A convergence result in the Emery topology and another proof of FTAP

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Introduction

Youri Kabanov's abstract setting

A guided tour through the proof of FTAP

(NUPBR) implies the (P-UT) property

How the P-UT property leads to convergence in the Emery topology

An extension towards large financial markets

FTAP and its history

- The fundamental theorem of asset pricing (FTAP) is the single most important result in mathematical Finance.
- It states the equivalence of an “absence of arbitrage” property (NFLVR) with the existence of an equivalent separating measure.
- The first complete proof has been presented by F. Delbaen and W. Schachermayer in [1, 2].
- The proof is beautiful, impressive and tricky. No essential simplification has been obtained since then, but it was realized soon that the presented proof is almost literally actually valid in a more general situation.
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- Discuss the proof in Y. Kabanov's setting.
- Present a general principle for sequences of semi-martingales, which allows to conclude from pathwise uniform convergence in probability ("up-convergence") the desired convergence in the Emery topology (this is an L^0-interpretation of BDG-inequalities).
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- We consider a finite time horizon $T = 1$ and a fixed probability space with usual conditions $(\Omega, \mathcal{F}, \mathbb{P})$.
- The set of semi-martingales on $[0, 1]$ starting at 0 is denoted by $\mathbb{S}$.
- We equip $\mathbb{S}$ with the Emery metric

$$\sup_{H \in b\mathcal{E}, \|H\| \leq 1} E[|(H \bullet (X - Y))|_1^* \wedge 1] = d_E(X, Y)\,,$$

making it a complete metric space.
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Definition

We consider a convex set $\mathcal{X}_1 \subset \mathbb{S}$ of semi-martingales starting at 0 and bounded from below by -1, which is closed in the Emery topology.

We assume that for all bounded, predictable strategies $H, G \geq 0$, $X, Y \in \mathcal{X}_1$ with $HG = 0$ and $Z = (H \bullet X) + (G \bullet Y) \geq -1$, it holds that $Z \in \mathcal{X}_1$ ("concatenation property").

We denote $\mathcal{X} = \cup_{\lambda > 0} \lambda \mathcal{X}_1$ and call its elements *admissible portfolio wealth processes*. We denote K_0, respectively K_0^1 the evaluations of elements of $\mathcal{X}$, respectively $\mathcal{X}_1$, at final time $T = 1$.

Notions of No Arbitrage

(NA) The set $\mathcal{X}$ is said to satisfy No Arbitrage if $K_0 \cap L^0_{\geq 0} = \{0\}$ which can be shown to be equivalent to $C \cap L^\infty_{\geq 0} = \{0\}$, with $C = (K_0 - L^0_{\geq 0}) \cap L^\infty$.

(NFLVR) The set $\mathcal{X}$ is said to satisfy No free lunch with vanishing risk if

$$\overline{C} \cap L^\infty_{\geq 0} = \{0\},$$

where $\overline{C}$ denotes the norm closure in L^∞.

(NFL) The set $\mathcal{X}$ is said to satisfy No free lunch if

$$\overline{C}^* \cap L^\infty_{\geq 0} = \{0\},$$

where $\overline{C}^*$ denotes the weak-$*$-closure in L^∞.

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Definition

The set $\mathcal{X}$ satisfies the (ESM) (equivalent separating measure) property if there exists an equivalent measure $Q \sim P$ such that $\mathbb{E}_Q[X_1] \leq 0$ for all $X \in \mathcal{X}$.

(NFL) implies (ESM)

- It is a consequence of Hahn-Banach's Theorem (the Kreps-Yan Theorem) that (NFL) implies the existence of an equivalent measure $Q \sim P$ such that $E_Q[f] \leq 0$ for all $f \in C$ and hence for all $f \in K_0$.

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- Apparently it holds that

$$(NFL) \Rightarrow (NFLVR) \Rightarrow (NA)\,,$$

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  but it is a deep insight that under (NFLVR) it holds that $C = \overline{C}^*$, i.e. the cone C is already weak-$*$-closed and (NFL) holds.
- The goal is to show (NFLVR) $\Rightarrow C = \overline{C}^*$.
  Recall (NFLVR) $\Leftrightarrow$ (NA) $+$ (NUPBR).

1. The convex cone C is closed with respect to the weak-$*$-topology if and only if C_0 is Fatou-closed, i.e. for any sequence (f_n) in C_0 bounded from below and converging almost surely to f_0 it holds that $f_0 \in C$.

2. Take now $-1 \leq f_n \in C_0$ converging almost surely to f. Then we can find $f_n \leq g_n = Y_1^n$ with $Y^n \in \mathcal{X}$.

3. By (NA) it follows that $Y^n \in \mathcal{X}_1$.

4. By (NUPBR) it follows that there are forward-convex combinations $\widetilde{Y^n} \in \mathrm{conv}(Y^n, Y^{n+1}, \dots)$ such that $\widetilde{Y_1^n} \to \widetilde{h}_0 \geq f$ almost surely.

5. Again by (NUPBR) it follows that we can find a sequence of semi-martingales $X^n \in \mathcal{X}_1$ such that $X_1^n \to h_0$ almost surely and h_0 is maximal above f with this property.

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2. It is now the goal to show that indeed $X^n \to X$ in the Emery topology, an apparently much stronger statement. Convergence in the Emery topology can be shown with respect to any equivalent measure $Q \sim P$, since this notion of convergence only depends on the equivalence class of probability measures.
3. By the basic convergence result (1) (and passing to a subsequence) we know that $\xi := \sup_n |X^n|_1^* \in L^0$. We can therefore find a measure $Q \sim P$ (take, e.g., $dQ/dP = c\exp(-\xi)$) such that $X^n \in L^2(Q)$, hence we can continue the analysis with L^2-methods, in order to prove Emery-convergence with respect to Q. Now the proof starts!

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Assume (NUPBR), take a sequence of (special) semi-martingales $X^n = A^n + M^n$ whose sup-processes are uniformly bounded in L^2.

1. First key Lemma: the sequence $|M^n|^*$ is bounded in L^0.
2. Second key Lemma: define $\tau_c^n := \inf\{t \mid |M^n|^* > c\}$ for some $c > 0$, $X_c^n := (1_{[\tau_c^n,\infty]} \bullet X^n)$, then for every $\epsilon > 0$ there is $c_0 > 0$ such that for all
$$\widetilde{X} \in \cup_{c \geq c_0} \operatorname{conv}(X_c^1, \ldots, X_c^n, \ldots)$$
it holds that $Q[|\widetilde{M}|^* > \epsilon] \leq \epsilon$.
3. Third key Lemma: for every $\delta > 0$ there is $c_0 > 0$ such that for all $\widetilde{X} \in \cup_{c \geq c_0} \operatorname{conv}(X_c^1, \ldots, X_c^n, \ldots)$ it holds that $d_E(\widetilde{M}, 0) \leq \delta$.
4. Fourth key Lemma: there exist $\widetilde{X^n} \in \operatorname{conv}(X_n, \ldots)$ such that $\widetilde{M^n} \to \widetilde{M}$ in the Emery topology.

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Proposition on the Emery convergence of the finite variation part

Assume (NUPBR). Let $\widetilde{X^n} = \widetilde{M^n} + \widetilde{A^n} \in \mathcal{X}_1$ be a sequence of special semi-martingales converging to a maximal element h_0 such that $\widetilde{M^n} \to \widetilde{M}$ converges in the Emery topology, then $\widetilde{A^n} \to \widetilde{A}$ in the Emery topology.

From this proposition it follows by the fact that *the set $\mathcal{X}_1$ is closed in the Emery topology* that $f_0 \in C_0$.

Discussion of the proof

- the proof is beautiful but quite tricky.
- the change of measure is technical and not fully motivated from the point of view of mathematical finance.
- it remains open within the proof if the forward convex combination passing from X^n to $\widetilde{X^n}$ are really necessary or if $X^n \to X$ already in the Emery topology.
- the series of key lemmas would deserve a theorem or property on its own.
- it would be interesting to obtain proofs, which can be easier communicated from a finance point of view.

We take the following important definition from Jacod/Shiryaev:

Definition

We say that a sequence $(X^n)_{n\geq 0}$ of adapted, càdlàg satisfies the P-UT property (predictably uniformly tight) if the family of random variables $\{(H \bullet X^n)_1 \,:\, H \in b\mathcal{E}, \|H\| \leq 1, n \geq 0\}$ is bounded in L^0, that is,

$$\sup_{H\in b\mathcal{E},\|H\|\leq 1} \sup_{n\geq 0} P[|(H \bullet X^n)|_t \geq c] \to 0 \,.$$

as $c \to \infty$.

The heart of our considerations now consists in proving that (NUPBR) implies P-UT for sequences of semi-martingales $X^n \to X$ converging uniformly along paths in probability. From this it will be (relatively) short way towards the existence of an equivalent separating measure.

Denote by $\check{X}$ the process of jumps, whose absolute values are greater than some $C > 0$, that is,

$$\check{X}_t = \sum_{s \le t} \Delta X_s 1_{\{|\Delta X_s| > C\}} \,. \tag{1}$$

Lemma

Let $(X^n)_{n \ge 0}$ together with an adapted, càdlàg process X such that $|X^n - X|_1^* \to 0$ in probability as $n \to \infty$. Then the sequence $(\mathrm{TV}(\check{X}_1^n))_{n \ge 0}$ of total variations of $\check{X}^n$ is bounded in L^0, i.e., for every $\varepsilon > 0$ there exists some $c > 0$ such that

$$\sup_n \mathbb{P}\left[\sum_{s \le 1} |\Delta X_s^n| 1_{\{|\Delta X_s^n| > C\}} \ge c\right] \le \varepsilon.$$

Moreover, the sequence $(\check{X}^n)_{n \ge 0}$ satisfies the P-UT property.

Theorem

Assume (NUPBR). Let $(X^n)_{n\geq 0}$ together with an adapted, càdlàg process X such that $|X^n - X|_1^* \to 0$ in probability as $n \to \infty$ be a sequence in $\mathcal{X}_1$.

1. *Then for every $C > 0$ there exists a decomposition $X^n = M^n + B^n + \check{X}^n$ into a local martingale M^n, a predictable, finite variation process B^n and the finite variation process $\check{X}^n$, for $n \geq 0$, such that jumps of M^n and B^n are bounded by $2C$ uniformly in n.*
2. *The sequence $(|M^n|_1^*)_{n\geq 0}$ is bounded in L^0 and $(M^n)_{n\geq 0}$ satisfies P-UT (first key lemma).*
3. *The sequence $(\mathrm{TV}(B^n)_1)_{n\geq 0}$ of total variations of B^n is bounded in L^0 and $(B^n)_{n\geq 0}$ satisfies P-UT (the analogous statement on the finite variation part).*
4. *The sequence $(X^n)_{n\geq 0}$ satisfies P-UT.*

Proof

In contrast to the previous key lemmas, the proofs here have some straight forward aspect:

- (NUPBR) implies P-UT is based on the first key lemma with an additional analysis of the finite variation part.
- the P-UT property is a natural boundedness property in the Emery topology. It is therefore natural to investigate this property first.

YAP – a finance view point

Definition

A positive càdlàg adapted process D is called supermartingale deflator for $1+\mathcal{X}_1$ if D is strictly positive, $D_0 \leq 1$ and $D(1+X)$ is a supermartigale for all $X \in \mathcal{X}_1$.

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Theorem (Karatzas and Kardaras (2007)/ Kardaras (2013))

Assume (NUPBR) for $\mathcal{X}$, then there exists a supermartingale deflator D.

(P-UT) property for supermartingales

Lemma

Let (Z^n) be a sequence of non-negative supermartingales such that $Z_0^n \leq K$ for all $n \in \mathbb{N}$ and some $K > 0$. Then (Z^n) satisfies the P-UT property.

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Let (Z^n) be a sequence of non-negative supermartingales such that $Z_0^n \leq K$ for all $n \in \mathbb{N}$ and some $K > 0$. Then (Z^n) satisfies the P-UT property.

Proof.

By an inequality of Burkholder for non-negative supermartingales S and processes $H \in b\mathcal{E}$ with $\|H\| \leq 1$ it holds that

$$cP[|(H \bullet S)|_1^* \geq c] \leq 9\mathbb{E}[|S_0|]$$

for all $c \geq 0$. Applying this inequality to Z^n and letting $c \to \infty$ yields the P-UT property. $\square$

(P-UT) property for sequences in $\mathcal{X}_1$

Proposition

Let $\mathcal{X}$ satisfy (NUPBR) and let $X^n \in \mathcal{X}_1$ be a sequence of semimartingales. Then (X^n) satisfies the P-UT property.

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Let $\mathcal{X}$ satisfy (NUPBR) and let $X^n \in \mathcal{X}_1$ be a sequence of semimartingales. Then (X^n) satisfies the P-UT property.

Proof.

The (P-UT) property of the supermartingales $(Z^n) := (D(1+X^n))$ can be easily transferred to the sequence (X^n). It relies on Itô's integration by parts formula and the fact that $(H^n_- \bullet S^n)$ satisfies (P-UT), if (S^n) is a sequence of semimartingales satisfying (P-UT) and (H^n) a sequence of adapted càdlàg processes such that $(|H^n|^*_1)_n$ is bounded in L^0. $\square$

Emery convergence for the local martingale and the big jump part under (P-UT) and up-convergence

For a sequence of semimartingales (X^n) with $X_0^n = 0$ and some $C > 0$ let us consider the following decomposition

$$X^n = B^{n,C} + M^{n,C} + \check{X}^{n,C}. \tag{2}$$

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Theorem (Memin and Slominski (1991))

Let (X^n) be a sequence of semimartingales with $X_0^n = 0$, which converges pathwise uniformly in probability to X and satisfies the (P-UT) property. Then there exists some $C > 0$ such that $M^{n,C} \to M^C$ and $\check{X}^{n,C} \to \check{X}^C$ in the Emery topology and $B^{n,C} \to B^C$ pathwise uniformly in probability.

Emery convergence for the finite variation part (without big jumps)

Proposition

Let $\mathcal{X}$ satisfy (NUPBR) and let (X^n) be a sequence in $\mathcal{X}_1$, which converges pathwise uniformly in probability to X such that X_1 is a maximal element in $\widehat{K_0^1}$. Assume that $M^{n,C} \to M^C$ and $\check{X}^{n,C} \to \check{X}^C$ in the Emery topology. Then $B^{n,C} \to B^C$ in the Emery topology.

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Proof.

This follows essentially the proposition on Emery convergence in FTAP proof if martingale parts are known to converge already. $\square$

A convergence result in the Emery topology

Combining the above assertions yields...

Theorem

Let $\mathcal{X}$ satisfy (NUPBR) and let (X^n) be a sequence in $\mathcal{X}_1$, which converges pathwise uniformly in probability to X such that X_1 is a maximal element in $\widehat{K_0^1}$. Then $X^n \to X$ in the Emery topology.

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Proof.

This follows from ((NUPBR) $\Rightarrow$ (P-UT)), Memin and Slominski's theorem together with the previous result. $\square$

Proof variant of FTAP

The previous considerations lead to the following structure of the proof:

- Portfolios of the form 1 plus 1-admissible admit a supermartingale deflator under (NUPBR).
- A set of non-negative semimartingales admitting a supermartingale deflator satisfies (P-UT).
- Take a sequence (X^n) of 1-admissible portfolios satisfying (P-UT) and converging uniformly pathwise in probability to a semi-martingale with maximal terminal value, then (X^n) converges in the Emery topology (Burkholder-Davis-Gundy type of conclusion beyond martingales!).
- This allows to conclude that C is already weak-$*$-closed if uniformly closed!

An extension towards large financial markets

Definition

We consider an increasing sequence of convex set $\mathcal{X}_1^n \subset \mathbb{S}$ of semi-martingales starting at 0 and bounded from below by -1.

For each fixed n it holds that for all bounded, predictable strategies $H, G \geq 0$, $X, Y \in \mathcal{X}_1^n$ with $HG = 0$ and $Z = (H \bullet X) + (G \bullet Y) \geq -1$, it holds that $Z \in \mathcal{X}_1^n$ ("concatenation property" for each n).

Define $\mathcal{X}_1 = \overline{\cup_{n \geq 1} \mathcal{X}_1^n}$ as the Emery closure of the union.

We denote $\mathcal{X} = \cup_{\lambda > 0} \lambda \mathcal{X}_1$ and call its elements *asymptotically admissible (portfolio) wealth processes*. We denote K_0, respectively K_0^1 the evaluations of elements of $\mathcal{X}$, respectively $\mathcal{X}_1$, at final time $T = 1$.

FTAP for large financial markets

In complete analogy to small financial markets we define $C \cap L^\infty_{\geq 0} = \{0\}$, with $C = (K_0 - L^0_{\geq 0}) \cap L^\infty$. for a set of asymptotically admissible portfolio wealth processes.

The set $\mathcal{X}$ is said to satisfy No (asymptotic) free lunch with vanishing risk if

$$\overline{C} \cap L^\infty_{\geq 0} = \{0\},$$

where $\overline{C}$ denotes the norm closure in L^∞.

Theorem

If (NAFLVR) holds true, then $C = \overline{C}^*$ and there exists an equivalent separating measure Q such that $E_Q[X_1] \leq 0$ for all $X \in \mathcal{X}$, in particular for terminal values of portfolios stemming from small markets.

Conclusions

- It appears that the conclusions of the key lemmas can be replaced by the P-UT property for converging sequences in $\mathcal{X}_1$ – the P-UT property summarizes their mathematical contents.
- Given a super-martingale deflator, which is a quite natural object for $1 + \mathcal{X}_1$ provided (NUPBR) holds true, the P-UT property is an easy consequence of a Burkholder's inequality for super-martingales.
- the middle part appears as an L^0 version of BDG inequalities for semi-martingales.
- characterization of existence of ESM for (some) large financial markets (compare De Donno/Guasoni/Pratelli).

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