frac{1}{2\sqrt{12}}[\varepsilon_1,\varepsilon_0]}e^{\frac{1}{24}[\varepsilon_1,\varepsilon_1,\varepsilon_0]}e^{-\frac{1}{240}[\varepsilon_1,\varepsilon_1,\varepsilon_1,\varepsilon_0]} \]

with \(\lambda_\pm = \frac{1}{2}\). Use the Hopf algebra structure to simplify the computations!

\(^a\Pi: \text{truncation operator with respect to the grading.}\)
Book

Pierre Henry-Labordère

Analysis, Geometry, and Modeling in Finance
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