An Asymptotic Expansion Approach in Finance

Akihiko Takahashi
Graduate School of Economics, University of Tokyo

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Abstract

This presentation reviews an asymptotic expansion approach to numerical problems for pricing and hedging derivatives.

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1 Motivation

- Recently, there are many liquid products in OTC (over-the-counter) derivative markets:
  - For European type contracts, “Plain vanilla” call/put options (equity, foreign exchange), bond options (treasury), cap/floor, swaptions (interest rates), and average options (commodities).

- Simple and liquid contracts, but models are sometimes complicating due to smiles or/and long-term contracts.
  - Examples:
    - Average options (commodities):
      - Pricing under a model with stochastic volatility
    - Long-term foreign exchange options:
      - Pricing under a model with stochastic volatility, and term structures of domestic/foreign interest rates.

- It is difficult to obtain explicit (“closed-form”) formulas in many cases.

- On the other hand, trading liquid products requires fast computation of values and risk-indicators (“Greeks”), as well as calibration to the market prices.
  - Greeks (Delta, Gamma, Vega, Theta...)
  - Calibration: search parameters in a model so that the model reproduces liquid market prices.

- Fast and accurate approximation scheme is desired.

- An asymptotic expansion approach
  - This method is applicable in the unified manner to pricing of above products in the economy evolved by broad class of Itô processes; almost the same procedure can be applied.
2 Related Literatures

- Pricing Average options: Kunitomo and Takahashi(1992), Yoshida(1992b)

- Mathematical Validity based on Watanabe theory(Watanabe(1987)) in Malliavin calculus: Yoshida(1992a,b)
  In applications to finance; Takahashi(1995,1999), Kunitomo-Takahashi(2003),

  Takahashi-Saito(2003)


- Dynamic Optimal Portfolio based on Clark-Ocone formula:


- Pricing in Jump-diffusion and Levy processes: Kunitomo-Takahashi(2004),
  Takahashi(2007)

- Pricing in Market Model: Kawai(2003), Takahashi-Takehara(2007a,b,c)

- Pricing in Credit model: Muroi(2005)

- Pricing in Cross-currency models: Takahashi(1995), Osajima(2006),Takahashi-
  Takehara(2007a,b,c)

- Random limit(expansion around a log-normal distribution):
  e.g. Fournie-Lasry-Touzi(1997), Lütkebohmert(2004),
3 Outline of an Asymptotic Expansion Approach

- First, I consider a $d$-dimensional diffusion process $X^{(\epsilon)}$, which is the strong solution to the following stochastic differential equation:

$$dX_t^{(\epsilon)} = V_0(X_t^{(\epsilon)}, \epsilon)dt + \epsilon V(X_t^{(\epsilon)})dW_t; \quad X_0^{(\epsilon)} = x_0, \quad t \in [0, T],$$

where $W$ denotes a $m$-dimensional standard Wiener process and $\epsilon \in [0, 1]$ is a known parameter.

Suppose that coefficients $V_0 : \mathbb{R}^d \times [0, 1] \mapsto \mathbb{R}^d$, $V : \mathbb{R}^d \mapsto \mathbb{R}^d \times \mathbb{R}^m$ are smooth and satisfy regularity conditions.

- Next, suppose that a function $g : \mathbb{R}^d \mapsto \mathbb{R}$ to be smooth and all derivatives have polynomial growth orders. Then, for $\epsilon \downarrow 0$, $g(X_T^{(\epsilon)})$ has its asymptotic expansion:

$$g(X_T^{(\epsilon)}) = g_{0T} + \epsilon g_{1T} + \epsilon^2 g_{2T} + \epsilon^3 g_{3T} + o(\epsilon^3).$$

The justification of this and the following asymptotic expansions was provided in Watanabe(1987) and Yoshida(1992a,b) based on Malliavin Calculus. (See Kunitomo-Takahashi(2003), Takahashi(1995), Takahashi-Yoshida(2004,2005) in the context of finance.)

- This talk concentrates on applications of the method.
• The coefficients in the expansion, $g_0T$, $g_1T$, $g_2T\ldots$ can be obtained by Taylor’s formula and represented based on multiple Wiener-Ito integrals.

• In particular, let $D_t = \frac{\partial X_t^{(i)}}{\partial x}|_{x=0}$, $E_t = \frac{\partial^2 X_t^{(i)}}{\partial x^2}|_{x=0}$ and $F_t = \frac{\partial^3 X_t^{(i)}}{\partial x^3}|_{x=0}$ then $g_0T, g_1T, g_2T$ and $g_3T$ can be written as

\[
g_0T = g(X_T^{(0)}), \quad g_1T = \sum_{i=1}^{d} \partial_i g(X_T^{(0)}) D_T^i,
\]

\[
g_2T = \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j g(X_T^{(0)}) D_T^i D_T^j + \frac{1}{2} \sum_{i=1}^{d} \partial_i g(X_T^{(0)}) E_T^i,
\]

\[
g_3T = \frac{1}{6} \sum_{i,j,k=1}^{d} \partial_i \partial_j \partial_k g(X_T^{(0)}) D_T^i D_T^j D_T^k + \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j g(X_T^{(0)}) E_T^i D_T^j
\]

where $\partial_k = \frac{\partial}{\partial x_k}$.

• Here, $D_t^i$, $E_t^i$ and $F_t^i$, $i = 1, \ldots, d$ denote the $i$-th elements of $D_t$, $E_t$ and $F_t$ respectively. $D_t$, $E_t$ and $F_t$ are represented by

\[
D_t = \int_{0}^{t} Y_u Y_u^{-1} [\partial\epsilon V_0(X_u^{(0)},0) du + V(X_u^{(0)}) dW_u],
\]

\[
E_t = \int_{0}^{t} Y_u Y_u^{-1} \left( \sum_{j,k=1}^{d} \partial_j \partial_k V_0(X_u^{(0)},0) D_u^j D_u^k du + \partial^2_\epsilon V_0(X_u^{(0)},0) du \right) + 2 \sum_{j=1}^{d} \partial_j \partial V_0(X_u^{(0)},0) D_u^j du
\]

\[
+ 2 \sum_{j=1}^{d} \partial_j V(X_u^{(0)}) D_u^j dW_u \right),
\]

\[
F_t = \int_{0}^{t} Y_u Y_u^{-1} \left( \sum_{j,k=1}^{d} \partial_j \partial_k \partial \epsilon V_0(X_u^{(0)},0) D_u^j D_u^k du + 3 \sum_{j,k=1}^{d} \partial_j \partial_k V_0(X_u^{(0)},0) E_u^j D_u^k du
\]

\[
+ 3 \sum_{j,k=1}^{d} \partial_j \partial_\epsilon V_0(X_u^{(0)},0) D_u^j D_u^k du + 3 \sum_{j=1}^{d} \partial_j \partial_\epsilon V_0(X_u^{(0)},0) E_u^j du
\]

\[
+ 3 \sum_{j=1}^{d} \partial_j \partial_\epsilon^2 V_0(X_u^{(0)},0) D_u^j du + \partial_\epsilon^3 V_0(X_u^{(0)},0) du
\]

\[
+ 3 \sum_{j,k=1}^{d} \partial_j \partial_\epsilon V(X_u^{(0)}) D_u^j D_u^k dW_u + 3 \sum_{j=1}^{d} \partial_j V(X_u^{(0)}) E_u^j dW_u \right).
\]

where $Y$ denotes the solution to the differential equation;

\[
dY_t = \partial V_0(X_t^{(0)},0) Y_t dt; \quad Y_0 = I_d,
\]

where $\partial V_0$ denotes the $d \times d$ matrix whose $(j,k)$-element is $\partial_k V_0^j$, $V_0^j$ is the $j$-th element of $V_0$, and $I_d$ denotes the $d \times d$ identity matrix.
Next, normalize $g(X_T^{(c)})$ as

$$G^{(c)} = \frac{g(X_T^{(c)}) - g_0}{\epsilon}$$

for $\epsilon \in (0, 1]$.

Moreover, let

$$a_t = (\partial g(X_T^{(0)}))^T [Y_t Y_t^{-1} V(X_t^{(0)})]$$

and make the following assumption:

(Assumption 1) \( \Sigma_T = \int_0^T a_t a_t^T dt > 0. \)

Since $\Sigma_T$ is the variance of the random variable $g_1T$, which follows a normal distribution, Assumption 1 means the condition that the distribution of $g_1T$ does not degenerate. In application, it is easy to check this condition in most cases.
• Next, the characteristic function of $G^{(\epsilon)}$, $\psi_{G^{(\epsilon)}}(\xi)$ is approximated by

$$
\psi_{G^{(\epsilon)}}(\xi) = E[\exp(i\xi G^{(\epsilon)})]
= E[\exp(i\xi g_{1T})] + \epsilon(i\xi) E[\exp(i\xi g_{1T}) g_{2T}]
+ \epsilon^2(i\xi) E[\exp(i\xi g_{1T}) g_{3T}] + o(\epsilon^2).
$$

Moreover,

$$
\psi_{G^{(\epsilon)}}(\xi) = \exp \left( \frac{(i\xi)^2 \Sigma_T}{2} \right) + \epsilon(i\xi) E[\exp(i\xi g_{1T})] E[g_{2T}|g_{1T}] + \epsilon^2(i\xi) E[\exp(i\xi g_{1T})] E[g_{2T}^2|g_{1T}] + o(\epsilon^2).
$$

Note that $E[g_{2T}|g_{1T}]$, $E[g_{2T}^2|g_{1T}]$ and $E[g_{3T}|g_{1T}]$ are polynomial functions of $g_{1T}$.

• Then, the inversion of the approximated characteristic function provides an approximated probability density function of $G^{\epsilon}$:

$$
f_{G^{(\epsilon)}}(x) = n[x; 0, \Sigma_T] + \epsilon \left[ -\frac{\partial}{\partial x} \{p_2(x)n[x; 0, \Sigma_T]\} \right]
+ \epsilon^2 \left[ -\frac{\partial}{\partial x} \{p_3(x)n[x; 0, \Sigma_T]\} \right] + \frac{1}{2} \epsilon^2 \left[ \frac{\partial^2}{\partial x^2} \{p_{22}(x)n[x; 0, \Sigma_T]\} \right] + o(\epsilon^2),
$$

where $p_2(x) = E[g_{2T}|g_{1T} = x]$, $p_{22}(x) = E[g_{2T}^2|g_{1T} = x]$ and $p_3(x) = E[g_{3T}|g_{1T} = x]$. Also, $n[x; 0, \Sigma_T]$ denotes the density function of the normal distribution with mean 0 and variance $\Sigma_T$:

$$
n[x; 0, \Sigma_T] = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp \left\{ -\frac{x^2}{2\Sigma_T} \right\}.
$$
Now, given a smooth function $\phi : \mathbb{R} \mapsto \mathbb{R}$ of which all derivatives have polynomial growth orders.

Then, the expectation $E[\phi(G^{(c)}(G^{(c)})] I_B(G^{(c)})]$ has its asymptotic expansion as

$$E[\phi(G^{(c)}(G^{(c)})] I_B(G^{(c)})] = \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + o(\epsilon^2).$$

($B$ denotes a Borel set in $\mathbb{R}$, and $I_B(G^{(c)}) = 1$ if $G^{(c)} \in B$ and $I_B(G^{(c)}) = 0$ otherwise.)

Here, each term of the expansion can be expressed based on the expectations of polynomials of coefficients in the asymptotic expansion of $g(X_T)$ conditional on a normal random variable.

For example, $\Phi_0$, $\Phi_1$ and $\Phi_2$ are written as

$$\Phi_0 = \int_B \phi(x) n[x; 0, \Sigma_T] dx,$$

$$\Phi_1 = -\int_B \phi(x) \partial_x \{E[g_{2T}|g_{1T} = x] n[x; 0, \Sigma_T]\} dx,$$

$$\Phi_2 = \int_B \left( \frac{1}{2} \phi(x) \partial_x^2 \{E[g_{2T}^2|g_{1T} = x] n[x; 0, \Sigma_T]\} - \phi(x) \partial_x \{E[g_3|g_{1T} = x] n[x; 0, \Sigma_T]\} \right) dx,$$

- Note that $E[g_{2T}|g_{1T} = x]$, $E[g_{2T}^2|g_{1T} = x]$ and $E[g_{3T}|g_{1T} = x]$ are polynomial functions of $x$ and hence the computations of the expectations become easier; the formula of the conditional expectations given in the following can be used.

(Example) For a call option on a security with maturity $T$, its maturity price $g(X_T^\epsilon)$ and a strike price $K = g(X_0^\epsilon) - \epsilon y$ for arbitrary $y \in \mathbb{R}$, the payoff is expressed as

$$\max\{g(X_T^\epsilon) - K, 0\} = \epsilon \phi(G^\epsilon) I_B(G^\epsilon),$$

where $\phi(x) = (x + y)$ and $B = \{G^\epsilon \geq -y\}$. 

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I list up some formulas of conditional expectations used in asymptotic expansions. $W = \{(W^1_t, \cdots, W^m_t) : 0 \leq t \leq T\}$ denotes a $m$-dimensional Wiener process. Let $q_i : [0, T] \mapsto \mathbb{R}^m, i = 1, 2, 3, 4, 5$ are non-random functions and we define $\Sigma$ as

$$\Sigma = \int_0^T q_{1v} q_{1v} dv,$$

where $z'$ is the transpose of $z$. We assume that $0 < \Sigma < \infty$ and integrability in the following formulas.

Moreover, $H_n(x; \Sigma)$ denotes the Hermite polynomial of degree $n$:

$$H_n(x; \Sigma) := (-\Sigma)^k x^{2k} \frac{d^k}{dx^k} e^{-x^2/2\Sigma}.$$

For example, $H_0(x; \Sigma) = 1, H_1(x; \Sigma) = x, H_2(x; \Sigma) = x^2 - \Sigma, H_3(x; \Sigma) = x^3 - 3\Sigma x, H_4(x; \Sigma) = x^4 - 6\Sigma x^2 + 3\Sigma^2$.

1. $$E \left[ \int_0^T q_{2t}^2 dW_t \int_0^T q_{1v} dW_v = x \right] = \left( \int_0^T q_{2t}^2 dt \right) \frac{H_1(x; \Sigma)}{\Sigma}$$

2. $$E \left[ \int_0^T \int_0^t q_{2a} q_{2d} q_{3d} dW_t \int_0^T q_{1v} dW_v = x \right] =$$

$$\left( \int_0^T q_{2a} q_{1d} dv \right) \left( \int_0^T q_{3a} dW_s \right) \frac{H_2(x; \Sigma)}{\Sigma^2}$$

3. $$E \left[ \left( \int_0^T q_{2a} q_{2a} dW_v \right) \left( \int_0^T q_{3a} dW_s \right) \left( \int_0^T q_{1v} dW_v = x \right) \right] =$$

$$\left( \int_0^T q_{2a} q_{1d} dv \right) \left( \int_0^T q_{2a} q_{1d} dv \right) \frac{H_2(x; \Sigma)}{\Sigma^2}$$

4. $$E \left[ \int_0^T \int_0^t \int_0^s q_{2a} q_{2a} dW_s q_{3a} dW_s q_{4t} dW_t \int_0^T q_{1v} dW_v = x \right] =$$

$$\left( \int_0^T q_{4t} q_{1t} dv \right) \left( \int_0^T q_{3a} q_{1s} dv \right) \left( \int_0^T q_{2a} q_{1d} dv \right) \frac{H_3(x; \Sigma)}{\Sigma^3}$$

5. $$E \left[ \int_0^T \left( \int_0^t q_{2a} q_{2a} dW_v \right) \left( \int_0^t q_{3a} dW_s \right) q_{4t} dW_t \int_0^T q_{1v} dW_v = x \right] =$$

$$\left\{ \int_0^T \left( \int_0^t q_{2a} q_{1a} dv \right) \left( \int_0^t q_{3a} q_{1s} dv \right) q_{4t} q_{1t} dt \right\} \frac{H_3(x; \Sigma)}{\Sigma^3}$$

$$+ \left( \int_0^T \int_0^t q_{2a} q_{3a} dW_s q_{4t} q_{1t} dt \right) \frac{H_1(x; \Sigma)}{\Sigma}$$
6. \[
E \left[ \left( \int_0^T \int_0^t q_{2x}^* dW_s q_{3t}^* dW_t \right) \left( \int_0^T q_{1u}^* dW_u \right) \left| \int_0^T q_{1v}^* dW_v = x \right. \right] = \\
\left( \int_0^T q_{2x} q_{4u} \int_0^t q_{2x}^* q_{1s}^* ds dt \right) \left( \int_0^T q_{4u}^* q_{1v}^* du \right) \frac{H_2(x; \Sigma)}{\Sigma^3} + \\
\left( \int_0^T q_{3t} q_{1u} \int_0^t q_{2x}^* q_{4u}^* ds dt + \int_0^T q_{3t} q_{4u} \int_0^t q_{2x}^* q_{1s}^* ds dt \right) \frac{H_1(x; \Sigma)}{\Sigma}
\]

7. \[
E \left[ \left( \int_0^T \int_0^t q_{2x}^* dW_s q_{3t}^* dW_t \right) \left( \int_0^T \int_0^r q_{4u}^* dW_u q_{5r}^* dW_r \right) \left| \int_0^T q_{1v}^* dW_v = x \right. \right] = \\
\left( \int_0^T q_{2x} q_{4u} \int_0^t q_{2x}^* q_{5r}^* dW_r \int_0^t q_{4u}^* q_{1u}^* dW_r \right) \frac{H_2(x; \Sigma)}{\Sigma^2} + \\
\int_0^T \int_0^t q_{2x}^* q_{4u}^* dW_s q_{3t}^* q_{5s}^* dt
\]

8. \[
E \left[ \left( \int_0^T q_{2x}^* dW_s \right) \left( \int_0^T q_{3s}^* dW_s \right) \left( \int_0^T \int_0^r q_{4u}^* dW_u q_{5s}^* dW_r \right) \left| \int_0^T q_{1v}^* dW_v = x \right. \right] = \\
\left( \int_0^T q_{2x} q_{3t} q_{4u} \int_0^t q_{3s}^* q_{1s}^* ds dt \right) \left( \int_0^T q_{4u}^* q_{5r}^* \int_0^t q_{4u}^* q_{1u}^* dW_r \right) \frac{H_4(x; \Sigma)}{\Sigma^4} + \\
\left( \int_0^T q_{2x} q_{5s} q_{3t} q_{4u} \int_0^t q_{3s}^* q_{1s}^* ds dt \right) + \left( \int_0^T q_{2x} q_{4u} q_{5r} q_{3t} q_{4u} \int_0^t q_{3s}^* q_{1s}^* ds dt \right) \frac{H_2(x; \Sigma)}{\Sigma^2} + \\
\int_0^T q_{2x} \int_0^t q_{3s}^* q_{4u}^* ds dt + \int_0^T q_{2x} q_{5s} q_{3t} q_{4u} \int_0^t q_{4u}^* q_{1u}^* dW_r
\]
4 Average Option

- Shiraya-Takahashi[2008]

4.1 $\lambda$-SABR model

- Examples of Stochastic Volatility Models
  Heston[1993], SABR (Hagan-Kumar-Lesniewski-Woodward[2002]), $\lambda$-SABR(Labordere[2005]): Pricing European Plain Vanilla Options.

- I show an approximation of an average option under $\lambda$-SABR model following Shiraya-Takahashi[2008].

- $\lambda$-SABR model

  The dynamics of the underlying asset’s price $S(t)$, $t \in [0, T]$ is described by a two-dimensional diffusion process $\{(S, \sigma) : (S(t), \sigma(t)), 0 \leq t \leq T\}$ that is a solution of the following stochastic differential equation under the equivalent martingale measure (EMM):

  $$
  S(t) = S(0) + \alpha \int_0^t S(u)du + \int_0^t \sigma(u)S(u)^{\beta}dW_1(u)
  $$

  $$
  \sigma(t) = \sigma(0) + \int_0^t \lambda(\theta - \sigma(u))du + \int_0^t \nu_1 \sigma(u)dW_1(u) + \int_0^t \nu_2 \sigma(u)dW_2(u)
  $$

  Here, $S(0)$ and $\sigma(0)$ are positive constants; $\beta \in [0, 1]$ is a constant; $\lambda$ and $\theta$ are nonnegative constants; $\alpha$ is a constant; $\nu_1 = \rho \nu$, $\nu_2 = (\sqrt{1 - \rho^2})\nu$, where $\nu \geq 0$, $\rho \in [-1, 1]$ are constants; $W = (W_1, W_2)$ is a two-dimensional Wiener Process.

- An average call option’s payoff with maturity $T$ and strike price $K$:

  $$
  C(T) = \max\{X(T) - K, 0\},
  $$

  where

  $$
  X(T) = \frac{1}{T} \int_0^T S(t)dt.
  $$

- Then, $C(0)$, the price at $t = 0$ of the average call option is expressed as:

  $$
  C(0) = e^{-rT}E[C(T)],
  $$

  where $r$ denotes the risk-free rate that is a nonnegative constant.
4.2 Asymptotic Expansion of an Average Call Option

- $\lambda$-SABR model is rewritten in the asymptotic expansion’s framework:
  
  For $\epsilon \in [0, 1]$, 
  
  $$
  S^{(\epsilon)}(t) = S(0) + \alpha \int_0^t S^{(\epsilon)}(u) du + \epsilon \int_0^t \sigma^{(\epsilon)}(u) S^{(\epsilon)}(u)^3 dW_1(u) \quad (1)
  $$
  
  $$
  \sigma^{(\epsilon)}(t) = \sigma(0) + \int_0^t \lambda(\theta - \sigma^{(\epsilon)}(u)) du 
  + \epsilon \left( \int_0^t \nu_1 \sigma^{(\epsilon)}(u) dW_1(u) + \int_0^t \nu_2 \sigma^{(\epsilon)}(u) dW_2(u) \right)
  $$

- An average call option’s payoff with maturity $T$ and strike price $K$ where $K = X^{(0)}_T - \epsilon y$ for arbitrary $y \in \mathbb{R}$:
  
  $$
  C(T) = \max \left\{ X^{(\epsilon)}(T) - K, 0 \right\},
  $$

  where
  
  $$
  X^{(\epsilon)}(T) := \frac{1}{T} \int_0^T S^{(\epsilon)}(t) dt.
  $$
• Then, the asymptotic expansion of $X^\alpha(T)$ when $\alpha \neq 0$ is given by:

$$X^\alpha(T) = X^{(0)}(T) + \epsilon X^{(1)}(T) + \epsilon^2 X^{(2)}(T) + \epsilon^3 X^{(3)}(T) + o(\epsilon^3),$$

where $X^{(0)}(T) = X^\alpha(T)|_{\epsilon = 0}$, $X^{(k)}(T) = \frac{1}{k!} \frac{\partial^k X^\alpha(T)}{\partial \epsilon^k}|_{\epsilon = 0}$, $k = 1, 2, 3$ and they are given as follows:

$$X^{(0)}(T) = e^{\alpha T} - 1 S(0),$$

$$X^{(1)}(T) = \int_0^T f_{11}(s') dW(s),$$

$$X^{(2)}(T) = \sum_{i=1}^2 \int_0^T \int_0^s f_{21}(u') dW(u) g_{22}(s') dW(s),$$

$$X^{(3)}(T) = \left( \sum_{i=1}^3 \int_0^T \int_0^s f_{31}(v') dW(v) g_{32}(u') dW(u) h_{33}(s') dW(s) \right.$$

$$\left. + \sum_{i=1}^2 \int_0^T \left( \int_0^s g_{41}(u') dW(u) \right) \left( \int_0^s f_{42}(u') dW(u) \right) h_{44}(s') dW(s) \right),$$

where $x'$ denotes the transpose of $x$.

$f_{11}(t)$, $f_{21}(t)(i = 1, 2)$, $f_{31}(t)(i = 1, 2, 3)$, $f_{41}(i = 1, 2)$, $g_{22}(t)(i = 1, 2)$, $g_{32}(t)(i = 1, 2, 3)$, $h_{33}(t)(i = 1, 2, 3)$, and $h_{44}(t)(i = 1, 2)$ are given as follows:

$$f_{21}(t) = f_{31}(t) = f_{41}(t) = \begin{pmatrix} e^{-\alpha t} (S(0)e^{\alpha t})^\beta \left( \theta + (\sigma(0) - \theta)e^{-\lambda t} \right) \end{pmatrix},$$

$$f_{11}(t) = \frac{e^{\alpha (T-t)} - 1}{\alpha T} e^{\alpha t} f_{21}(t),$$

$$f_{22}(t) = f_{32}(t) = f_{42}(t) = \begin{pmatrix} \nu_1 (\theta e^{\lambda t} + \sigma(0) - \theta) \\ \nu_2 (\theta e^{\lambda t} + \sigma(0) - \theta) \end{pmatrix},$$

$$g_{31}(t) = \begin{pmatrix} \beta (S(0)e^{\alpha t})^{\beta-1} \left( \theta + (\sigma(0) - \theta)e^{-\lambda t} \right) \end{pmatrix},$$

$$h_{32}(t) = \frac{e^{\alpha (T-t)} - 1}{\alpha T} g_{31}(t),$$

$$g_{21}(t) = h_{31}(t) = h_{32}(t),$$

$$g_{32}(t) = \begin{pmatrix} (S(0)e^{\alpha t})^\beta e^{-(\alpha + \lambda)t} \end{pmatrix},$$

$$g_{22}(t) = h_{33}(t) = \frac{e^{\alpha (T-t)} - 1}{\alpha T} g_{32}(t),$$

$$g_{33}(t) = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix},$$

$$h_{41}(t) = \begin{pmatrix} e^{\alpha (T-t-1)} \beta (\beta - 1) (S(0)e^{\alpha t})^{\beta-2} \left( \theta + (\sigma(0) - \theta)e^{-\lambda t} \right) e^{\alpha t} \end{pmatrix},$$

$$h_{42}(t) = \begin{pmatrix} e^{\alpha (T-t-1)} \beta (S(0)e^{\alpha t})^{\beta-1} e^{-\lambda t} \end{pmatrix},$$

(2)

**Remark 1** When $\alpha = 0$, the asymptotic expansion of $X^\alpha(T)$ is obtained as $\alpha \to 0$ above.
Then, \( C(0) \), the price at \( t = 0 \) of the average call option is expressed as follows:

\[
C(0) = e^{-rT} \left( \epsilon \left( y \int_{-y}^{\infty} n[x; 0, \Sigma] dx + \int_{-y}^{\infty} x n[x; 0, \Sigma] dx \right) \\
+ \epsilon^2 \int_{-y}^{\infty} \mathbb{E} \left[ X^{(2)}(T) | X^{(1)}(T) = x \right] n[x; 0, \Sigma] dx \\
+ \epsilon^3 \left( \int_{-y}^{\infty} \mathbb{E} \left[ X^{(3)}(T) | X^{(1)}(T) = x \right] n[x; 0, \Sigma] dx \\
+ \frac{1}{2} \mathbb{E} \left[ \left( X^{(2)}(T) \right)^2 | X^{(1)}(T) = -y \right] n[y; 0, \Sigma] \right) \\
+ o(\epsilon^3),
\]

where \( \Sigma = \int_0^T |f_{11}(s)|^2 ds \) and \( n[x; 0, \Sigma] = \frac{1}{\sqrt{2\pi\Sigma}} \exp \left\{ -\frac{x^2}{2\Sigma} \right\} \).
• More concrete approximation of the price is obtained by formulas of the conditional expectations:

**Theorem 1** Suppose that the underlying asset price follows (1) for $\epsilon \in (0, 1]$ under the equivalent martingale measure. Then, the asymptotic expansion up to the $\epsilon^3$-order of $C(0)$, the price of a average call option at the contract date with the maturity date $T$ and the strike price $K$ where $K = X_T^{(0)} - \epsilon y$ for arbitrary $y \in \mathbb{R}$ is given by:

$$C(0) = e^{-rT} \left[ \epsilon \left\{ yN \left( \frac{y}{\sqrt{\Sigma}} \right) + \Sigma n[y; 0, \Sigma] \right\} \right. $$

$$+ \epsilon^2 \int_{-\infty}^{\infty} C_1 H_2(x; \Sigma) \frac{n[x; 0, \Sigma]}{\Sigma^2} dx $$

$$+ \epsilon^3 \left\{ \int_{-\infty}^{\infty} C_2 \frac{H_3(x; \Sigma)}{\Sigma^3} n[x; 0, \Sigma] dx + C_3 n[y; 0, \Sigma] \right. $$

$$+ \left( C_4 \frac{H_4(y; \Sigma)}{\Sigma^4} + C_5 \frac{H_2(y; \Sigma)}{\Sigma^2} + C_6 \right) n[y; 0, \Sigma] \right\} $$

$$+ o(\epsilon^3), \quad (3)$$

where $N(x)$ denotes the distribution function of a standard normal distribution. Moreover,

$$C_1 = \sum_{i=1}^{2} \int_{0}^{T} f_{11}(s)^{1} g_{2i}(s) \int_{0}^{u} f_{11}(u)^{1} f_{2i}(u) duds$$

$$C_2 = \sum_{i=1}^{3} \int_{0}^{T} f_{11}(s)^{1} h_{3i}(s) \int_{0}^{u} f_{11}(u)^{1} g_{3i}(u) \int_{0}^{u} f_{11}(v)^{1} f_{3i}(v) dv dus$$

$$+ \frac{1}{2} \sum_{i=1}^{2} \int_{0}^{T} f_{11}(s)^{1} h_{3i}(s) \int_{0}^{u} f_{11}(u)^{1} g_{3i}(u) du \int_{0}^{u} f_{11}(v)^{1} f_{3i}(u) dv dus$$

$$C_3 = \sum_{i=1}^{3} \int_{0}^{T} f_{11}(s)^{1} h_{4i}(s) \int_{0}^{u} g_{4i}(u)^{1} f_{4i}(u) duds$$

$$C_4 = \sum_{i=1}^{3} \int_{0}^{T} f_{11}(s)^{1} k_{5i}(s) \int_{0}^{u} f_{11}(s)^{1} h_{5i}(u) duds$$

$$C_5 = \sum_{i=1}^{3} \int_{0}^{T} f_{11}(s)^{1} k_{5i}(s) \int_{0}^{u} f_{11}(u)^{1} g_{5i}(u) \int_{0}^{u} f_{5i}(v)^{1} h_{5i}(v) dv dus$$

$$+ \int_{0}^{T} f_{11}(s)^{1} g_{5i}(s) \int_{0}^{u} f_{11}(u)^{1} k_{5i}(u) \int_{0}^{u} f_{5i}(v)^{1} h_{5i}(v) dv dus$$

$$+ \int_{0}^{T} f_{11}(s)^{1} g_{5i}(s) \int_{0}^{u} f_{5i}(u)^{1} k_{5i}(u) \int_{0}^{u} f_{11}(v)^{1} h_{5i}(v) dv dus$$

$$+ \int_{0}^{T} g_{5i}(s)^{1} k_{5i}(s) \int_{0}^{u} f_{11}(u)^{1} h_{5i}(u) du \int_{0}^{u} f_{11}(v)^{1} f_{5i}(u) dv dus$$

$$+ \int_{0}^{T} f_{11}(s)^{1} k_{5i}(s) \int_{0}^{u} g_{5i}(u)^{1} h_{5i}(u) \int_{0}^{u} f_{11}(v)^{1} f_{5i}(v) dv dus$$

$$C_6 = \sum_{i=1}^{3} \int_{0}^{T} g_{5i}(s)^{1} k_{5i}(s) \int_{0}^{u} f_{11}(u)^{1} h_{5i}(u) duds.$$  

Here, $f_{11}(t)$, $f_{2i}(t)(i = 1, 2)$, $f_{3i}(t)(i = 1, 2, 3)$, $f_{4i}(i = 1, 2)$; $g_{2i}(t)(i =
4.3 Heston model

- The underlying asset price $S$:
  \begin{align*}
  dS(t) &= \alpha S(t)dt + S(t)\sqrt{V(t)}dW^1(t), \\
  dV(t) &= \kappa(\theta - V(t))dt + \nu_1 \sqrt{V(t)}dW^1(t) + \nu_2 \sqrt{V(t)}dW^2(t),
  \end{align*}
  where $\nu \ge 0$, $\nu_1 = \rho \nu$, $\nu_2 = (\sqrt{1 - \rho^2})\nu$ and $\rho \in [-1, 1]$.

- An asymptotic expansion of an average option price is obtained in the similar way as in $\lambda$-SABR model. In particular, changing the coefficients of the equations in theorem 1 to the following provides an approximation formula under Heston model.
  \begin{align*}
  f_{21}(t) &= f_{31}(t) = g_{31}(t) = f_{41}(t) = \begin{pmatrix}
  \sqrt{\theta + (V(0) - \theta)e^{-\kappa t}} \\
  0
  \end{pmatrix}, \\
  f_{11}(t) &= g_{21}(t) = h_{31}(t) = 2h_{32}(t) = \frac{e^{\alpha(T-t)} - 1}{\alpha T} S(0) f_{21}(t), \\
  f_{22}(t) &= f_{32}(t) = f_{42}(t) = g_{41}(t) = g_{42}(t) = \begin{pmatrix}
  \sqrt{\theta + (V(0) - \theta)e^{-\kappa t}} \nu_1 e^{\kappa t} \\
  \sqrt{\theta + (V(0) - \theta)e^{-\kappa t}} \nu_2 e^{\kappa t}
  \end{pmatrix}, \\
  g_{32}(t) &= \left( \frac{e^{-\kappa t}}{\sqrt{\theta + (V(0) - \theta)e^{-\kappa t}}} \right) ; g_{22}(t) = 2h_{33}(t) = h_{41}(t) = \frac{e^{\alpha(T-t)} - 1}{2\alpha T} S(0) g_{32}(t), \\
  g_{43}(t) &= \left( \frac{e^{-\kappa t}}{\sqrt{\theta + (V(0) - \theta)e^{-\kappa t}}} \right) ; h_{42}(t) = \begin{pmatrix}
  -\left(e^{\alpha(T-t)} - 1\right) e^{-2 \kappa t} S(0) \\
  8\alpha T \left(\sqrt{\theta + (V(0) - \theta)e^{-\kappa t}}\right)^3
  \end{pmatrix}.
  \end{align*}

Remark 2 The following relation may be useful to compute the integrals on the right hand side of the equation (3).

$$
\int_{-y}^{\infty} \frac{1}{\sqrt{\pi}} H_k(x; \Sigma) n[x; 0, \Sigma] dx = \frac{1}{\sqrt{\pi}} H_{k-1}(-y; \Sigma) n[y; 0, \Sigma] \quad (k \ge 1)
$$
4.4 Numerical Examples

- $\lambda$-SABR model
- Benchmark values:

Monte Carlo Simulation (5 million trials, 2000 time steps)

$S_0 = 100$, $\alpha = r = 0$, $\epsilon = 1$, and $\sigma(0)$ are determined such that the coefficient of the diffusion term is equivalent to that of log-normal process at time 0: for example in Case i where the log-normal volatility is 30%, $\sigma(0)S_0^{0.5} = 0.3S_0$.

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Figure 1: Approximated Density of $X(T)$ (Case ix)
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Computation with Calibrated Parameters

- \( F(t, T) \) denotes the futures price at time \( t \) with maturity \( T \).
- The process of \( F(t, T) \) under the Equivalent Maringale Measure (EMM):
  \[
  dF(t, T) = \mu(t, T)F(t, T)\sigma(t)dt + \sigma(t)dW(t),
  \]
  \[
  d\sigma(t) = \lambda(\theta - \sigma(t))dt + \nu_1\sigma(t)dW_1(t) + \nu_2\sigma(t)dW_2(t),
  \]
  where \( \mu(t, T) := 1 \{ t < T \} \).

- The call price at time 0 with maturity \( T^* \leq T \):
  \[
  C(0) = e^{-rT^*}E\left[ \max \left\{ \frac{1}{T} \int_0^{T^*} F(t, T)dt - K, 0 \right\} \right].
  \]

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Table 3:
**Average Option: $\lambda$-SABR model**

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21
## Average Option: Heston model

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5 Currency Option under a Libor Market Model of Interest Rates and a Stochastic Volatility of a Spot Exchange Rate

Takahashi and Takehara[2007]

5.1 Cross-Currency Libor Market Models

- Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$ be a filtered probability space satisfying the usual conditions, $\{\tilde{W}_t\}_{0 \leq t \leq T}$ be a $D$-dimensional $\{\mathcal{F}_t\}$-standard Wiener process.

- The following notations are firstly prepared.
  - $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = T$: Resetting time of forward Libor rates
  - $\{f_{dj}(t)\}_{j=0}^N$: Domestic forward Libor rates on the period $[T_j, T_{j+1}]$
  - $\{f_{fj}(t)\}_{j=0}^N$: Foreign forward Libor rates on the period $[T_j, T_{j+1}]$
  - $P_d(t,s)$: A domestic zero coupon bond with maturity $s > t$
  - $P_f(t,s)$: A domestic zero coupon bond with maturity $s > t$
  - $S(t)$: A spot exchange rate

- Then, I consider the pricing problem

$$
V(0) = \frac{P_d(0,T)\mathbb{E}^{P_d}[\max\{S(T) - K, 0\}]}{P_d(0,T)\mathbb{E}^{P_d}[\max\{F_T(T) - K, 0\}]} 
$$

where

$$
F_T(t) := \frac{P_f(t,T)}{P_d(t,T)} S(t) 
$$

is the forward exchange rate(remainder forex) with maturity $T$ and $P_d$ is the EMM whose numeraire is given by $P_d(0,T) = P_d(0,T_{N+1})$.

- First, under $P_d$, $\{f_{dj}(t)\}$ are assumed to follow

$$
f_{dj}(t) = f_{dj}(0) + \sum_{i=j+1}^{N} \int_0^t f_{dj}(u) \gamma_{dj}(u) f_{dj}(u) du + \int_0^t f_{dj}(u) \gamma_{dj}(u) dW_u, 
$$

where $\{W_t\}$ is a $D$-dimensional standard Wiener process under $P_d$, $g_{dj}(t) := \frac{-\gamma_{dj}(t)}{1+\tau_{fj}(t)} f_{dj}(t)$, $\tau_j := T_{j+1} - T_j$ and $\{\gamma_{dj}(t)\}_{j=0}^N$ are deterministic functions of $t$.

- Similarly, under the EMM $P_f$ whose numeraire is $P_f(0,T)$, $\{f_{fj}(t)\}$ are assumed to follow

$$
f_{fj}(t) = f_{fj}(0) + \sum_{i=j+1}^{N} \int_0^t g_{fj}(u) \gamma_{fj}(u) f_{fj}(u) du + \int_0^t f_{fj}(u) \gamma_{fj}(u) dW^f_u, 
$$

where $\{W^f_t\}$ is a $D$-dimensional standard Wiener process under $P_f$, $g_{fj}(t) := \frac{-\gamma_{fj}(t)}{1+\tau_{fj}(t)} f_{fj}(t)$ and $\{\gamma_{fj}(t)\}_{j=0}^N$ are also deterministic functions of $t$. 

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Finally, it is assumed that under the domestic risk-neutral measure the spot forex \( S(t) \) and its volatility \( \sigma(t) \) follow

\[
S(t) = S(0) + \int_0^t (r_d(u) - r_f(u)) S(u) du + \int_0^t f(\sigma(u)) \sigma' S(u) d\tilde{W}_u
\]

\[
\sigma(t) = \sigma(0) + \int_0^t \bar{\mu}(u, \sigma(u)) du + \int_0^t \omega'(u, \sigma(u)) d\tilde{W}_u
\]

where \( \{\tilde{W}_t\} \) is a \( D \)-dimensional standard Wiener process under that measure, and \( \bar{\sigma} \) is some constant vector satisfying \( \|\bar{\sigma}\| = 1 \).

Unifying (7), (8) and (10) into the processes under the same EMM \( \mathbb{P}_N^d \), \( \{f_{dj}(t)\} \), \( \{f_{fj}(t)\} \) and \( \sigma(t) \) are the solutions of the following graded system of stochastic differential equations (SDEs):

\[
f_{d_j}(t) = f_{d_j}(0) + \sum_{i=j+1}^N \int_0^t g_{ai}^0(u) \gamma_{d_i}(u) f_{d_i}(u) du + \int_0^t f_{d_j}(u) \gamma_{d_j}(u) dW_u,
\]

\[
\sigma(t) = \sigma(0) + \int_0^t \mu(u, \sigma(u), \{f_{d_j}(u)\}) du + \int_0^t \omega'(u, \sigma(u)) dW_u,
\]

\[
f_{f_j}(t) = f_{f_j}(0) - \sum_{i=0}^j \int_0^t g_{ai}^0(u) \gamma_{f_i}(u) f_{f_i}(u) du + \sum_{i=0}^N \int_0^t g_{ai}^0(u) \gamma_{f_j}(u) f_{f_j}(u) du - \int_0^t f(\sigma(u)) \bar{\sigma}' \gamma_{f_j}(u) f_{f_j}(u) du + \int_0^t f_{f_j}(u) \gamma_{f_j}(u) dW_u,
\]

where

\[
\mu(t, \sigma(t), \{f_{d_j}(t)\}) := \tilde{\mu}(t, \sigma(t)) + \omega'(t, \sigma(t)) \sum_{i=0}^N g_{ai}^0(t).
\]

Then, the forward forex \( F_T(t) \) is a martingale under \( \mathbb{P}_N^d \) and follows

\[
F_T(t) = F_T(0) + \int_0^t \sigma_F'(u) F(u) dW_u
\]

where

\[
\sigma_F(t) := \sum_{j=0}^N (g_{f_j}^0(t) - g_{d_j}^0(t)) + f(\sigma(t)).
\]
5.2 An Asymptotic Expansion of the Forward Forex

According to the framework in the previous section, I expand the underlying forward forex $F_T(t)$.

- The underlying system of (11), (12), (13) and (14) are rewritten in an asymptotic expansion’s framework with $\epsilon \in [0, 1]$:

\[
\begin{align*}
    f_{dj}^{(c)}(t) &= f_{dj}(0) + \epsilon \sum_{i=j+1}^{N} \int_{0}^{t} g^{0,(c)}_{dj}(u) \gamma_{dj}(u) f_{dj}^{(c)}(u) du + \epsilon \int_{0}^{t} f_{dj}^{(c)}(u) \gamma_{dj}(u) dW_u, \\
    \sigma(t) &= \sigma(0) + \int_{0}^{t} \mu(u, \sigma^{(c)}(u), \{f_{dj}^{(c)}(u)\}) du + \epsilon \int_{0}^{t} \omega(u, \sigma^{(c)}(u)) dW_u, \ \\
    f_{fj}^{(c)}(t) &= f_{fj}(0) - \epsilon^2 \sum_{i=0}^{j} \int_{0}^{t} g_{fj}^{0,(c)}(u) \gamma_{fj}(u) f_{fj}^{(c)}(u) du + \epsilon^2 \sum_{i=0}^{N} \int_{0}^{t} g_{fj}^{0,(c)}(u) \gamma_{fj}(u) f_{fj}^{(c)}(u) du \\
    &\quad - \epsilon^2 \int_{0}^{t} f(\sigma^{(c)}(u)) \sigma^{(c)}(u) \gamma_{fj}(u) f_{fj}^{(c)}(u) du + \epsilon \int_{0}^{t} f_{fj}^{(c)}(u) \gamma_{fj}(u) dW_u, \\
    F_T^{(c)}(t) &= F_T(0) + \epsilon \int_{0}^{t} \sigma_{F}^{(c)}(u) F^{(c)}(u) du dW_u
\end{align*}
\]

where

\[
\begin{align*}
    \sigma_{F}^{(c)}(t) &:= \sum_{j=0}^{N} \left( g_{fj}^{0,(c)}(t) - g_{dj}^{0,(c)}(t) \right) + f(\sigma^{(c)}(t)), \\
    g_{dj}^{0,(c)}(t) &:= \frac{-\tau_{fj} f_{dj}^{(c)}(t)}{1 + \tau_{fj} f_{dj}^{(c)}(t)} \gamma_{dj}(t), \\
    g_{fj}^{0,(c)}(t) &:= \frac{-\tau_{fj} r_{fj}^{(c)}(t)}{1 + \tau_{fj} r_{fj}^{(c)}(t)} \gamma_{fj}(t).
\end{align*}
\]

- Then, the explicit expansion of $F_T^{(c)}(T)$ up to the third order of $\epsilon$ can be obtained by the methodolgy already presented in the previous section:

\[
F_T^{(c)}(T) = F_T(0) + \epsilon A_T^{1} + \epsilon^2 A_T^{2} + \epsilon^3 A_T^{3} + o(\epsilon^3).
\]

- Assume $\epsilon \in (0, 1]$ and let $K_\epsilon := F_T(0) - \epsilon y$ for arbitrary $y \in \mathbb{R}$. Then from the expansion above, under the condition $\Sigma_T > 0$ the price of the call currency option with maturity $T$ and strike rate $K_\epsilon$ is given by

\[
V(0; T, K) = P^{D}(0, T) \left[ \epsilon \int_{-\infty}^{\infty} (x + y)n[x; 0, \Sigma_T] dx + \epsilon^2 \int_{-\infty}^{\infty} E[A_T^2 | A_T = x] n[x; 0, \Sigma_T] dx \\
+ \epsilon^3 \int_{-\infty}^{\infty} E[A_T^3 | A_T = x] n[x; 0, \Sigma_T] dx + \frac{\epsilon^3}{2} E[(A_T^2)^2 | A_T = -y] n[y; 0, \Sigma_T] \right] + o(\epsilon^3)
\]

(21)

where $\Sigma_T := \int_{0}^{T} \|\sigma^{(c)}(u)\|^2 du$ and $n[x; m, v] := \frac{1}{\sqrt{2\pi v}} \exp \left( -\frac{(x-m)^2}{2v} \right)$.

- Note that since the conditional expectations such as $E[A_T^2 | A_T = x]$ are given by linear combinations of Hermite polynomials of $x$ with variance $\Sigma_T$, this pricing formula can be solved explicitly.
Table 4: Initial domestic/foreign forward interest rates and their volatilities

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<tr>
<th>Case (i)</th>
<th>f_d</th>
<th>γ_d</th>
<th>f_f</th>
<th>γ_f</th>
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<td>0.05</td>
<td>0.2</td>
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<td>Case (iii)</td>
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5.3 Numerical Examples

- In this subsection I confirm the accuracy of the method in this cross-currency framework. Especially, the process of the volatility of the spot forex is specified by

\[
\begin{align*}
\tilde{\mu}(t, x) &= \kappa(\theta - x), \\
\omega(t, x) &= \omega \sqrt{t},
\end{align*}
\]

with some constants \(\kappa, \theta\) and a constant vector \(\omega\).

- Moreover, the parameters are set as follows. \(D = 4, \epsilon = 1, \sigma(0) = \theta = 0.1,\) and \(\kappa = 0.1, \omega = \omega^* V_\sigma\) where \(\omega^* = 0.1\) and \(V_\sigma\) denotes a four dimensional constant vector given below.

- For \(f(x)\) it is assumed to be \(f(x) = \min\{x, K_{\text{max}}\}\) with some constant \(K_{\text{max}}\), while the approximation with an asymptotic expansion is exercised with replacement of \(f(x)\) by \(\tilde{f}(x) := x\) to avoid complexity in calculation; \(K_{\text{max}}\) is set to be \(10^6\) in numerical examples below.

- I further suppose that initial term structures of domestic and foreign forward interest rates are flat, and their volatilities have flat structures and are constant over time: that is, for all \(j, f_d(j) = f_d\), \(f_f(j) = f_f\), \(\gamma_d(j) = \gamma_d V_d 1_{\{t \leq T_j\}}(t)\) and \(\gamma_f(j) = \gamma_f V_f 1_{\{t \leq T_j\}}(t)\). Here, \(\gamma_d\) and \(\gamma_f\) are constant scalars, and \(V_d\) and \(V_f\) denote four dimensional constant vectors.

- Moreover, given a correlation matrix \(C\) among all four factors, I can determine the constant vectors \(V_d, V_f, V_S\) and \(V_\sigma\) to satisfy \(\|V_d\| = \|V_f\| = \|V_S\| = \|V_\sigma\| = 1\) and \(V'V = C\) where \(V := (V_d, V_f, V_S, V_\sigma)\), and \(V_S \equiv \bar{\sigma}\); \(\bar{\sigma}\) was defined right after the equation (9).

- Additionally, I make an assumption that \(\gamma_{dt(t)-1}(t)\) and \(\gamma_{ft(t)-1}(t)\), volatilities of the domestic and foreign interest rates applied to the period from \(t\) to the next fixing date \(T_{t(n(t))}\), are set to be zero for arbitrary \(t \in [t, T_{t(n(t))}]\) where \(n(t) := \min\{j; t \leq T_j\}\).

- Finally, for correlations the following four sets of parameters are considered:
  - “Corr.1”: All the factors are independent:
  - “Corr.2”: There exists only the correlation of -0.5 between the spot exchange rate and its volatility (i.e. \(V_d'V_\sigma = -0.5\)) while there are no correlations among the others:
  - “Corr.3”: The correlation between interest rates and the spot exchange rate are allowed while there are no correlations among the others: the correlation between domestic ones and the spot forex is 0.5(\(V_d'V_S = 0.5\)) and the correlation between foreign ones and the spot forex is -0.5(\(V_f'V_S = -0.5\)).
“Corr.4”: Correlations among most factors are considered; \( V_d V_f = 0.3 \) between the domestic and foreign interest rates; \( V_d V_S = 0.5 \), \( V_f V_S = -0.5 \) between interest rates and the spot forex; and \( V_S V_f = -0.5 \) between the spot forex and its volatility.

- Monte Carlo simulations are exercised with the following settings.
  - Using Euler-Maruyama scheme with time step of 0.05
  - Also using the Antithetic Variable Method
  - 1,000,000 trials

<table>
<thead>
<tr>
<th>T</th>
<th>Int.</th>
<th>Corr.</th>
<th>( K/F_0 )</th>
<th>Estimated Values</th>
<th>Differences</th>
<th>Relative Differences</th>
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<td>1st 2nd 3rd</td>
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Table 5: A comparison of estimators by asymptotic expansions to by monte carlo simulations: 5y.
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<th>Corr.</th>
<th>$K/F_0$</th>
<th>Estimated Values</th>
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<td>4.24</td>
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<td>Case (iii)</td>
<td>Corr.1</td>
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<td>5.15</td>
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<td>1.4</td>
<td>4.08</td>
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<td>Corr.3</td>
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<td>1.5</td>
<td>6.43</td>
<td>3.75 5.53 6.50</td>
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<td>Corr.4</td>
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<td>1.08 -0.25 0.32</td>
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<td>-1.83 -0.50 0.08</td>
<td>-37.0% -10.0% 1.6%</td>
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Table 6: A comparison of estimators by asymptotic expansions to by monte carlo simulations: 10y.
6 Computation of Higher-Orders

6.1 Expansion around a Normal Distribution

Takahashi-Toda(2008)

- A $N$-dimensional process $S_t^{(ε)} = (S_t^{(ε),1}, \ldots, S_t^{(ε),N})$:

\[
dS_t^{(ε),i} = \epsilon \sum_{j=1}^{d} V_j^{(ε)}(S_t^{(ε)},t)dW_t^j \quad (i = 1, \ldots, N)
\]

where $W = (W^1, \ldots, W^d)$ is a $d$-dimensional standard Wiener process, and $V_j = (V_j^1, \ldots, V_j^N)$: $\mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}^N$ satisfies some regularity conditions.

**Remark 3**

\[
dS_t^{(ε),i} = \kappa^i(t)(\theta^i(t) - S_t^{(ε),i})dt + \epsilon \sum_{j=1}^{d} V_j^{(ε)}(S_t^{(ε)},t)dW_t^j \quad (i = 1, \ldots, N)
\]

where $\kappa^i, \theta^i (i = 1, \ldots, N)$ are deterministic functions of $t$.

Define $\tilde{S}_t^{(ε)} = (\tilde{S}_t^{(ε),1}, \ldots, \tilde{S}_t^{(ε),N})$ as

\[
\tilde{S}_t^{(ε),i} := e^{\int_0^t \kappa^i(s)ds}S_t^{(ε),i} - \int_0^t e^{\int_0^u \kappa^i(s)ds} \kappa^i \theta^i ds \quad (i = 1, \ldots, N).
\]

Then,

\[
d\tilde{S}_t^{(ε),i} = \epsilon \sum_{j=1}^{d} \tilde{V}_j^{(ε)}(\tilde{S}_t^{(ε)},t)dW_t^j \quad (i = 1, \ldots, N),
\]

\[
\tilde{V}_j^{(ε)}(\tilde{S}_t^{(ε)},t) := e^{\int_0^t \kappa^i(s)ds}V_j^{(ε)}(S_t^{(ε)},t).
\]

- Asymptotic expansion of $S_t^{(ε)}$:

\[
S_t^{(ε)} = S_t^{(0)} + \epsilon A_{1t} + \frac{\epsilon^2}{2} A_{2t} + \frac{\epsilon^3}{6} A_{3t} + \frac{\epsilon^4}{24} A_{4t} + o(\epsilon^4), \text{ as } \epsilon \downarrow 0,
\]

where $S_t^{(0)} = S_0$ and

\[
da_{1t} = \sum_{j=1}^{d} V_j(S_t^{(0)},t)dW_t^j
\]

\[
da_{2t} = 2 \sum_{j=1}^{d} \sum_{i=1}^{N} A_{1t}^i \frac{\partial}{\partial x_i} V_j(S_t^{(0)},t)dW_t^j
\]

\[
da_{3t} = 3 \sum_{i=1}^{N} A_{2t}^i \frac{\partial}{\partial x_i} V_j(S_t^{(0)},t) + \sum_{i=1}^{N} \sum_{k=1}^{N} A_{1t}^i A_{1t}^k \frac{\partial^2}{\partial x_i \partial x_k} V_j(S_t^{(0)},t)\right) dW_t^j
\]

\[
da_{4t} = 4 \sum_{i=1}^{N} A_{3t}^i \frac{\partial}{\partial x_i} V_j(S_t^{(0)},t) + 3 \sum_{i=1}^{N} \sum_{k=1}^{N} A_{1t}^i A_{2t}^k \frac{\partial^2}{\partial x_i \partial x_k} V_j(S_t^{(0)},t)
\]

\[
+ \sum_{i=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} A_{1t}^i A_{1t}^k A_{1t}^l \frac{\partial^3}{\partial x_i \partial x_k \partial x_l} V_j(S_t^{(0)},t)\right) dW_t^j.
\]
• For $\xi \in \mathbb{R}$, define a complex-valued stochastic process $Z_t := Z_t^{(\xi)}$ as
  
  \[ Z_t^{(\xi)} := \exp\{(i\xi)A_{1t}^1 - \frac{1}{2}(i\xi)^2(A_{1t}^1)_t\}. \]

  Then, $Z_t^{(\xi)}$ is a martingale and

  \[ dZ_t^{(\xi)} = (i\xi) \sum_{j=1}^{d} V_j^1(S_t^{(0)}, t) Z_t^{(\xi)} dW_t^j \]

  and note that

  \[ \exp\{(i\xi)A_{1t}^1\} = Z_t^{(\xi)} \exp\{-\frac{\xi^2}{2} \Sigma_t\}, \]

  where

  \[ \Sigma_t := \int_0^t \sum_{j=1}^{d} V_j^1(S_s^{(0)}, t)^2 ds. \]

• Normalization of $S^{(\epsilon),1}$:

  \[ X_t^{(\epsilon)} := \frac{S_t^{(\epsilon),1} - S_t^{(0),1}}{\epsilon} = A_{1t} + \frac{\epsilon}{2} A_{2t} + \frac{\epsilon^2}{6} A_{3t} + \frac{\epsilon^3}{24} A_{4t} + o(\epsilon^3) \]
• Asymptotic expansion of the characteristic function of \( X^{(c)}_T \) up to the \( \epsilon^3 \)-order:

\[
\psi_{X^{(c)}_T}(\xi) := E[\exp\{(i\xi)X^{(c)}_T\}] \\
= E[\exp\{(i\xi)A^1_{1T}\}] \exp\left\{ \frac{\epsilon}{2}(i\xi)A^1_{1T} + \frac{\epsilon^2}{6}(i\xi)A^1_{2T} + \frac{\epsilon^3}{24}(i\xi)A^1_{3T} + o(\epsilon^3) \right\} \\
= E[\exp\{(i\xi)A^1_{1T}\}] + \epsilon \left\{ \frac{(i\xi)}{2} E[A^1_{2T}] \exp\{(i\xi)A^1_{1T}\} \right. \\
+ \frac{\epsilon^2}{6} \left\{ \frac{(i\xi)}{6} E[A^1_{3T}] \exp\{(i\xi)A^1_{1T}\} + \frac{(i\xi)^2}{8} E[(A^1_{2T})^2] \exp\{(i\xi)A^1_{1T}\} \right\} \\
\left. + \frac{\epsilon^3}{24} \left\{ \frac{(i\xi)^3}{24} E[A^1_{3T}] \exp\{(i\xi)A^1_{1T}\} + o(\epsilon^3) \right\} \right] \\
+ \epsilon^3 \left\{ \frac{(i\xi)^2}{12} E[A^1_{2T}A^1_{3T}] + \frac{(i\xi)^3}{48} E[(A^1_{2T})^3] \right\} \right] \exp\{-\frac{\epsilon^2}{2} \Sigma_T\} \\
+ o(\epsilon^3)
\]

• Define \( \eta_{1,1}^i, \eta_{2,1}^{i,k}, \eta_{3,1}^{i,k}, \eta_{4,1}^{i,k}, \eta_{3,1}^{i,k}, \eta_{4,2,1}^{i,k}, \eta_{4,2,2}^{i,k} \), and \( \eta_{6,3}^{i,k,l} \) as

\[
\eta_{1,1}^i(t) := E[A^i_{1t}Z_t] \\
\eta_{2,1}^{i,k}(t) := E[A^i_{2t}Z_t] \\
\eta_{2,2}^{i,k}(t) := E[A^i_{1t}A^k_{1t}Z_t] \\
\eta_{3,1}^{i,k}(t) := E[A^i_{3t}Z_t] \\
\eta_{3,2}^{i,k}(t) := E[A^i_{1t}A^k_{2t}Z_t] \\
\eta_{3,3}^{i,k,l}(t) := E[A^i_{1t}A^k_{1t}A^l_{2t}Z_t] \\
\eta_{4,1}^{i,k}(t) := E[A^i_{4t}Z_t] \\
\eta_{4,2,1}^{i,k}(t) := E[A^i_{1t}A^k_{3t}Z_t] \\
\eta_{4,2,2}^{i,k}(t) := E[A^i_{2t}A^k_{3t}Z_t] \\
\eta_{4,3}^{i,k,l}(t) := E[A^i_{1t}A^k_{1t}A^l_{3t}Z_t] \\
\eta_{5,2}^{i,k}(t) := E[A^i_{2t}A^k_{4t}Z_t] \\
\eta_{5,3}^{i,k,l}(t) := E[A^i_{1t}A^k_{2t}A^l_{3t}Z_t] \\
\eta_{6,3}^{i,k,l}(t) := E[A^i_{2t}A^k_{3t}A^l_{3t}Z_t].
\]
Consider the evaluation of $\eta_{2,1}(T) = E[A_{2T}^{1}Z_{T}]$ which appears in the $\epsilon$-order.

$$d(A_{2t}^{1}Z_{t}) = A_{2t}^{1}dZ_{t} + Z_{t}dA_{2t}^{1} + dA_{2t}^{1}dZ_{t}$$

$$= \left\{ 2(i\xi) \sum_{j=1}^{d} V_{j}^{1}(S_{t}^{(0)}, t)Z_{t} \sum_{i=1}^{N} A_{it}^{1} \frac{\partial}{\partial x_{i}} V_{j}^{1}(S_{t}^{(0)}, t) \right\} dt$$

$$+ \sum_{j=1}^{d} \left\{ (i\xi)V_{j}^{1}(S_{t}^{(0)}, t)A_{2t}^{1}Z_{t} + 2 \sum_{i=1}^{N} \frac{\partial}{\partial x_{i}} V_{j}^{1}(S_{t}^{(0)}, t)A_{it}^{1}Z_{t} \right\} dW_{j}^{i}$$

Since the second and third terms are martingales, taking the expectation on both sides, we have the following ordinary differential equation of $\eta_{2,1}$:

$$\frac{d}{dt}\eta_{2,1}(t) = (i\xi) \sum_{j=1}^{d} \sum_{i=1}^{N} V_{j}^{1}(S_{t}^{(0)}, t) \frac{\partial}{\partial x_{i}} V_{j}^{1}(S_{t}^{(0)}, t) \eta_{1,1}(t)$$

Here, $\eta_{1,1}(i = 1, \cdots, N)$ appearing in the right hand side of above ODE is evaluated in the similar manner:

$$d(A_{1t}^{1}Z_{t}) = A_{1t}^{1}dZ_{t} + Z_{t}dA_{1t}^{1} + dA_{1t}^{1}dZ_{t}$$

$$= (i\xi) \sum_{j=1}^{d} V_{j}^{1}(S_{t}^{(0)}, t)V_{j}^{1}(S_{t}(0), t)Z_{t}dt$$

$$+ \sum_{j=1}^{d} \left\{ (i\xi)V_{j}^{1}(S_{t}^{(0)}, t)A_{1t}^{1}Z_{t} + V_{j}^{1}(S_{t}^{(0)}, t)Z_{t} \right\} dW_{j}^{i}$$

Hence,

$$\frac{d}{dt}\eta_{1,1}(t) = (i\xi) \sum_{j=1}^{d} V_{j}^{1}(S_{t}^{(0)}, t)V_{j}^{1}(S_{t}^{(0)}, t)E[Z_{t}]$$

Since $E[Z_{t}] = 1$, we have

$$\eta_{1,1}(t) = (i\xi) \int_{0}^{t} \sum_{j=1}^{d} V_{j}^{1}(S_{s}^{(0)}, s)V_{j}^{1}(S_{s}^{(0)}, s)ds.$$

Higher order terms can be evaluated in the similar way.

The key observation is that each ODE does not involve any higher order terms, and only lower terms appears in the r.h.s. of the ODE. So, one can easily solve the system of ODEs and evaluate expectations.
• Proposition 1 The asymptotic expansion of the characteristic function of $X_T^{(c)}$ up to the $\epsilon^3$-order is expressed as

$$
\psi_{X_T^{(c)}}(\xi) = \left\{ 1 + \frac{(i\xi)^2}{2} \eta_{2,1}(T) + \epsilon^2 \left\{ \frac{(i\xi)^1}{6} \eta_{3,1}(T) + \frac{(i\xi)^2}{8} \eta_{4,1,2}(T) \right\} 
+ \epsilon^3 \left\{ \frac{(i\xi)^1}{24} \eta_{3,1}(T) + \frac{(i\xi)^2}{12} \eta_{4,2,1}(T) + \frac{(i\xi)^3}{48} \eta_{6,1,1}(T) \right\} \right\} \times \exp\left\{ -\frac{\epsilon^2}{2} \Sigma_T \right\} + o(\epsilon^3),
$$

where $\eta_{2,1}(T), \eta_{3,1}(T), \eta_{4,1,2}(T), \eta_{4,1,1}(T), \eta_{5,2}(T)$ and $\eta_{6,1,1}(T)$ are obtained by the following system of ordinary differential equations:

$$
\eta_{1,1}^j(t) = (i\xi) \int_0^t \sum_{j=1}^d \sum_{s=1}^d V_j^1(S_s^0, t) V_j^1(S_s^0, s) ds
$$

$$
\eta_{2,1}^j(t) = (i\xi) \int_0^t \sum_{j=1}^{d-1} \sum_{s=1}^d V_j^1(S_s^0, t) \frac{\partial}{\partial x_j} V_j^1(S_s^0, t) \eta_{1,1}^j(s) ds
$$

$$
\eta_{3,1}^j(t) = (i\xi) \int_0^t \sum_{j=1}^{d-2} \sum_{s=1}^d V_j^1(S_s^0, t) \frac{\partial}{\partial x_j} V_j^1(S_s^0, t) \eta_{2,1}^j(s)
+ 3 \sum_{j=1}^{d-1} \sum_{s=1}^d V_j^1(S_s^0, s) \frac{\partial^2}{\partial x_j \partial x_j} V_j^1(S_s^0, s) \eta_{2,1}^j(s) ds
$$

$$
\eta_{2,2}^j(t) = (i\xi) \int_0^t \sum_{j=1}^{d-1} \sum_{s=1}^d V_j^1(S_s^0, s) V_j^1(S_s^0, s) \eta_{2,1}^j(s)
+ 2 \sum_{j=1}^{d-1} \sum_{s=1}^d V_j^1(S_s^0, s) \frac{\partial}{\partial x_j} V_j^1(S_s^0, s) \eta_{2,2}^j(s) ds
$$

$$
\eta_{3,3}^j(t) = (i\xi) \int_0^t \sum_{j=1}^{d-2} \sum_{s=1}^d V_j^1(S_s^0, s) V_j^1(S_s^0, s) \eta_{2,1}^j(s)
+ V_j^1(S_s^0, s) V_j^1(S_s^0, s) \eta_{2,2}^j(s) + V_j^1(S_s^0, s) V_j^1(S_s^0, s) \eta_{2,2}^j(s) ds
+ \int_0^t \sum_{j=1}^{d-1} \sum_{s=1}^d V_j^1(S_s^0, s) V_j^1(S_s^0, s) \eta_{2,1}^j(s)
+ V_j^1(S_s^0, s) V_j^1(S_s^0, s) \eta_{2,2}^j(s) \eta_{2,2}^j(s) ds
$$

$$
\eta_{3,3}^j(t) = (i\xi) \int_0^t \sum_{j=1}^{d-2} \sum_{s=1}^d V_j^1(S_s^0, s) V_j^1(S_s^0, s) \eta_{2,1}^j(s)
+ V_j^1(S_s^0, s) V_j^1(S_s^0, s) \eta_{2,2}^j(s) + V_j^1(S_s^0, s) V_j^1(S_s^0, s) \eta_{2,2}^j(s) ds
+ \int_0^t \sum_{j=1}^{d-1} \sum_{s=1}^d V_j^1(S_s^0, s) V_j^1(S_s^0, s) \eta_{2,1}^j(s)
+ V_j^1(S_s^0, s) V_j^1(S_s^0, s) \eta_{2,2}^j(s) \eta_{2,2}^j(s) ds
$$

$$
$$
\[ \eta_{4,1}(t) = (i\xi) \int_0^t \sum_{l=1}^d \left\{ 3 \sum_{i',k'=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) \frac{\partial}{\partial x_{i'}} V_{j}^{i} \left( S_s^{(0)}, t \right) \eta_{3,1}^{i,i',k'}(s) \right. \\
+ 12 \sum_{l=1}^d \sum_{k'=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) \frac{\partial^2}{\partial x_{i'} \partial x_{k'}} V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{3,2}^{i,i',k'}(s) \\
\left. + 3 \sum_{i',k'=1}^N \sum_{l=1}^d \sum_{i''=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) \frac{\partial^2}{\partial x_{i'} \partial x_{i''} \partial x_{k'}} V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{3,3}^{i,i',i'',k'}(s) \right\} ds \]

\[ \eta_{4,2,1}(t) = (i\xi) \int_0^t \sum_{j=1}^d \left\{ 3 \sum_{i',k'=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{3,1}^{i,i',k'}(s) + 3 \sum_{i',k'=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) \frac{\partial}{\partial x_{i'}} V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{3,2}^{i,i',k'}(s) \right. \\
\left. + \sum_{i',k'=1}^N \sum_{j=1}^d \left\{ 3 \sum_{i',k'=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) \frac{\partial^2}{\partial x_{i'} \partial x_{k'}} V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{2,1}^{i,k}(s) \right. \\
\left. + 3 \sum_{i',k'=1}^N \sum_{j=1}^d \sum_{i''=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) \frac{\partial^2}{\partial x_{i'} \partial x_{i''} \partial x_{k'}} V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{2,2}^{i,k,i''}(s) \right\} ds \]

\[ \eta_{4,2,2}(t) = (i\xi) \int_0^t \sum_{j=1}^d \left\{ 2 \sum_{i',k'=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) \frac{\partial}{\partial x_{i'}} V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{3,2}^{i,i',k'}(s) \right. \\
\left. + 2 \sum_{i',k'=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) \frac{\partial}{\partial x_{i'}} V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{3,3}^{i,i',k'}(s) \right\} ds \]

\[ \eta_{4,3}(t) = (i\xi) \int_0^t \sum_{j=1}^d \left\{ 2 \sum_{i',k'=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{3,2}^{i,i',k'}(s) + V_{j}^{i} \left( S_s^{(0)}, s \right) V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{3,2}^{i,i',k'}(s) \right. \\
\left. + 2 \sum_{i',k'=1}^N \sum_{j=1}^d \left\{ \sum_{i',k'=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{2,1}^{i,k}(s) \right. \\
\left. + 2 \sum_{i',k'=1}^N \sum_{j=1}^d \sum_{i''=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{2,2}^{i,k,i''}(s) + 2 \sum_{i',k'=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) \frac{\partial}{\partial x_{i'}} V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{2,2}^{i,k,i''}(s) \right\} ds \]

\[ \eta_{5,2}(t) = (i\xi) \int_0^t \sum_{j=1}^d \left\{ 2 \sum_{i',k'=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) \frac{\partial}{\partial x_{i'}} V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{4,2,1}^{i,i',k'}(s) \right. \\
\left. + 3 \sum_{i',k'=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) \frac{\partial}{\partial x_{i'}} V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{4,2,2}^{i,i',k'}(s) \right. \\
\left. + 3 \sum_{i',k'=1}^N \sum_{j=1}^d \sum_{i''=1}^N V_{j}^{i} \left( S_s^{(0)}, s \right) \frac{\partial^2}{\partial x_{i'} \partial x_{i''} \partial x_{k'}} V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{4,3}^{i,i',i'',k'}(s) \right\} ds \]

\[ + \int_0^t \sum_{j=1}^d \left\{ 6 \sum_{i',k'=1}^N \sum_{j=1}^d \frac{\partial}{\partial x_{i'}} V_{j}^{i} \left( S_s^{(0)}, s \right) \frac{\partial}{\partial x_{i'}} V_{j}^{i} \left( S_s^{(0)}, s \right) \eta_{5,2}^{i,i',k'}(s) \right\} ds \]
\[ +6 \sum_{\nu=1}^{N} \sum_{\nu'=1}^{N} \sum_{\nu''=1}^{N} \frac{\partial}{\partial x_{\nu'}} V_j^1(S_s^{(0)}, s) \frac{\partial^2}{\partial x_{\nu''} \partial x_{\nu'}} V_j^k(S_s^{(0)}, s) \eta_{v,v',v''}^i \eta_{v',v''}^{i,k'} (s) \} ds \]

\[ \eta_{i,3}^{k,3}(t) = (i \xi) \int_0^t \sum_{j=1}^{d} \left\{ V_j^1(S_s^{(0)}, s) V_j^1(S_s^{(0)}, s) \eta_{i,3}^{k,3}(s) \right\} ds \]

\[ +2 \sum_{\nu=1}^{N} V_j^1(S_s^{(0)}, s) \frac{\partial}{\partial x_{\nu'}} V_j^1(S_s^{(0)}, s) \eta_{i,3}^{i,k} (s) \]

\[ +2 \sum_{\nu=1}^{N} V_j^1(S_s^{(0)}, s) \frac{\partial}{\partial x_{\nu'}} V_j^k(S_s^{(0)}, s) \eta_{i,3}^{i,k} (s) \} ds \]

\[ + \int_0^t \sum_{j=1}^{d} \left\{ 2 \sum_{\nu'=1}^{N} V_j^1(S_s^{(0)}, s) \frac{\partial}{\partial x_{\nu'}} V_j^1(S_s^{(0)}, s) \eta_{i,3}^{i,k} (s) \right\} ds \]

\[ +2 \sum_{\nu=1}^{N} V_j^1(S_s^{(0)}, s) \frac{\partial}{\partial x_{\nu'}} V_j^1(S_s^{(0)}, s) \eta_{i,3}^{i,k} (s) \]

\[ +4 \sum_{\nu=1}^{N} \sum_{\nu'=1}^{N} \frac{\partial}{\partial x_{\nu'}} V_j^1(S_s^{(0)}, s) \frac{\partial}{\partial x_{\nu''}} V_j^1(S_s^{(0)}, s) \eta_{i,3}^{i,k'} (s) \} ds \]

\[ \eta_{i,5}^{k,3}(t) = (i \xi) \int_0^t \sum_{j=1}^{d} \left\{ 2 \sum_{\nu'=1}^{N} V_j^1(S_s^{(0)}, s) \frac{\partial}{\partial x_{\nu'}} V_j^1(S_s^{(0)}, s) \eta_{i,3}^{i,k} (s) \right\} ds \]

\[ +2 \sum_{\nu=1}^{N} V_j^1(S_s^{(0)}, s) \frac{\partial}{\partial x_{\nu'}} V_j^k(S_s^{(0)}, s) \eta_{i,3}^{i,k} (s) \}

\[ +2 \sum_{\nu=1}^{N} V_j^1(S_s^{(0)}, s) \frac{\partial}{\partial x_{\nu'}} V_j^1(S_s^{(0)}, s) \eta_{i,3}^{i,k} (s) \} ds \]

\[ + \int_0^t \sum_{j=1}^{d} \left\{ 4 \sum_{\nu'=1}^{N} \sum_{\nu''=1}^{N} \frac{\partial}{\partial x_{\nu''}} V_j^1(S_s^{(0)}, s) \frac{\partial}{\partial x_{\nu'}} V_j^1(S_s^{(0)}, s) \eta_{i,3}^{i,k'} (s) \right\} ds \]

\[ +4 \sum_{\nu=1}^{N} \sum_{\nu'=1}^{N} \frac{\partial}{\partial x_{\nu'}} V_j^1(S_s^{(0)}, s) \frac{\partial}{\partial x_{\nu''}} V_j^1(S_s^{(0)}, s) \eta_{i,3}^{i,k'} (s) \} ds \]

\[ +4 \sum_{\nu=1}^{N} \sum_{\nu'=1}^{N} \frac{\partial}{\partial x_{\nu'}} V_j^1(S_s^{(0)}, s) \frac{\partial}{\partial x_{\nu''}} V_j^1(S_s^{(0)}, s) \eta_{i,3}^{i,k'} (s) \} ds \]
From the discussion above, it is easily shown that the asymptotic expansion of the characteristic function can be expressed as a product of the Gaussian characteristic function and a polynomial of $i\xi$. Indeed, the asymptotic expansion of the characteristic function of $X^{(r)}_{\epsilon}$ up to the $\epsilon^3$-order is given by

$$
\psi_{X^{(r)}_{\epsilon}}(\xi) = \{1 + \epsilon \{ C_{23}(i\xi)^3 \} 
+ \epsilon^2 \{ C_{32}(i\xi)^2 + C_{34}(i\xi)^4 + C_{36}(i\xi)^6 \} 
+ \epsilon^3 \{ C_{43}(i\xi)^3 + C_{45}(i\xi)^5 + C_{47}(i\xi)^7 + C_{49}(i\xi)^9 \} \exp \{- \frac{\xi^2}{2} \Sigma_{T} \} \} + o(\epsilon^3)
$$

for some constants $C_{23}, C_{32}, C_{34}, C_{36}, C_{43}, C_{45}, C_{47}, C_{49}$. (Precisely, each $\eta_n, \ldots$ is an $n$th-order polynomial of $(i\xi)$.)

To obtain the asymptotic expansion of density function, we need to invert $\psi_{X^{(r)}_{\epsilon}}$ using the inverse Fourier transformation:

$$
F^{-1}[(i\xi)^n e^{-\frac{\xi^2}{2} \Sigma}] (x) = \frac{1}{\Sigma^n} H_n(x; \Sigma) n[x; 0, \Sigma],
$$

where

$$
H_n(x; \Sigma) := (-\Sigma)^n \frac{d^n}{dx^n} e^{-x^2/2\Sigma}
$$

and

$$
n[x; \mu, \Sigma] := \frac{1}{\sqrt{2\pi\Sigma}} \exp \{- \frac{(x - \mu)^2}{2\Sigma} \}.
$$

For example, for $n = 0, 1, 2, 3, 4$,

$$
H_0(x; \Sigma) = 1, \quad H_1(x; \Sigma) = x, \quad H_2(x; \Sigma) = x^2 - \Sigma,
$$

$$
H_3(x; \Sigma) = x^3 - 3\Sigma x, \quad H_4(x; \Sigma) = x^4 - 6\Sigma x^2 + 3\Sigma^2.
$$

Using Hermite polynomials, we obtain the asymptotic expansion of the density function of $X^{(r)}_{\epsilon}$ up to the $\epsilon^3$-order as

$$
f_{X^{(r)}_{\epsilon}}(x) = n[x; 0, \Sigma_T]
+ \epsilon \left\{ \frac{C_{23}}{\Sigma_T} H_3(x; \Sigma_T) \right\} n[x; 0, \Sigma_T]
+ \epsilon^2 \left\{ \frac{C_{32}}{\Sigma_T} H_2(x; \Sigma_T) + \frac{C_{34}}{\Sigma_T} H_4(x; \Sigma_T) + \frac{C_{36}}{\Sigma_T} H_6(x; \Sigma_T) \right\} n[x; 0, \Sigma_T]
+ \epsilon^3 \left\{ \frac{C_{43}}{\Sigma_T} H_3(x; \Sigma_T) + \frac{C_{45}}{\Sigma_T} H_5(x; \Sigma_T) + \frac{C_{47}}{\Sigma_T} H_7(x; \Sigma_T) + \frac{C_{49}}{\Sigma_T} H_9(x; \Sigma_T) \right\} n[x; 0, \Sigma_T]
+ o(\epsilon^3).
$$
On Computation of Average Options

• $\tilde{S}(\epsilon)$ is defined by

$$\tilde{S}(\epsilon) := \int_0^t S(\epsilon) \, ds.$$

• The asymptotic expansion of $S(\epsilon)$ as $\epsilon \downarrow 0$ is given by:

$$\tilde{S}(\epsilon) = \tilde{S}(0) + \epsilon \tilde{A}_1 + \epsilon^2 \tilde{A}_2 + \epsilon^3 \tilde{A}_3 + \epsilon^4 \tilde{A}_4 + o(\epsilon^4),$$

where $\tilde{S}(0), \tilde{A}_i, (i = 1, 2, 3, 4)$ are obtained as follows:

$$\tilde{S}(0) = \int_0^t S(0) \, ds,$$

$$d\tilde{A}_i = A_i \, dt.$$

• Fix $T > 0$ and apply Fubini’s theorem to $\tilde{A}_1$:

$$\tilde{A}_1 = \sum_{j=1}^d \int_0^T \int_0^t V_j^1(S(0), s) \, dW^j_s \, dt$$

$$= \sum_{j=1}^d \int_0^T \tilde{V}_j^1(S(0), t) \, dW^j_t,$$

where

$$\tilde{V}_j^1(S(0), t) := (T - t) V_j^1(S(0), t).$$

• Define $\tilde{A}_1$ as

$$d\tilde{A}_1 = \sum_{j=1}^d \tilde{V}_j^1(S(0), t) \, dW^j_t, \quad \tilde{A}_1^0 = 0.$$ 

Note that $\tilde{A}_1 = \tilde{A}_1^T$ at $t = T$.

• Using $\tilde{A}_1$, define $Z_t := Z_t^{(\xi)}$ for $\xi \in \mathbb{R}$ as

$$Z_t^{(\xi)} := \exp \left\{ (i\xi) \tilde{A}_1^T - \frac{1}{2} (i\xi)^2 (\tilde{A}_1^T)^2 \right\}.$$ 

Then, $Z_t^{(\xi)}$ is a martingale and satisfies the following SDE:

$$dZ_t^{(\xi)} = (i\xi) \sum_{j=1}^d \tilde{V}_j^1(S(0), t) Z_t^{(\xi)} \, dW^j_t.$$ 

• It also holds that

$$\exp\{ (i\xi) \tilde{A}_1^T \} = Z_T^{(\xi)} \exp\{ -\frac{\xi^2}{2} \tilde{\Sigma}_T \},$$

where

$$\tilde{\Sigma}_T := \int_0^T \sum_{j=1}^d \tilde{V}_j^1(S(0), t)^2 \, dt.$$
The asymptotic expansion of the characteristic function of $\tilde{X}_T^{(\epsilon)} = \frac{\tilde{S}_T^{(\epsilon)} - \tilde{S}_T^{(0)} \epsilon}{\epsilon}$ is expressed as follows:

$$\psi_{\tilde{X}_T^{(\epsilon)}}(\xi) = \left\{ 1 + \epsilon \frac{(i\xi)}{2} E[\tilde{A}_{2T}^{1} Z_T^{T,(\xi)}] \right. \\
+ \epsilon^2 \left\{ \frac{(i\xi)^2}{6} E[\tilde{A}_{3T}^{1} Z_T^{T,(\xi)}] + \frac{(i\xi)^2}{8} E[(\tilde{A}_{2T}^{1})^2 Z_T^{T,(\xi)}] \right\} \\
+ \epsilon^3 \left\{ \frac{(i\xi)^3}{24} E[\tilde{A}_{4T}^{1} Z_T^{T,(\xi)}] + \frac{(i\xi)^3}{12} E[\tilde{A}_{2T}^{1} \tilde{A}_{1T}^{1} Z_T^{T,(\xi)}] + \frac{(i\xi)^3}{48} E[(\tilde{A}_{2T}^{1})^3 Z_T^{T,(\xi)}] \right\} \\
\times \exp\{-\frac{\xi^2}{2} \tilde{\Sigma}_T\} + o(\epsilon^3).$$

Each expectation on the right hand side can be obtained in the similar manner as before.
Numerical Examples: $\lambda$-SABR model

- $\lambda$-SABR model:

$$
\begin{align*}
    dS(t) &= \sigma(t) S(t)^{\beta} dW^1(t), \\
    d\sigma(t) &= \lambda (\theta - \sigma(t)) dt + \nu_1 \sigma(t) dW^1(t) + \nu_2 \sigma(t) dW^2(t),
\end{align*}
$$

where $\nu_1 = \rho \nu$, $\nu_2 = (\sqrt{1 - \rho^2}) \nu$ (The correlation between $S$ and $\sigma$ is $\rho \in [-1, 1]$.)

- European plain-vanilla call and put prices (interest rate=0\%):
  Approximated prices by the asymptotic expansion method upto the fifth-order:
  All the solutions to differential equations are obtained analytically.
  Benchmark values are computed by Monte Carlo simulations.

<table>
<thead>
<tr>
<th>Case</th>
<th>Parameter</th>
<th>$\lambda$</th>
<th>$\sigma(0)$</th>
<th>$\beta$</th>
<th>$\rho$</th>
<th>$\theta$</th>
<th>$\nu$</th>
<th>$T$</th>
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Case i: Euler-Maruyama scheme, 1024 time steps
Case ii: Euler-Maruyama scheme, 1024 time steps
Case iii: Euler-Maruyama scheme, 512 time steps
The number of trials: $10^8$
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<th>3rd</th>
<th>4th</th>
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<td>4.289</td>
<td>0.244</td>
<td>0.074</td>
<td>-0.015</td>
<td>0.009</td>
<td>0.004</td>
<td>5.70%</td>
<td>1.74%</td>
<td>-0.36%</td>
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<td>0.239</td>
<td>0.046</td>
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<tr>
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<td>0.192</td>
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<td>0.002</td>
<td>17.77%</td>
<td>1.62%</td>
<td>-3.30%</td>
<td>0.88%</td>
</tr>
<tr>
<td>150 Call</td>
<td>0.466</td>
<td>0.129</td>
<td>-0.004</td>
<td>-0.036</td>
<td>0.010</td>
<td>0.001</td>
<td>27.62%</td>
<td>-0.75%</td>
<td>-7.81%</td>
<td>2.13%</td>
</tr>
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6.2 Case of Random Limit 
(Expansion around a Log-normal Distribution)


- The underlying asset price \( S^{(c)} \) follows:
  \[
  dS^{(c)}_t = g(X^{(c)}_t)S^{(c)}_t \sigma dW_t; \quad S^{(c)}_0 = s_0
  \]
  \[
  dX^{(c)}_t = V_0(X^{(c)}_t, \epsilon)dt + \epsilon V(X^{(c)}_t)dW_t; \quad X^{(c)}_0 = x_0 \in \mathbb{R}^d,
  \]
  where \( W \) denotes a \( m \)-dimensional standard Wiener process, \( \epsilon \in [0, 1] \) is a known parameter, and \( \bar{\sigma} \in \mathbb{R}^m \) is a constant vector.

Suppose that coefficients \( h : \mathbb{R}^d \rightarrow \mathbb{R}, V_0 : \mathbb{R}^d \times [0, 1] \rightarrow \mathbb{R}^d, V : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^m \) are smooth and satisfy regularity conditions.

- Define \( \hat{X}^{(c)} \) as
  \[
  \hat{X}^{(c)}_t = \log \left( \frac{S^{(c)}_t}{s_0} \right).
  \]
  Then,
  \[
  \hat{X}^{(c)}_t = -|\bar{\sigma}|^2 \int_0^t g(X^{(c)}_u)^2 du + \int_0^t g(X^{(c)}_u)\bar{\sigma} dW_u,
  \]
  and note that
  \[
  \hat{X}^{(0)}_T \sim N(\mu_T, \Sigma_T),
  \]
  where
  \[
  \mu_T = -|\bar{\sigma}|^2 \int_0^T g(X^{(0)}_u)^2 du = -\frac{1}{2} \Sigma_T
  \]
  \[
  \Sigma_T = |\sigma|^2 \int_0^T g(X^{(0)}_u)^2 du.
  \]

- Then, an asymptotic expansion of \( \hat{X}^{(c)}_T \) up to \( \epsilon^2 \) is expressed as
  \[
  \hat{X}^{(c)}_T = \epsilon \hat{D}_T + \frac{\epsilon^2}{2} \hat{E}_T + o(\epsilon^2),
  \]
  where \( \hat{D}_t \) and \( \hat{E}_t \) are defined as
  \[
  \hat{D}_t = \frac{\partial \hat{X}^{(c)}_t}{\partial \epsilon}|_{\epsilon=0},
  \]
  \[
  \hat{E}_t = \frac{\partial^2 \hat{X}^{(c)}_t}{\partial \epsilon^2}|_{\epsilon=0}.
  \]

That is, \( \hat{D} \) and \( \hat{E} \) are given by

\[
\hat{D}_t = -|\bar{\sigma}|^2 \int_0^t \partial_x g^{(0)}_u D_u g^{(0)}_u du + \int_0^t \partial_x g^{(0)}_u D_u \bar{\sigma} dW_u
\]
\[
\hat{E}_t = -|\bar{\sigma}|^2 \int_0^t \sum_{j,k=1}^d \partial_j \partial_k g^{(0)}_u D_u^j \bar{\sigma}^{(0)}_u \partial_k g^{(0)}_u du - |\bar{\sigma}|^2 \int_0^t \partial_x g^{(0)}_u E_u h^{(0)}_u du - |\bar{\sigma}|^2 \int_0^t \{ \partial_x g^{(0)}_u D_u \}^2 du
\]
\[
+ \int_0^t \sum_{j,k=1}^d \partial_j \partial_k g^{(0)}_u D_u^j \bar{\sigma} dW_u + \int_0^t \partial_x g^{(0)}_u E_u \bar{\sigma} dW_u,
\]
where \( g^{(0)}_t = g(X^{(0)}_t) \).
• An asymptotic expansion of the characteristic function of $X_T^{(\epsilon)}$ upto $\epsilon^2$ is obtained as

\[
\psi_{X_T^{(\epsilon)}}(\xi) = E[\exp(i\xi X_T^{(\epsilon)})] = \exp\left(i\xi\mu - \frac{\xi^2\Sigma}{2}\right) + \epsilon(i\xi)e^{i\xi\mu}E[e^{i\xi Z_T}E[\hat{D}_T|Z_T]]
\]
\[+ \frac{\epsilon^2}{2}(i\xi)e^{i\xi\mu}E[e^{i\xi Z_T}E[\hat{E}_T|Z_T]] + \frac{\epsilon^2}{2}(i\xi)^2e^{i\xi\mu}E[e^{i\xi Z_T}E[(\hat{D}_T)^2|Z_T]] + o(\epsilon^2),
\]

where

\[Z_T = \int_0^T g(X_t^{(\epsilon)})\sigma dW_t.
\]

Note that $E[\hat{D}_T|Z_T = x] = \hat{p}_{2}(x)$, $E[\hat{E}_T|Z_T = x] = \hat{p}_{3}(x)$ and $E[(\hat{D}_T)^2|Z_T = x] = \hat{p}_{22}(x)$ are polynomial functions of $x$.

• Hence, an asymptotic expansion of the density function of $X_T^{(\epsilon)}$ upto $\epsilon^2$ is obtained as

\[
f_{X_T^{(\epsilon)}}(x) = n(x; \mu_T, \Sigma_T) + \epsilon \left[-\frac{\partial}{\partial x}\{\hat{p}_{2}(x - \mu_T)n(x; \mu_T, \Sigma_T)\} + \frac{\partial^2}{\partial x^2}\{\hat{p}_{22}(x - \mu_T)n(x; \mu_T, \Sigma_T)\}\right] + o(\epsilon^2).
\]

• Call option price with strike price $K$ and maturity $T$:

\[
\frac{C(0)}{P(0,T)} = \int_k^\infty (s_0e^{x} - K) f_{X_T^{(\epsilon)}}(x)dx,
\]

where $k = \log \frac{K}{s_0}$, and $P(0,T)$ denotes the price at time 0 of a zero coupon bond with maturity $T$. 

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An Log-Normal Asymptotic Expansion for Stochastic Volatility Models

- The underlying process under:
  \[ dS_t^{(c)} = g(\sigma_t^{(c)}, t) S_t^{(c)} dW_t^1, \]
  \[ d\sigma_t^{(c)} = a(\sigma_t^{(c)}, t)dt + \nu_1(\sigma_t^{(c)}, t)dW_t^1 + \nu_2(\sigma_t^{(c)}, t)dW_t^2, \]
  \[ S_0^{(c)} = s_0(>0), \quad \sigma_0^{(c)} = \sigma_0(>0). \]

where \( W = (W^1, W^2) \) is a two dimensional standard Wiener process and \( g, a, \nu_1, \nu_2 \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+; \mathbb{R}_+) \).

- Define \( X_t^{(c)} := \log \frac{S_t^{(c)}}{S_0}, \) then
  \[ dX_t^{(c)} = -\frac{1}{2} g(\sigma_t^{(c)}, t)^2 dt + g(\sigma_t^{(c)}, t)dW_t^1, \]
  \[ X_0^{(c)} = 0. \]

- The asymptotic expansion of \( X_t^{(c)} \) and \( \sigma_t^{(c)} \):
  \[ X_t^{(c)} = X_t^{(0)} + \epsilon A_{1t} + \frac{\epsilon^2}{2} A_{2t} + \frac{\epsilon^3}{6} A_{3t} + o(\epsilon^3), \]
  \[ \sigma_t^{(c)} = \sigma_t^{(0)} + \epsilon B_{1t} + \frac{\epsilon^2}{2} B_{2t} + \frac{\epsilon^3}{6} B_{3t} + o(\epsilon^3). \]

where

\[
X_t^{(0)} = -\frac{1}{2} \int_0^t g(\sigma_s^{(0)}, s)^2 ds + \int_0^t g(\sigma_s^{(0)}, s)dW_s^1,
\]

\[
dA_{1t} = -g(\sigma_t^{(0)}, t) \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t) B_{1t} dt + \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t) B_{1t} dW_t^1,
\]

\[
dA_{2t} = \left\{ -g(\sigma_t^{(0)}, t) \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t) B_{2t} - \left\{ g(\sigma_t^{(0)}, t) \frac{\partial^2}{\partial \sigma^2} g(\sigma_t^{(0)}, t) + \left( \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t) \right)^2 B_{2t}^1 \right\} dt \right\} dW_t^1,
\]

and

\[
d\sigma_t^{(0)} = a(\sigma_t^{(0)}, t) dt
\]

\[
dB_{1t} = \frac{\partial}{\partial \sigma} a(\sigma_t^{(0)}, t) B_{1t} dt + \nu_1(\sigma_t^{(0)}, t)dW_t^1 + \nu_2(\sigma_t^{(0)}, t)dW_t^2,
\]

\[
dB_{2t} = \left\{ \frac{\partial}{\partial \sigma} a(\sigma_t^{(0)}, t) B_{2t} + \frac{\partial^2}{\partial \sigma^2} a(\sigma_t^{(0)}, t) B_{2t}^1 \right\} dt
+ 2 \frac{\partial}{\partial \sigma} \nu_1(\sigma_t^{(0)}, t) B_{1t} dW_t^1 + 2 \frac{\partial}{\partial \sigma} \nu_2(\sigma_t^{(0)}, t) B_{1t} dW_t^2,
\]

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The asymptotic expansion of the characteristic function of $X_T^{(\epsilon)}$ up to the $\epsilon^2$-order:

$$
\psi_{X_T^{(\epsilon)}}(\xi) := E[\exp\{(i\xi)X_T^{(\epsilon)}\}]
= E[\exp\{(i\xi)X_T^{(0)}\}]
+ \epsilon \left\{ (i\xi)E[A_{1T}\exp\{(i\xi)X_T^{(0)}\}] \right\}
+ \epsilon^2 \left\{ \frac{(i\xi)^2}{2} E[A_{2T}\exp\{(i\xi)X_T^{(0)}\}] + \frac{(i\xi)^2}{2} E[A_{1T}^2\exp\{(i\xi)X_T^{(0)}\}] \right\}
+ o(\epsilon^2)
$$

For $\xi \in \mathbb{R}$, we define a complex-valued stochastic process $Z_t := Z_t^{(\xi)}$ as

$$
Z_t^{(\xi)} := \exp\left\{ (i\xi) \int_0^t g(\sigma_s^{(0)}, s) dW_s^1 - \frac{1}{2} (i\xi)^2 \int_0^t g(\sigma_s^{(0)}, s)^2 ds \right\}
$$

then, $Z_t^{(\xi)}$ is a martingale and

$$
dZ_t^{(\xi)} = (i\xi)g(\sigma_t^{(0)}, t)Z_t^{(\xi)} dW_t^1
$$

and we have

$$
\exp\{(i\xi)X_T^{(0)}\} = Z_T^{(\xi)} \exp\{\frac{(i\xi)^2}{2} - (i\xi)\Sigma_T\}
$$

where

$$
\Sigma_T := \int_0^T g(\sigma_t^{(0)}, t)^2 dt
$$

Then, $\psi_{X_T^{(\epsilon)}}$ can be expressed as

$$
\psi_{X_T^{(\epsilon)}}(\xi) = \left\{ 1 + \epsilon \left\{ (i\xi)E[A_{1T}Z_T^{(\xi)}] \right\} 
+ \epsilon^2 \left\{ \frac{(i\xi)^2}{2} E[A_{2T}Z_T^{(\xi)}] + \frac{(i\xi)^2}{2} E[A_{1T}^2Z_T^{(\xi)}] \right\} \right\}
\times \exp\{\frac{(i\xi)^2}{2} - (i\xi)\Sigma_T\} + o(\epsilon^2).
$$
Define $\eta$ as

\[
\eta_{1,1}(t) := E[A_1 t Z_t] \\
\eta_{1,2}(t) := E[B_1 t Z_t] \\
\eta_{2,1}(t) := E[A_2 t Z_t] \\
\eta_{2,2}(t) := E[B_2 t Z_t] \\
\eta_{2,3}(t) := E[A_1 B_{11} Z_t] \\
\eta_{2,4}(t) := E[B_{11}^2 Z_t]
\]

Then, we can derive ordinary differential equations as follows:

\[
\begin{align*}
A_1 dZ_t &- Z_t dA_1 + dA_1 dZ_t \\
&= \left\{ (i\xi)g(\sigma_{1}^{(0)}, t) \frac{\partial}{\partial \sigma} g(\sigma_{1}^{(0)}, t) B_{11} Z_t - g(\sigma_{1}^{(0)}, t) \frac{\partial}{\partial \sigma} g(\sigma_{1}^{(0)}, t) B_{11} Z_t \right\} dt \\
&\quad + \left\{ (i\xi)g(\sigma_{1}^{(0)}, t) A_{11} Z_t + \frac{\partial}{\partial \sigma} g(\sigma_{1}^{(0)}, t) B_{11} Z_t \right\} dW_t^1
\end{align*}
\]

Taking the expectation on both sides, we have the following ordinary differential equation of $\eta_{1,1}$:

\[
\frac{d}{dt} \eta_{1,1}(t) = (i\xi)g(\sigma_{1}^{(0)}, t) \frac{\partial}{\partial \sigma} g(\sigma_{1}^{(0)}, t) \eta_{1,1}(t) - g(\sigma_{1}^{(0)}, t) \frac{\partial}{\partial \sigma} g(\sigma_{1}^{(0)}, t) \eta_{1,2}(t).
\]

In the similar manner, we have

\[
\frac{d}{dt} \eta_{1,2}(t) = (i\xi)g(\sigma_{1}^{(0)}, t) \nu_1(\sigma_{1}^{(0)}, t) + a(\sigma_{1}^{(0)}, t) \eta_{1,2}(t).
\]
- Proposition 2 The asymptotic expansion of the characteristic function of \( X_T^{(c)} \) up to the \( \epsilon^2 \)-order is expressed as

\[
\psi_{X_T^{(c)}}(\xi) = \left\{ 1 + \epsilon \left\{ (i\xi)\eta_{1,1}(T) \right\} \right. \\
+ \epsilon^2 \left\{ \frac{(i\xi)}{2} \eta_{2,1,1}(T) + \frac{(i\xi)^2}{2} \eta_{2,2,1}(T) \right\} \right\} \\
\times \exp\left\{ \frac{(i\xi)^2 - (i\xi)\Sigma_T}{2} \right\} + o(\epsilon^2).
\]

where \( \eta_{1,1}(T) \), \( \eta_{2,1,1}(T) \) and \( \eta_{2,2,1}(T) \) are obtained by the following system of ordinary differential equations:

\[
\frac{d}{dt} \eta_{1,1}(t) = (i\xi)g(\sigma_t^{(0)}, t) \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t)\eta_{1,1}(t) - g(\sigma_t^{(0)}, t) \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t)\eta_{1,1}(t) \\
\frac{d}{dt} \eta_{1,2}(t) = (i\xi)g(\sigma_t^{(0)}, t)\nu_1(\sigma_t^{(0)}, t) + \frac{\partial}{\partial \sigma} a(\sigma_t^{(0)}, t)\eta_{1,1}(t) \\
\frac{d}{dt} \eta_{2,1,1}(t) = (i\xi) \left\{ g(\sigma_t^{(0)}, t) \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t)\eta_{2,1,2}(t) + g(\sigma_t^{(0)}, t) \frac{\partial^2}{\partial \sigma^2} g(\sigma_t^{(0)}, t)\eta_{2,2,3}(t) \right\} \\
- g(\sigma_t^{(0)}, t) \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t)\eta_{2,1,2}(t) \\
- \left\{ g(\sigma_t^{(0)}, t) \frac{\partial^2}{\partial \sigma^2} g(\sigma_t^{(0)}, t) + \left( \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t) \right)^2 \right\} \eta_{2,1,2}(t) \\
\frac{d}{dt} \eta_{2,2,2}(t) = 2(i\xi)g(\sigma_t^{(0)}, t) \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t)\eta_{2,2,2}(t) \\
- 2g(\sigma_t^{(0)}, t) \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t)\eta_{2,2,2}(t) + \left( \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t) \right)^2 \eta_{2,2,3}(t) \\
\frac{d}{dt} \eta_{2,2,3}(t) = (i\xi) \left\{ g(\sigma_t^{(0)}, t) \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t)\eta_{2,2,3}(t) + g(\sigma_t^{(0)}, t)\nu_1(\sigma_t^{(0)}, t)\eta_{1,1}(t) \right\} \\
+ \frac{\partial}{\partial \sigma} a(\sigma_t^{(0)}, t)\eta_{2,2,2}(t) + g(\sigma_t^{(0)}, t) \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t)\eta_{2,2,3}(t) \\
+ g(\sigma_t^{(0)}, t)\nu_1(\sigma_t^{(0)}, t)\eta_{1,1}(t) + \frac{\partial}{\partial \sigma} g(\sigma_t^{(0)}, t)\nu_1(\sigma_t^{(0)}, t)\eta_{1,1}(t) \\
\frac{d}{dt} \eta_{2,3,3}(t) = 2(i\xi)g(\sigma_t^{(0)}, t)\nu_1(\sigma_t^{(0)}, t)\eta_{1,1}(t) \\
+ 2\frac{\partial}{\partial \sigma} a(\sigma_t^{(0)}, t)\eta_{2,2,3}(t) + \nu_1(\sigma_t^{(0)}, t)^2 + \nu_1(\sigma_t^{(0)}, t)^2
\]
• It is easily shown that the asymptotic expansion of the characteristic function can be expressed as a product of the Gaussian characteristic function and a polynomial of $(i\xi)$.

• To obtain the asymptotic expansion of density function, we need to invert $\psi_{X_T^{(c)}}$ using the inverse Fourier transformation:

\[
\mathcal{F}^{-1}[(i\xi)^n e^{\mu(i\xi) - \frac{\xi^2}{2}\Sigma}](x) = \frac{1}{\sqrt{2\pi\Sigma^n}} H_n(x - \mu; \Sigma) n[x - \mu; 0, \Sigma],
\]

where $\Sigma \equiv \Sigma_T$, $\mu = -\frac{1}{2} \Sigma$ and

\[
H_n(x; \Sigma) := (-\Sigma)^n e^{x^2/2\Sigma} \frac{d^n}{dx^n} e^{-x^2/2\Sigma}
\]

and

\[
n[x; \mu, \Sigma] := \frac{1}{\sqrt{2\pi\Sigma}} \exp\left\{-\frac{(x - \mu)^2}{2\Sigma}\right\}.
\]

• The asymptotic expansion of the density function of $X_T^{(c)}$ up to the $\epsilon^2$-order is obtained as:

\[
f_{X_T^{(c)}}(x) = n[x; \mu, \Sigma]
+ \epsilon \left\{ \eta_{1,1}(T) \frac{H_1(x - \mu; \Sigma)}{\Sigma} \right\} n[x; \mu, \Sigma]
+ \frac{\epsilon^2}{2} \left\{ \eta_{2,1}(T) \frac{H_1(x - \mu; \Sigma)}{\Sigma} + \eta_{2,2}(T) \frac{H_2(x - \mu; \Sigma)}{\Sigma^2} \right\} n[x; \mu, \Sigma] + o(\epsilon^2).
\]

• Call option price with strike price $K$ and maturity $T$:

\[
\frac{C(0)}{P(0,T)} = \int_k^\infty (s_0 e^x - K) f_{X_T^{(c)}}(x) dx,
\]

where $k = \log \frac{K}{s_0}$, and $P(0,T)$ denotes the price at time 0 of a zero coupon bond with maturity $T$. 
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An Expansion of Implied Volatility

I follow Fouque-Papanicolaou-Sircar(2000) using the result in the previous subsection.

• An implied volatility $\sigma_I$ is determined so that:

$$C^{BS}(K,T,\sigma_I) = C(K,T).$$

Here, $C^{BS}(K,T,\sigma)$ denotes the call price with strike $K$, maturity $T$ and volatility $\sigma$ under the Black-Scholes model; $C(K,T)$ denotes the call price with strike $K$ and maturity $T$ under a stochastic volatility model in the previous section.

• Define $\bar{\sigma}$ as $\bar{\sigma} = \Sigma_T^{\frac{1}{2}}$.

• Suppose that an implied volatility is expanded around $\bar{\sigma}$:

$$\sigma_I = \bar{\sigma} + \epsilon \sigma_1 + \epsilon^2 \sigma_2 + o(\epsilon^2)$$

• Then, $C^{BS}(K,T,\sigma_I)$ is expanded as:

$$C^{BS}(K,T,\sigma_I) = C^{BS}(K,T,\bar{\sigma}) + \epsilon \left\{ \left. \partial C^{BS}(K,T,\sigma) / \partial \sigma \right|_{\sigma=\bar{\sigma}} \right\} \sigma_1 + \epsilon^2 \left\{ \left. \partial C^{BS}(K,T,\sigma) / \partial \sigma \right|_{\sigma=\bar{\sigma}} \right\} \sigma_2 + \epsilon^2 \left\{ \left. \partial^2 C^{BS}(K,T,\sigma) / \partial \sigma^2 \right|_{\sigma=\bar{\sigma}} \right\} (\sigma_1)^2.$$

• On the other hand, by an asymptotic expansion,

$$C(K,T) = C^{BS}(K,T,\bar{\sigma}) + \epsilon C_1 + \epsilon^2 C_2.$$

• Hence, $\sigma_1$ and $\sigma_2$ are obtained as

$$\begin{align*}
\sigma_1 &= \frac{C_1 / \left\{ \left. \partial C^{BS}(K,T,\sigma) / \partial \sigma \right|_{\sigma=\bar{\sigma}} \right\}}{\left\{ \left. \partial^2 C^{BS}(K,T,\sigma) / \partial \sigma^2 \right|_{\sigma=\bar{\sigma}} \right\} \left( \sigma_1 \right)^2} \\
\sigma_2 &= \frac{C_2 - \frac{1}{2} \left\{ \left. \partial^2 C^{BS}(K,T,\sigma) / \partial \sigma^2 \right|_{\sigma=\bar{\sigma}} \right\} \sigma_1}{\left\{ \left. \partial C^{BS}(K,T,\sigma) / \partial \sigma \right|_{\sigma=\bar{\sigma}} \right\}}.
\end{align*}$$

• The higher order expansion is obtained in the similar way.
• 4-th order expansion:

\[ C = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \epsilon^3 C_3 + \epsilon^4 C_4 + O(\epsilon^5). \]

\[ \sigma^{\text{implied. vol}} = \bar{\sigma} + \epsilon \sigma_1 + \epsilon^2 \sigma_2 + \epsilon^3 \sigma_3 + \epsilon^4 \sigma_4 + O(\epsilon^5) \]

\[ \sigma_1 = \frac{C_1}{C^{BS}_\sigma(\bar{\sigma})}, \]

\[ \sigma_2 = \frac{C_2}{C^{BS}_\sigma(\bar{\sigma})} - \frac{1}{2} \frac{C^2_1}{C^{BS}_\sigma(\bar{\sigma})^3} C^{BS}_{\sigma \sigma}(\bar{\sigma}), \]

\[ \sigma_3 = \frac{C_3}{C^{BS}_\sigma(\bar{\sigma})} - \left( \frac{C_1}{C^{BS}_\sigma(\bar{\sigma})^2} \right) \left( \frac{C_2}{C^{BS}_\sigma(\bar{\sigma})} - \frac{1}{2} \frac{C^2_1}{C^{BS}_\sigma(\bar{\sigma})^3} C^{BS}_{\sigma \sigma}(\bar{\sigma}) \right) C^{BS}_{\sigma \sigma}(\bar{\sigma}) - \frac{1}{3!} \frac{C^3_1}{C^{BS}_\sigma(\bar{\sigma})^4} C^{BS}_{\sigma \sigma \sigma}(\bar{\sigma}). \]

\[ \sigma_4 = \frac{1}{C^{BS}_\sigma(\bar{\sigma})} \left\{ C_4 - \frac{1}{2} C^{BS}_{\sigma \sigma}(\bar{\sigma}) \sigma_2^2 - C^{BS}_{\sigma \sigma \sigma}(\bar{\sigma}) \sigma_1 \sigma_3 - \frac{1}{4} C^{BS}_{\sigma \sigma \sigma}(\bar{\sigma}) \sigma_1^2 \sigma_2 - \frac{1}{3!} C^{BS}_{\sigma \sigma \sigma}(\bar{\sigma}) \sigma_1^3 \right\}. \]

\[ C^{BS}_\sigma = f \sqrt{T} n(d_1), \]

\[ C^{BS}_{\sigma \sigma} = \frac{f \sqrt{T}}{\sigma} n(d_1) d_1 d_2, \]

\[ C^{BS}_{\sigma \sigma \sigma} = f \sqrt{T} \frac{1}{\sigma^3} n(d_1) \{ d_1^3 d_2 - d_1 d_2 - d_1^2 - d_2^2 \} \]

\[ C^{BS}_{\sigma \sigma \sigma \sigma} = f \sqrt{T} \frac{1}{\sigma^4} n(d_1) \{ d_1^4 d_2^2 - 3 d_1 d_2^3 - 3 d_1^3 d_2 - 3 d_1^2 d_2^2 + 3 d_1^2 + 3 d_2^2 + 6 d_1 d_2 \} \]

\[ n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2}, \]

\[ d_1 = \frac{\log \left( \frac{K}{\delta} \right)}{\sqrt{\sigma^2 T}} + \frac{1}{2} \sqrt{\sigma^2 T}, \]

\[ d_2 = \frac{\log \left( \frac{K}{\delta} \right)}{\sqrt{\sigma^2 T}} - \frac{1}{2} \sqrt{\sigma^2 T}. \]
T=30 4th Order Asymptotic Expansion of Implied Volatility

ImpVol

Implied Vol

Moneyness

AE-ImpVol4
7 Future Research

- Detailed examination of approximations
- Comparison with other methods
- Dynamic optimal portfolio in incomplete markets
- Barrier type derivatives
- Levy processes