Implied volatility at long maturities

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Black–Scholes: The price of a European call option with strike $K$ and maturity $T$

$$C_t(K, T) = S_t \Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

- $S_t$ underlying stock price (no dividends)
- $r$ risk-free yield
- $d_\pm = \frac{\log(S_t/K)}{\sigma \sqrt{T-t}} + \left(\frac{r}{\sigma} \pm \frac{\sigma}{2}\right) \sqrt{T-t}$,
- $\Phi(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt$
\( \sigma \) is the volatility of the underlying stock. Unlike other parameters, not directly observed.

\[
\sigma^2 = \frac{1}{T} \text{Var}(\log S_T)
\]

- Liquid options priced by market already.
- Options often quoted in terms of implied volatility.
The assumptions

- No arbitrage (existence of martingale measure)
- Calls of all maturities and strikes liquidly traded
- Zero interest rate

Let \((S_t)_{t \geq 0}\) be a non-negative local martingale.

Price of European call option with strike \(K\) and maturity \(T\)

\[ C_t(K, T) = S_t - \mathbb{E}[S_T \wedge K | \mathcal{F}_t] \]
Warning:

\[
\mathbb{E}[(S_T - K)^+|\mathcal{F}_t] = \mathbb{E}[S_T - S_T \wedge K|\mathcal{F}_t] \\
\leq S_t - \mathbb{E}[S_T \wedge K|\mathcal{F}_t] \\
= C_t(T, K)
\]

with equality if and only if $S$ is a true martingale.
Black–Scholes call price function

$$BS(k, \nu) = \Phi \left( -\frac{k}{\sqrt{\nu}} + \frac{\sqrt{\nu}}{2} \right) - e^k \Phi \left( -\frac{k}{\sqrt{\nu}} - \frac{\sqrt{\nu}}{2} \right)$$

**Definition**

The random variable $\Sigma_t(k, \tau)$ is defined on $\{S_t > 0\}$ by

$$\mathbb{E} \left[ \frac{S_{t+\tau}}{S_t} \wedge e^k \mid \mathcal{F}_t \right] = 1 - BS(k, \tau \Sigma_t(k, \tau)^2)$$
**Assumption:** \( P(S_t > 0) > 0 \) all \( t \geq 0 \), but \( S_t \to 0 \) almost surely.

Equivalently:

- There exists a \( k \in \mathbb{R} \) such that \( \tau \Sigma(k, \tau)^2 \uparrow \infty \)
- \( \tau \Sigma(k, \tau)^2 \uparrow \infty \) for all \( k \in \mathbb{R} \).
- \( C(K, T) \uparrow S_0 \) for all \( K > 0 \).
- There exists a \( K > 0 \) such that \( C(K, T) \uparrow S_0 \)
With no loss, let $S_0 = 1$.

**Theorem**

\[
\tau \Sigma(k, \tau)^2 = -8 \log \mathbb{E}(S_\tau \wedge e^k) - 4 \log[- \log \mathbb{E}(S_\tau \wedge e^k)] + 4k - 4 \log \pi + \epsilon(k, \tau)
\]

where

\[
\sup_{-M \leq k \leq M} |\epsilon(k, \tau)| + \sup_{-M \leq k_1 < k_2 \leq M} \frac{|\epsilon(k_2, \tau) - \epsilon(k_1, \tau)|}{k_2 - k_1} \rightarrow 0
\]

for all $M > 0$. 

Corollary

\[
\sup_{k \in [-M,M]} \left| \frac{\tau \sum(k, \tau)^2}{-8 \log \mathbb{E}(S_{\tau} \wedge 1) - 1} - 1 \right| \to 0
\]

as \( \tau \uparrow \infty \) for all \( M > 0 \).
Long implied volatility can never fall

Theorem (Rogers–T 2008)

For any $k_1, k_2 \in \mathbb{R}$ we have

$$\limsup_{\tau \uparrow \infty} \Sigma_t(k_1, \tau) - \Sigma_s(k_2, \tau) \geq 0$$

for $t \geq s \geq 0$. There exist examples for which the inequality is strict.
Theorem (Dybvig–Ingersoll–Ross 1996)

Let $f_t(\tau)$ be the instantaneous forward interest rate with long rate

$$\limsup_{\tau \uparrow \infty} f_t(\tau) = \ell_t.$$

Then

$$\ell_s \leq \ell_t$$

for $0 \leq s \leq t$.

Theorem (Rogers–T. 2008)

Suppose

\[ \Sigma_t(k, \tau) = \Sigma_0(k, \tau) + \xi_t \]

for some process \((\xi_t)_{t \geq 0}\).

- Then \( \xi_t \geq 0 \).
- If \( \log \mathbb{E}(S_{s}^{1/2}) + \log \mathbb{E}(S_{t}^{1/2}) \leq \log \mathbb{E}(S_{s+t}^{1/2}) \), then \( \xi_t = 0 \).
Theorem (Balland 2002)

If $\xi_t = 0$ for all $t \geq 0$ then $\log S$ has independent, stationary increments.
Corollary

Let $Q$ be the measure locally equivalent to $P$ with density 

$$
\frac{dQ_t}{dP_t} = S_t.
$$

Then

$$
\frac{\partial}{\partial k} \tau \Sigma(k, \tau)^2 = 4 \left( \frac{Q(S_\tau < e^k) - e^k P(S_\tau \geq e^k)}{Q(S_\tau < e^k) + e^k P(S_\tau \geq e^k)} \right) + \epsilon'(k, \tau)
$$

if the distribution of $S_\tau$ continuous at $e^k$. In particular,

$$
\limsup_{\tau \uparrow \infty} \sup_{k \in [-M, M]} \left| \frac{\partial}{\partial k} \tau \Sigma(k, \tau)^2 \right| \leq 4
$$

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For comparison:

\[ \frac{\partial}{\partial k} \Sigma(k, \tau)^2 < \frac{4}{\tau} \quad \text{for all } k \geq 0 \]

\[ \frac{\partial}{\partial k} \Sigma(k, \tau)^2 > -\frac{4}{\tau} \quad \text{for all } k \leq 0 \]

\[ \frac{\partial}{\partial k} \Sigma(k, \tau)^2 < \frac{2}{\tau} \quad \text{for all } k \geq k_+(\tau), \text{ for some } k_+ \]

\[ \frac{\partial}{\partial k} \Sigma(k, \tau)^2 \geq -\frac{2}{\tau} \quad \text{for all } k \leq k_-(\tau) \text{ for some } k_- \]

Motivation:

\[ S \wedge 1 \leq S^p \]

for all \( 0 \leq p \leq 1 \) and \( S \geq 0 \), implies

\[
\liminf_{\tau \to \infty} \frac{\tau \sum (k, \tau)^2}{-8 \inf_{0 \leq p \leq 1} \log \mathbb{E}(S^p_\tau)} \geq 1.
\]
Let

$$\psi_t(p) = \log \mathbb{E}(S_t^p \mathbb{1}_{\{S_t>0\}}).$$

Properties of $\psi_t$

- $\psi_t(0) = \log \mathbb{P}(S_t > 0) \leq 0$, $\psi_t(1) = \log \mathbb{E}(S_t) \leq 0$
- finite-valued on $(0, 1) \Rightarrow$ real-analytic
- convex
**Intuition:** For many models

\[
\frac{1}{t} \psi_t(p) \to \bar{\psi}(p)
\]

Let \( p^* \) be the minimizer of \( \bar{\psi} \). Three cases

- \( 0 < p^* < 1 \)
- \( p^* = 0 \) or \( p^* = 1 \)
- \( p^* < 0 \) or \( p^* > 1 \)
**Assumption:** There exists a $0 < p^* < 1$ and a positive increasing function $C$ with $C(\tau) \uparrow \infty$ such that

$$\psi_{\tau} \left( p^* + i \frac{\theta}{C(\tau)} \right) - \psi_{\tau}(p^*) \to -\theta^2 / 2$$

as $\tau \uparrow \infty$ for all real $\theta$
Theorem

\[
\sup_{k \in [-M,M]} \left| \frac{\tau \Sigma(k,\tau)^2}{-8 \psi(\tau^{p^*})} - 1 \right| \to 0
\]

Proof: Cramér’s large deviation principle.

\[ \phi_{\tau}(\theta) = \frac{1}{\sqrt{2\pi}} \exp \left[ \psi_{\tau} \left( p^* + i \frac{\theta}{C(\tau)} \right) - \psi_{\tau}(p^*) \right]. \]

**Theorem**

If

\[ \int_{-\infty}^{\infty} \frac{|\phi_{\tau}(\theta)|}{1 + \theta^2/C(\tau)^2} d\theta \to 1 \]

then

\[ \tau \Sigma(k, \tau)^2 = -8\psi_{\tau}(p^*) + 4k(2p^*-1) + 8 \log \left( \frac{C(\tau)p^*(1-p^*)}{\sqrt{-\psi_{\tau}(p^*)/2}} \right) + \delta(k, \tau) \]

where \( \sup_{k \in [-M, M]} |\delta(k, \tau)| \to 0 \) as \( \tau \uparrow \infty \) for each \( M > 0 \).
**Assumption:** There exists a $p^* \in \{0, 1\}$ and a positive increasing function $C$ with $C(\tau) \uparrow \infty$ such that

$$\psi_\tau \left( p^* + i \frac{\theta}{C(\tau)} \right) - \psi_\tau(p^*) \rightarrow -\theta^2 / 2$$

as $\tau \uparrow \infty$ for all real $\theta$
Theorem
If $p^* = 1$ then

$$
\tau \Sigma(k, \tau)^2 = -8\psi_\tau(1) - 4 \log[-\psi_\tau(1)] + 4k - 4 \log(\pi/4) + \delta(k, \tau),
$$

and if $p^* = 0$, then

$$
\tau \Sigma(k, \tau)^2 = -8\psi_\tau(0) - 4 \log[-\psi_\tau(0)] - 4k - 4 \log(\pi/4) + \delta(k, \tau)
$$

where $\sup_{k \in [-M, M]} |\delta(k, \tau)| \to 0$ for all $M > 0$. 
Assumption: There exists a $p^*$ such that either

1. $p^* > 1$ and $\psi_{\tau}(p^*) - \psi_{\tau}(1) \to -\infty$, or
2. $p^* < 0$ and $\psi_{\tau}(p^*) - \psi_{\tau}(0) \to -\infty$. 
Theorem

If \( p^* > 1 \) then

\[ \tau \Sigma(\tau, k)^2 = -8\psi_\tau(1) - 4 \log[-\psi_\tau(1)] + 4k - 4 \log \pi + \delta(k, \tau), \]

and if \( p^* < 0 \) then

\[ \tau \Sigma(\tau, k)^2 = -8\psi_\tau(0) - 4 \log[-\psi_\tau(0)] - 4k - 4 \log \pi + \delta(k, \tau), \]

where \( \sup_{k \in [-M, M]} |\delta(k, \tau)| \to 0 \) for all \( M > 0 \).
Example: Independent, stationary increments

\[
\frac{1}{t} \psi_t(p) = \psi_1(p)
\]

If

\[
\inf_{q \neq 0} \frac{\Re \psi_1(p^* + iq) - \psi_1(p^*)}{q^2 \wedge 1} < 0,
\]

where \( \Re \) denotes the real part of a complex number, then the full asymptotic formula holds.
Example: Binomial model. Suppose \( S_{\tau+1} = \xi_{\tau+1} S_\tau \) where 
\[
P(\xi_\tau = e^b) = \frac{1}{e^b+1} = 1 - P(\xi_\tau = e^{-b})
\]
so
\[
\psi_1(p) = \log \left( \frac{\cosh[b(p - 1/2)]}{\cosh(b/2)} \right).
\]

Integrability fails:
\[
\tau \Sigma(k, \tau)^2 = 8\tau \log \cosh(b/2) + 4 \log \left( \frac{b^2 F(k, \tau)^2}{8 \log \cosh(c/2)} \right) + \delta(k, \tau)
\]

where
\[
F(k) = \sum_{n \in \mathbb{Z}} \frac{(-1)^{n\tau} \cos(kn\pi/b)}{1 + 4n^2\pi^2/b^2} \neq 1.
\]
Example: CEV model.

\[ dS_t = S_t^2 dW_t. \]

Note

\[ \mathbb{E}(S_t) = 2\Phi \left( \frac{1}{\sqrt{t}} \right) - 1, \]

Irregular case with \( p^* > 1 \)

\[ \tau \Sigma(k, \tau)^2 = 4 \log \tau - 4 \log \log \tau + 4k + \delta(k, \tau). \]